



INTERDISCIPLINARY
MATHEMATICS
INSTITUTE

2018:02

Gaussian Bounds for the Heat
Kernels on the Ball and Simplex:
Classical Approach

G. Kerkyacharian, P. Petrushev and
Y. Xu

IMI

PREPRINT SERIES

COLLEGE OF ARTS AND SCIENCES
UNIVERSITY OF SOUTH CAROLINA

GAUSSIAN BOUNDS FOR THE HEAT KERNELS ON THE BALL AND SIMPLEX: CLASSICAL APPROACH

GERARD KERKYACHARIAN, PENCHO PETRUSHEV, AND YUAN XU

ABSTRACT. Two-sided Gaussian bounds are established for the weighted heat kernels on the unit ball and simplex in \mathbb{R}^d generated by classical differential operators whose eigenfunctions are algebraic polynomials.

1. INTRODUCTION

Two-sided Gaussian bounds have been established for heat kernels in various settings. For example, Gaussian bounds for the Jacobi heat kernel on $[-1, 1]$ with weight $(1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, are obtained in [2, Theorem 7.2] and [8, Theorem 5.1], and also in [9] in the case when $\alpha, \beta \geq -1/2$ (see (1.24) below).

In this article we establish two-sided Gaussian estimates for the heat kernels generated by classical differential operators whose eigenfunctions are algebraic polynomials in the weighted cases on the unit ball and simplex in \mathbb{R}^d . Such estimates are also established in [8] using a general method that utilizes known two-sided Gaussian estimates for the heat kernels generated by weighted Laplace operators on Riemannian manifolds. Here we derive these results directly from the Gaussian bounds for the Jacobi heat kernel. Such a direct method leads to working in somewhat restricted range for the parameters of the weights (commonly used in the literature). We next describe our results in detail.

We shall use standard notation. In particular, positive constants will be denoted by $c, c', \tilde{c}, c_1, c_2, \dots$ and they may vary at every occurrence. Most constants will depend on parameters that will be clear from the context. The notation $a \sim b$ will stand for $c_1 \leq a/b \leq c_2$. The functions that we deal with in this article are assumed to be real-valued.

1.1. Heat kernel on the unit ball. Consider the operator

$$(1.1) \quad \mathcal{D}_\mu := \sum_{i=1}^d (1-x_i^2) \partial_i^2 - 2 \sum_{1 \leq i < j \leq d} x_i x_j \partial_i \partial_j - (d+2\mu) \sum_{i=1}^d x_i \partial_i,$$

acting on sufficiently smooth functions on the unit ball $\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| < 1\}$ in \mathbb{R}^d equipped with the measure

$$(1.2) \quad d\nu_\mu = w_\mu(x) dx := (1 - \|x\|^2)^{\mu-1/2} dx, \quad \mu \geq 0,$$

and the distance

$$(1.3) \quad d_{\mathbb{B}}(x, y) := \arccos(\langle x, y \rangle + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2}),$$

Date: January 21, 2018.

2010 Mathematics Subject Classification. 42C05, 35K08.

Key words and phrases. Heat kernel, Gaussian bounds, orthogonal polynomials, ball, simplex.

where $\langle x, y \rangle$ is the inner product of $x, y \in \mathbb{R}^d$ and $\|x\| := \sqrt{\langle x, x \rangle}$. As will be shown the operator \mathcal{D}_μ is symmetric and $-\mathcal{D}_\mu$ is positive in $L^2(\mathbb{B}, w_\mu)$. Furthermore, \mathcal{D}_μ is essentially self-adjoint.

Denote

$$(1.4) \quad B_{\mathbb{B}}(x, r) := \{y \in \mathbb{R}^d : d_{\mathbb{B}}(x, y) < r\} \quad \text{and} \quad V_{\mathbb{B}}(x, r) := \nu_\mu(B_{\mathbb{B}}(x, r)).$$

As is well known (see, e.g. [3, Lemma 11.3.6])

$$(1.5) \quad V_{\mathbb{B}}(x, r) \sim r^d(1 - \|x\|^2 + r^2)^\mu.$$

Denote by $\mathcal{V}_n(w_\mu)$ the set of all algebraic polynomials of degree n in d variables that are orthogonal to lower degree polynomials in $L^2(\mathbb{B}^d, w_\mu)$, and let $\mathcal{V}_0(w_\mu)$ be the set of all constants. As is well known (see e.g. [5, §2.3.2]) $\mathcal{V}_n(w_\mu)$, $n = 0, 1, \dots$, are eigenspaces of the operator \mathcal{D}_μ , more precisely,

$$(1.6) \quad \mathcal{D}_\mu P = -n(n + d + 2\mu - 1)P, \quad \forall P \in \mathcal{V}_n(w_\mu).$$

Let $P_n(w_\mu; x, y)$ be the kernel of the orthogonal projector onto $\mathcal{V}_n(w_\mu)$. Then the semigroup $e^{t\mathcal{D}_\mu}$, $t > 0$, generated by \mathcal{D}_μ has a (heat) kernel $e^{t\mathcal{D}_\mu}(x, y)$ of the form

$$(1.7) \quad e^{t\mathcal{D}_\mu}(x, y) = \sum_{n=0}^{\infty} e^{-tn(n+2\lambda)} P_n(w_\mu; x, y), \quad \lambda := \mu + (d-1)/2.$$

We establish two-sided Gaussian bounds on $e^{t\mathcal{D}_\mu}(x, y)$:

Theorem 1.1. *For any $\mu \geq 0$ there exist constants $c_1, c_2, c_3, c_4 > 0$ such that for all $x, y \in \mathbb{B}^d$ and $t > 0$*

$$(1.8) \quad \frac{c_1 \exp\{-\frac{d_{\mathbb{B}}(x, y)^2}{c_2 t}\}}{[V_{\mathbb{B}}(x, \sqrt{t})V_{\mathbb{B}}(y, \sqrt{t})]^{1/2}} \leq e^{t\mathcal{D}_\mu}(x, y) \leq \frac{c_3 \exp\{-\frac{d_{\mathbb{B}}(x, y)^2}{c_4 t}\}}{[V_{\mathbb{B}}(x, \sqrt{t})V_{\mathbb{B}}(y, \sqrt{t})]^{1/2}}.$$

1.2. Heat kernel on the simplex. We also establish two-sided Gaussian bounds for the heat kernel generated by the operator

$$(1.9) \quad \mathcal{D}_\kappa := \sum_{i=1}^d x_i \partial_i^2 - \sum_{i=1}^d \sum_{j=1}^d x_i x_j \partial_i \partial_j + \sum_{i=1}^d (\kappa_i + \frac{1}{2} - (|\kappa| + \frac{n+1}{2})x_i) \partial_i$$

with $|\kappa| := \kappa_1 + \dots + \kappa_{d+1}$ acting on sufficiently smooth functions on the simplex

$$\mathbb{T}^d := \left\{ x \in \mathbb{R}^d : x_1 \geq 0, \dots, x_d \geq 0, |x| \leq 1 \right\}, \quad |x| := x_1 + \dots + x_d,$$

in \mathbb{R}^d , $d \geq 1$, equipped with the measure

$$(1.10) \quad d\nu_\kappa(x) = w_\kappa(x) dx := \prod_{i=1}^d x_i^{\kappa_i - 1/2} (1 - |x|)^{\kappa_{d+1} - 1/2} dx, \quad \kappa_i \geq 0,$$

and the distance

$$(1.11) \quad d_{\mathbb{T}}(x, y) := \arccos \left(\sum_{i=1}^d \sqrt{x_i y_i} + \sqrt{1 - |x|} \sqrt{1 - |y|} \right).$$

As will be shown the operator \mathcal{D}_κ is symmetric and $-\mathcal{D}_\kappa$ is positive in the weighted space $L^2(\mathbb{T}, w_\kappa)$, furthermore, \mathcal{D}_κ is essentially self-adjoint.

We shall use the notation:

$$(1.12) \quad B_{\mathbb{T}}(x, r) := \{y \in \mathbb{T}^d : \rho(x, y) < r\} \quad \text{and} \quad V_{\mathbb{T}}(x, r) := \nu_\kappa(B(x, r)).$$

It is known that

$$(1.13) \quad V_{\mathbb{T}}(x, r) \sim r^d (1 - |x| + r^2)^{\kappa_{d+1}} \prod_{i=1}^d (x_i + r^2)^{\kappa_i}.$$

This equivalence follows e.g. from [3, (5.1.10)], see also (4.23)-(4.24) in [8].

Denote by $\mathcal{V}_n(w_\kappa)$ the set of all algebraic polynomials of degree n in d variables that are orthogonal to lower degree polynomials in $L^2(\mathbb{T}^d, w_\kappa)$, and let $\mathcal{V}_0(w_\kappa)$ be the set of all constants. As is well known (e.g. [5, §2.3.3]) $\mathcal{V}_n(w_\kappa)$, $n = 0, 1, \dots$, are eigenspaces of the operator \mathcal{D}_κ , namely,

$$(1.14) \quad \mathcal{D}_\kappa P = -n(n + |\kappa| + (d-1)/2)P, \quad \forall P \in \mathcal{V}_n(w_\kappa), \quad n = 0, 1, \dots$$

Let $P_n(w_\kappa; x, y)$ be the kernel of the orthogonal projector onto $\mathcal{V}_n(w_\kappa)$ in $L^2(\mathbb{T}^d, w_\kappa)$. The heat kernel $e^{t\mathcal{D}_\kappa}(x, y)$, $t > 0$, takes the form

$$(1.15) \quad e^{t\mathcal{D}_\kappa}(x, y) = \sum_{n=0}^{\infty} e^{-tn(n+\lambda_\kappa)} P_n(w_\kappa; x, y), \quad \lambda_\kappa := |\kappa| + (d-1)/2.$$

Theorem 1.2. *For any $\kappa_i \geq 0$, $i = 1, \dots, n+1$, there are constants $c_1, c_2, c_3, c_4 > 0$ such that for all $x, y \in \mathbb{T}^d$ and $t > 0$*

$$(1.16) \quad \frac{c_1 \exp\{-\frac{d_{\mathbb{T}}(x, y)^2}{c_2 t}\}}{[V_{\mathbb{T}}(x, \sqrt{t})V_{\mathbb{T}}(y, \sqrt{t})]^{1/2}} \leq e^{t\mathcal{D}_\kappa}(x, y) \leq \frac{c_3 \exp\{-\frac{d_{\mathbb{T}}(x, y)^2}{c_4 t}\}}{[V_{\mathbb{T}}(x, \sqrt{t})V_{\mathbb{T}}(y, \sqrt{t})]^{1/2}}.$$

1.3. Method of proof and discussion. We shall prove Theorems 1.1 and 1.2 by using the known two-sided Gaussian bounds on the Jacobi heat kernel on $[-1, 1]$. We next describe this result. The classical Jacobi operator is defined by

$$(1.17) \quad L_{\alpha, \beta} f(x) := \frac{[w_{\alpha, \beta}(x)(1-x^2)f'(x)]'}{w_{\alpha, \beta}(x)},$$

where

$$w_{\alpha, \beta}(x) := (1-x)^\alpha (1+x)^\beta, \quad \alpha, \beta > -1.$$

We consider $L_{\alpha, \beta}$ with domain $D(L) := \mathcal{P}[-1, 1]$ the set of all algebraic polynomials restricted to $[-1, 1]$. We also consider $[-1, 1]$ equipped with the weighted measure

$$(1.18) \quad d\nu_{\alpha, \beta}(x) := w_{\alpha, \beta}(x)dx = (1-x)^\alpha (1+x)^\beta dx$$

and the distance

$$(1.19) \quad \rho(x, y) := |\arccos x - \arccos y|.$$

It is not hard to see that the Jacobi operator $L_{\alpha, \beta}$ in the setting described above is essentially self-adjoint and $-L_{\alpha, \beta}$ is positive in $L^2([-1, 1], w_{\alpha, \beta})$.

We shall use the notation

$$(1.20) \quad B(x, r) := \{y \in [-1, 1] : \rho(x, y) < r\} \quad \text{and} \quad V(x, r) := \nu_{\alpha, \beta}(B(x, r)).$$

As is well known (see e.g. [2, (7.1)])

$$(1.21) \quad V(x, r) \sim r(1-x+r^2)^{\alpha+1/2}(1+x+r^2)^{\beta+1/2}, \quad x \in [-1, 1], \quad 0 < r \leq \pi.$$

It is well known [12] that the Jacobi polynomials $P_n^{(\alpha, \beta)}$, $n = 0, 1, \dots$, are eigenfunctions of the operator $L_{\alpha, \beta}$, namely,

$$(1.22) \quad L_{\alpha, \beta} P_n^{(\alpha, \beta)} = -n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}, \quad n = 0, 1, \dots$$

We consider the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}$ normalised in $L^2([-1, 1], w_{\alpha,\beta})$. Then the Jacobi heat kernel $e^{tL_{\alpha,\beta}}(x, y)$, $t > 0$, takes the form

$$(1.23) \quad e^{tL_{\alpha,\beta}}(x, y) = \sum_{n=0}^{\infty} e^{-tn(n+\lambda)} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y), \quad \lambda := \alpha + \beta + 1.$$

Theorem 1.3. *For any $\alpha, \beta > -1$ there exist constants $c_1, c_2, c_3, c_4 > 0$ such that for all $x, y \in [-1, 1]$ and $t > 0$*

$$(1.24) \quad \frac{c_1 \exp\{-\frac{\rho(x,y)^2}{c_2 t}\}}{[V(x, \sqrt{t})V(y, \sqrt{t})]^{1/2}} \leq e^{tL_{\alpha,\beta}}(x, y) \leq \frac{c_3 \exp\{-\frac{\rho(x,y)^2}{c_4 t}\}}{[V(x, \sqrt{t})V(y, \sqrt{t})]^{1/2}}.$$

This theorem is established in [2, Theorem 7.2] using a general result on heat kernels in Dirichlet spaces with a doubling measure and local Poincaré inequality. The same theorem is also proved in [8, Theorem 5.1]. In [9] Nowak and Sjögren obtained this result in the case when $\alpha, \beta \geq -1/2$ via a direct method using special functions.

For the proof of Theorem 1.1 it will be critical that the kernel $P(w_\mu; x, y)$ of the orthogonal projector onto $\mathcal{V}_n(w_\mu)$ in $L^2(\mathbb{B}, w_\mu)$ has an explicit representation in terms of the univariate Gegenbauer polynomials (see (2.5)-(2.7)). For the proof of Theorem 1.2 we deploy the well known representation of the kernel $P_n(w_\kappa; x, y)$ in terms of Jacobi polynomials (see (3.3)).

It should be pointed out that our method of proof of estimates (1.8) and (1.16) works only in the range $\mu \geq 0$ for the weight parameter in the case of the ball and in the range $\kappa_i \geq 0$, $i = 1, \dots, n$, in the case of the simplex. These restrictions on the range of the parameters are determined by the range for the parameters in the representations of the kernels $P(w_\mu; x, y)$ and $P_n(w_\kappa; x, y)$.

Observe that the two-sided estimates on the heat kernels from (1.8),(1.16) coupled with the general results from [2, 7] entail smooth functional calculus in the settings on the ball and simplex (see [6, 11]), in particular, the finite speed propagation property is valid. For more details, see [8, §3.1].

2. PROOF OF GAUSSIAN BOUNDS FOR THE HEAT KERNEL ON THE BALL

We adhere to the notation from §1.1. Define

$$D_{i,j} := x_i \partial_j - x_j \partial_i, \quad 1 \leq i \neq j \leq d.$$

It is easy to see that

$$(2.1) \quad D_{i,j} = \partial_{\theta_{i,j}} \quad \text{with} \quad (x_i, x_j) = r_{i,j}(\cos \theta_{i,j}, \sin \theta_{i,j}).$$

Further, define the second order differential operators

$$D_{i,i}^2 := [w_\mu(x)]^{-1} \partial_i [(1 - \|x\|^2) w_\mu(x)] \partial_i, \quad 1 \leq i \leq d.$$

It turns out that the differential operator D_μ from (1.1) can be decomposed as a sum of second order differential operators [3, Proposition 7.1]:

$$(2.2) \quad D_\mu = \sum_{i=1}^d D_{i,i}^2 + \sum_{1 \leq i < j \leq d} D_{i,j}^2 = \sum_{1 \leq i \leq j \leq d} D_{i,j}^2.$$

The basic properties of the operator D_μ are given in the following

Theorem 2.1. For $f \in C^2(\mathbb{B}^d)$ and $g \in C^1(\mathbb{B}^d)$,

$$(2.3) \quad \int_{\mathbb{B}^d} \mathcal{D}_\mu f(x)g(x)w_\mu(x)dx \\ = - \int_{\mathbb{B}^d} \left[\sum_{i=1}^d \partial_i f(x)\partial_i g(x)(1 - \|x\|)^2 + \sum_{1 \leq i < j \leq d} D_{i,j} f(x)D_{i,j} g(x) \right] w_\mu(x)dx.$$

Consequently, the operator D_μ is essentially self-adjoint and $-D_\mu$ is positive in $L^2(\mathbb{B}^d, w_\mu)$.

Proof. Applying integration by parts in the variable x_i we obtain

$$\int_{\mathbb{B}^d} (D_{i,i}^2 f(x))g(x)w_\mu(x)dx = \int_{\mathbb{B}^d} (\partial_i [(1 - \|x\|^2)w_\mu(x)\partial_i f(x)])g(x)dx \\ = - \int_{\mathbb{B}^d} \partial_i f(x)\partial_i g(x)(1 - \|x\|)^2 w_\mu(x)dx.$$

We now handle $D_{i,j}^2$. It is sufficient to consider $D_{1,2}$. If $d = 2$ we switch to polar coordinates and use (2.1) and integration by parts for 2π -periodic functions to obtain

$$\int_{\mathbb{B}^2} (D_{i,j}^2 f(x))g(x)w_\mu(x)dx = \int_0^1 r(1 - r^2)^{\mu-1} \int_0^{2\pi} (\partial_\theta^2 f)gd\theta dr \\ = - \int_0^1 r(1 - r^2)^{\mu-1} \int_0^{2\pi} \partial_\theta f \partial_\theta g d\theta dr \\ = - \int_{\mathbb{B}^2} D_{i,j} f(x)D_{i,j} g(x)w_\mu(x)dx.$$

In dimension $d > 2$ we apply the following integration identity that follows by a simple change of variables,

$$(2.4) \quad \int_{\mathbb{B}^d} f(x)dx = \int_{\mathbb{B}^{d-2}} \left[\int_{\mathbb{B}^2} f(\sqrt{1 - \|v\|^2}u, v)du \right] (1 - \|v\|^2)dv,$$

and parametrizing the integral over \mathbb{B}^2 by polar coordinates we arrive at

$$\int_{\mathbb{B}^d} (D_{i,j}^2 f(x))g(x)w_\mu(x)dx = - \int_{\mathbb{B}^d} D_{i,j} f(x)D_{i,j} g(x)w_\mu(x)dx.$$

The above identities imply (2.3).

We consider the operator \mathcal{D}_μ with domain $D(\mathcal{D}_\mu) = \mathcal{P}(\mathbb{B}^d)$ the set of all polynomial on \mathbb{B}^d , which is obviously dense in $L^2(\mathbb{B}^d, w_\mu)$. From (2.3) it readily follows that the operator \mathcal{D}_μ is symmetric and $-\mathcal{D}_\mu$ is positive.

We next show that the operator \mathcal{D}_μ is essentially self-adjoint, that is, the completion $\overline{\mathcal{D}_\mu}$ of the operator \mathcal{D}_μ is self-adjoint. Let $\{P_{nj} : j = 1, \dots, \dim \mathcal{V}_n\}$ be an orthonormal basis of $\mathcal{V}_n = \mathcal{V}_n(w_\mu)$ consisting of real-valued polynomials. Clearly

$$D(\mathcal{D}_\mu) = \left\{ f = \sum_{n,j} a_{nj} P_{nj} : a_{nj} \in \mathbb{R}, \{a_{nj}\} \text{ compactly supported} \right\}, \text{ and}$$

$$\mathcal{D}_\mu f = - \sum_{n,j} a_{nj} n(n + 2\lambda) P_{nj} \quad \text{if} \quad f = \sum_j a_{nj} P_{nj} \in D(\mathcal{D}_\mu).$$

We define $\overline{\mathcal{D}}_\mu$ and its domain $D(\overline{\mathcal{D}}_\mu)$ by

$$D(\overline{\mathcal{D}}_\mu) := \left\{ f = \sum_{n=0}^{\infty} \sum_{j=1}^{\dim \mathcal{V}_n} a_{nj} P_{nj} : \sum_{n,j} |a_{nj}|^2 < \infty, \sum_{n,j} |a_{nj}|^2 (n(n+2\lambda))^2 < \infty \right\}$$

and

$$\overline{\mathcal{D}}_\mu f := - \sum_{n,j} a_{nj} n(n+2\lambda) P_{nj} \quad \text{if } f = \sum_{n,j} a_{nj} P_{nj} \in D(\overline{\mathcal{D}}_\mu).$$

It is easily to show that $\overline{\mathcal{D}}_\mu$ is the closure of \mathcal{D}_μ and that $\overline{\mathcal{D}}_\mu$ is self-adjoint. \square

Remark 2.2. Identity (2.3) is the weighted Green's formula on \mathbb{B}^d (see [8]).

Proof of Theorem 1.1. We shall assume that $0 < t \leq 1$. In the case $t > 1$ the Gaussian bounds (1.8) obviously follow from (1.8) in the case $t = 1$.

It is known (see [5, Thm. 5.2.8]) that for $\mu > 0$ the kernel $P_n(w_\mu; x, y)$ of the orthogonal projector onto $\mathcal{V}_n(w_\mu)$ in $L^2(\mathbb{B}, w_\mu)$ has the representation (2.5)

$$P_n(w_\mu; x, y) = c_\lambda \frac{n+\lambda}{\lambda} \int_{-1}^1 C_n^\lambda \left(\langle x, y \rangle + u \sqrt{1-\|x\|^2} \sqrt{1-\|y\|^2} \right) (1-u^2)^{\mu-1} du,$$

where C_n^λ is the Gegenbauer polynomial of degree n and $c_\lambda > 0$ is a constant depending only on λ and d . The Gegenbauer polynomials $\{C_n^\lambda\}$ are orthogonal in the weighted space $L^2([-1, 1], w_\lambda)$ with $w_\lambda(u) := (1-u^2)^{\lambda-1/2}$ and can be defined by the generating function

$$(1-2uz-z^2)^{-\mu} = \sum_{n=0}^{\infty} C_n^\lambda(u) z^n, \quad |z| < 1, |u| < 1.$$

Using that $C_n^\lambda(1) = \binom{n+2\lambda-1}{n}$ it is easy to show that

$$(2.6) \quad \int_{-1}^1 |C_n^\lambda(u)|^2 w_\lambda(u) du = \frac{\lambda}{n+\lambda} C_n^\lambda(1).$$

In the limiting case $\mu = 0$ the representation of $P_n(w_\mu; x, y)$ takes the form

$$(2.7) \quad P_n(w_0; x, y) = c_d \frac{\lambda+n}{\lambda} \left[C_n^\lambda \left(\langle x, y \rangle + \sqrt{1-\|x\|^2} \sqrt{1-\|y\|^2} \right) + C_n^\lambda \left(\langle x, y \rangle - \sqrt{1-\|x\|^2} \sqrt{1-\|y\|^2} \right) \right].$$

If $\alpha = \beta = \lambda - 1/2$ we denote the Jacobi operator by $L_\lambda := L_{\lambda-1/2, \lambda-1/2}$ and we have $L_\lambda f(x) = (1-x^2)f''(x) - (2\lambda+1)f'(x)$. We denote by $e^{tL_\lambda}(u, v)$ the Jacobi heat kernel in this case and by (1.23) and (2.6) we obtain

$$(2.8) \quad e^{tL_\lambda}(u, v) = \sum_{n=0}^{\infty} e^{-tn(n+2\lambda)} \frac{n+\lambda}{\lambda} \frac{C_n^\lambda(u)C_n^\lambda(v)}{C_n^\lambda(1)}, \quad \lambda := \mu + (d-1)/2.$$

Assume $\mu > 0$. The above, (1.7), and (2.5) lead to the representation

$$(2.9) \quad e^{t\mathcal{D}^\mu}(x, y) = c_\lambda \int_{-1}^1 e^{tL_\lambda} \left(1, \langle x, y \rangle + u \sqrt{1-\|x\|^2} \sqrt{1-\|y\|^2} \right) (1-u^2)^{\mu-1} du.$$

Note that in the case of Gegenbauer polynomials ($\alpha = \beta = \lambda - 1/2$) by (1.21) it follows that $V(x, r) \sim r(1-x^2+r^2)^\lambda$, $-1 \leq x \leq 1$, and hence

$$(2.10) \quad V(1, \sqrt{t}) \sim t^{\lambda+1/2} \quad \text{and} \quad V(z, \sqrt{t}) \sim t^{\lambda+1/2} (1+(1-z^2)/t)^\lambda, \quad |z| \leq 1.$$

If $x = \cos \theta$, then $1 - x = 2 \sin^2 \frac{\theta}{2} \sim \theta^2$ and hence

$$(2.11) \quad \rho(1, z) = |\arccos 1 - \arccos z| = \arccos z \sim \sqrt{1 - z}, \quad -1 \leq z \leq 1.$$

From this, (2.9), (1.24), and (2.10) we obtain

$$(2.12) \quad e^{t\mathcal{D}^\mu}(x, y) \leq c_1 \int_{-1}^1 \frac{\exp\left\{-\frac{1-z(u;x,y)}{c_2 t}\right\}}{t^{\lambda+1/2}\left(1+\frac{1-z(u;x,y)^2}{t}\right)^\lambda} (1-u^2)^{\mu-1} du$$

and

$$(2.13) \quad e^{t\mathcal{D}^\mu}(x, y) \geq c_3 \int_{-1}^1 \frac{\exp\left\{-\frac{1-z(u;x,y)}{c_4 t}\right\}}{t^{\lambda+1/2}\left(1+\frac{1-z(u;x,y)^2}{t}\right)^\lambda} (1-u^2)^{\mu-1} du,$$

where $z(u; x, y) := \langle x, y \rangle + u\sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}$.

Since $1 + b \leq e^b$ for $b \geq 0$, we have

$$1 \leq \left(1 + \frac{1-z^2}{t}\right)^\lambda \leq \left(1 + 2\frac{1-z}{t}\right)^\lambda \leq \exp\left\{2\lambda\frac{1-z}{t}\right\}, \quad |z| \leq 1.$$

Therefore, by replacing the constant c_4 in (2.13) by a smaller constant c'_4 we get

$$(2.14) \quad e^{t\mathcal{D}^\mu}(x, y) \geq \frac{c'_3}{t^{\lambda+1/2}} \int_{-1}^1 \exp\left\{-\frac{1-z(u;x,y)}{c'_4 t}\right\} (1-u^2)^{\mu-1} du.$$

Obviously, from (2.12) it follows that

$$(2.15) \quad e^{t\mathcal{D}^\mu}(x, y) \leq \frac{c_1}{t^{\lambda+1/2}} \int_{-1}^1 \exp\left\{-\frac{1-z(u;x,y)}{c_2 t}\right\} (1-u^2)^{\mu-1} du.$$

We have

$$1 - z(u; x, y) = 1 - \langle x, y \rangle - \sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2} + (1-u)\sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}$$

and using the definition of $d_{\mathbb{B}}(x, y)$ in (1.3) we get

$$1 - z(1; x, y) = 1 - \cos d_{\mathbb{B}}(x, y) = 2 \sin^2 \frac{d_{\mathbb{B}}(x, y)}{2} \sim d_{\mathbb{B}}(x, y)^2.$$

Hence,

$$1 - z(u; x, y) \sim d_{\mathbb{B}}(x, y)^2 + (1-u)H(x, y), \quad H(x, y) := \sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}.$$

Consequently,

$$\exp\left\{-\frac{1-z(u;x,y)}{c_2 t}\right\} \leq \exp\left\{-\frac{d_{\mathbb{B}}(x,y)^2}{c' t}\right\} \exp\left\{-\frac{(1-u)H(x,y)}{c' t}\right\}$$

and

$$\exp\left\{-\frac{1-z(u;x,y)}{2c_4 t}\right\} \geq \exp\left\{-\frac{d_{\mathbb{B}}(x,y)^2}{c'' t}\right\} \exp\left\{-\frac{(1-u)H(x,y)}{c'' t}\right\}.$$

These two inequalities along with (2.14)-(2.15) imply that in order to obtain the two-sided Gaussian bounds in (1.8) it suffices to show that the quantity

$$(2.16) \quad A_t(x, y) := \frac{1}{t^{\lambda+1/2}} \int_{-1}^1 \exp\left\{-\frac{(1-u)H(x,y)}{ct}\right\} (1-u^2)^{\mu-1} du$$

satisfies the following inequalities, for any $\varepsilon > 0$,

$$(2.17) \quad \frac{c^*}{[V_{\mathbb{B}}(x, \sqrt{t})V_{\mathbb{B}}(y, \sqrt{t})]^{1/2}} \leq A_t(x, y) \leq \frac{c^{**} \exp\left\{\varepsilon \frac{d_{\mathbb{B}}(x,y)^2}{t}\right\}}{[V_{\mathbb{B}}(x, \sqrt{t})V_{\mathbb{B}}(y, \sqrt{t})]^{1/2}}.$$

Here the constant $c^{**} > 0$ depends on ε .

Lower bound estimate. First, assume that $H(x, y)/t \geq 1$. Then we have

$$\begin{aligned}
(2.18) \quad A_t(x, y) &\geq \frac{\tilde{c}}{t^{\lambda+1/2}} \int_0^1 \exp\left\{-\frac{(1-u)H(x, y)}{ct}\right\} (1-u)^{\mu-1} du \\
&= \frac{\tilde{c}t^\mu}{t^{\lambda+1/2}H(x, y)^\mu} \int_0^{H(x, y)/t} v^{\mu-1} e^{-v/c} dv \\
&\geq \frac{c_*}{t^{d/2}H(x, y)^\mu} \quad \text{with} \quad c_* = \tilde{c} \int_0^1 v^{\mu-1} e^{-v/c} dv,
\end{aligned}$$

where we applied the substitution $v = (1-u)H(x, y)/t$ and used that $\lambda + 1/2 = \mu + d/2$. However, by (1.5), $V_{\mathbb{B}}(x, r) \geq cr^d(1 - \|x\|^2)^\mu$, which implies

$$t^{d/2}H(x, y)^\mu = t^{d/2}(\sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2})^\mu \leq [V_{\mathbb{B}}(x, \sqrt{t})V_{\mathbb{B}}(y, \sqrt{t})]^{1/2}.$$

Putting the above together we conclude that $A_t(x, y)$ obeys the lower bound in (2.17) in this case.

Now, assume that $H(x, y)/t \leq 1$. Then $\exp\left\{-\frac{(1-u)H(x, y)}{ct}\right\} \geq e^{-1/c}$ and we have

$$A_t(x, y) \geq \frac{\tilde{c}}{t^{\lambda+1/2}} \geq \frac{c^*}{[V_{\mathbb{B}}(x, \sqrt{t})V_{\mathbb{B}}(y, \sqrt{t})]^{1/2}}.$$

Here we used that, by (1.5), $V_{\mathbb{B}}(x, r) \geq cr^{d+2\mu} = cr^{2\lambda+1}$. Thus, $A_t(x, y)$ again obeys the lower bound estimate in (2.17) and this completes its proof.

Upper bound estimate. Obviously $\exp\left\{-\frac{(1-u)H(x, y)}{ct}\right\} \leq 1$ and hence

$$(2.19) \quad A_t(x, y) \leq \frac{c_*}{t^{\lambda+1/2}} = \frac{c_*}{t^{d/2+\mu}}.$$

We shall obtain another estimate on $A_t(x, y)$ by breaking the integral in (2.16) into two parts: one over $[0, 1]$ and the other over $[-1, 0]$. Just as in (2.18) applying the substitution $v = (1-u)H(x, y)/t$ we obtain

$$\frac{1}{t^{\lambda+1/2}} \int_0^1 \exp\left\{-\frac{(1-u)H(x, y)}{ct}\right\} (1-u^2)^{\mu-1} du \leq \frac{c^* \max\{1, 2^{\mu-1}\}}{t^{d/2}H(x, y)^\mu}$$

with $c^* = \int_0^\infty v^{\mu-1} e^{-v/c} dv$. Here we used that $(1+u)^{\mu-1} \leq \max\{1, 2^{\mu-1}\}$.

For the integral over $[-1, 0]$ we use the fact that $1-u \geq 1$ for $u \in [-1, 0]$ to obtain

$$\begin{aligned}
\frac{1}{t^{\lambda+1/2}} \int_{-1}^0 \exp\left\{-\frac{(1-u)H(x, y)}{ct}\right\} (1-u^2)^{\mu-1} du &\leq \frac{c_*}{t^{\lambda+1/2}} \exp\left\{-\frac{H(x, y)}{ct}\right\} \\
&\leq \frac{\tilde{c}}{t^{\lambda+1/2}} \left(\frac{t}{H(x, y)}\right)^\mu = \frac{\tilde{c}}{t^{\lambda+1/2}H(x, y)^\mu}.
\end{aligned}$$

Here we used that $v^\mu \leq [\mu + 1]!e^v$, $\forall v > 0$, and $\lambda = \mu + (d-1)/2$.

Together, the above inequalities imply

$$(2.20) \quad A_t(x, y) \leq \frac{c^*}{t^{d/2}H(x, y)^\mu}.$$

In turn, (2.19) and (2.20) yield

$$(2.21) \quad A_t(x, y) \leq \frac{c_\diamond}{t^{d/2}(t + H(x, y))^\mu}.$$

It remains to show that the above estimate implies the upper bound estimate in (2.17). To this end we need the following simple inequalities:

$$(2.22) \quad (u+a)(u+b) \leq 3(u^2+ab)(1+u^{-1}|a-b|), \quad a, b \geq 0, \quad 0 < u \leq 1,$$

(see, e.g. [10, (2.21)] and (see [11, (4.9)]))

$$|\sqrt{1-\|x\|^2} - \sqrt{1-\|y\|^2}| \leq \sqrt{2}d_{\mathbb{B}}(x, y), \quad x, y \in \mathbb{B}^d.$$

Together, these two inequalities yield

$$(2.23) \quad (\sqrt{t} + \sqrt{1-\|x\|^2})(\sqrt{t} + \sqrt{1-\|y\|^2}) \leq c(t + H(x, y)) \left(1 + \frac{d_{\mathbb{B}}(x, y)}{\sqrt{t}}\right).$$

Evidently $1+u \leq \varepsilon^{-1}e^{\varepsilon u}$ for $u \geq 0$ and $0 < \varepsilon \leq 1$, and hence

$$(2.24) \quad (1+b)^\mu \leq 2^\mu(1+b^2)^{\mu/2} \leq 2^\mu \varepsilon^{-\mu/2} e^{\mu \varepsilon b^2}, \quad \forall b \geq 0, \quad 0 < \varepsilon \leq 1.$$

Also, from (1.5) it follows that $V_{\mathbb{B}}(x, r) \sim r^d(r + \sqrt{1-\|x\|^2})^{2\mu}$. From this, (2.23), and (2.24) it follows that

$$\begin{aligned} [V_{\mathbb{B}}(x, \sqrt{t})V_{\mathbb{B}}(y, \sqrt{t})]^{1/2} &\leq ct^{d/2}(\sqrt{t} + \sqrt{1-\|x\|^2})^\mu (\sqrt{t} + \sqrt{1-\|y\|^2})^\mu \\ &\leq c_\varepsilon t^{d/2}(t + H(x, y))^\mu \exp\left\{\mu \varepsilon \frac{d_{\mathbb{B}}(x, y)^2}{t}\right\}. \end{aligned}$$

In turn, this and (2.21) yield the upper bound estimate in (2.17).

We next consider the case when $\mu = 0$. Now, (1.7), (2.7), and (2.8) yield the representation

$$(2.25) \quad \begin{aligned} e^{t\mathcal{D}^0}(x, y) &= c_\lambda e^{tL_\lambda} \left(1, \langle x, y \rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\right) \\ &\quad + c_\lambda e^{tL_\lambda} \left(1, \langle x, y \rangle - \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\right). \end{aligned}$$

From this point on the proof follows in the footsteps of the proof when $\mu > 0$ from above, but is much simpler because the integral in (2.9) is replaced in (2.25) by two terms. We omit the further details. The proof of Theorem 1.1 is complete. \square

3. PROOF OF GAUSSIAN BOUNDS FOR THE HEAT KERNEL ON THE SIMPLEX

In this part we adhere to the notation from §1.2. The differential operator \mathcal{D}_κ from (1.9) can be written in the more symmetric form

$$(3.1) \quad \mathcal{D}_\kappa = \sum_{i=1}^d U_i + \sum_{1 \leq i < j \leq d} U_{i,j},$$

where, with the notation $\partial_{i,j} := \partial_i - \partial_j$,

$$\begin{aligned} U_i &:= \frac{1}{w_\kappa(x)} \partial_i(x_i(1-|x|)w_\kappa(x)) \partial_i, \\ U_{i,j} &:= \frac{1}{w_\kappa(x)} \partial_{i,j}(x_i x_j w_\kappa(x)) \partial_{i,j}, \quad 1 \leq i \leq d. \end{aligned}$$

This decomposition was first established in [1] for $w_\kappa(x) = 1$ and later used in [4] for w_κ . It is easy to verify it directly. The following basic property of the operator \mathcal{D}_κ follows immediately from (3.1) by integration by parts:

Proposition 3.1. *For any $f \in C^2(\mathbb{T}^d)$ and $g \in C^1(\mathbb{T}^d)$,*

$$(3.2) \quad \int_{\mathbb{T}^d} \mathcal{D}_\kappa f(x) \cdot g(x) w_\kappa(x) dx = - \int_{\mathbb{T}^d} \left[\sum_{i=1}^d \partial_i f(x) \partial_i g(x) x_i (1 - |x|) \right. \\ \left. + \sum_{1 \leq i < j \leq d} \partial_{i,j} f(x) \partial_{i,j} g(x) x_i x_j \right] w_\kappa(x) dx.$$

Observe that identity (3.2) is the weighted Green's formula on the simplex \mathbb{T}^d (see [8]).

We consider the operator \mathcal{D}_κ defined on the set $D(\mathcal{D}_\kappa) = \mathcal{P}(\mathbb{T}^d)$ of all algebraic polynomials on \mathbb{T}^d , which is obviously dense in $L^2(\mathbb{T}^d, w_\kappa)$. From (3.2) it readily follows that the operator \mathcal{D}_κ is symmetric and $-\mathcal{D}_\kappa$ is positive in $L^2(\mathbb{T}^d, w_\kappa)$. Furthermore, just as in the proof of Theorem 2.1 it follows that the operator \mathcal{D}_κ is essentially self-adjoint.

Proof of Theorem 1.2. We may assume that $0 < t \leq 1$, because the case $t > 1$ follow immediately from the case $t = 1$.

Recall that we consider in this article the Jacobi polynomials $P_n^{(\alpha, \beta)}$, $n = 0, 1, \dots$, normalized in $L^2([-1, 1], w_{\alpha, \beta})$. It is known (see [5, Theorem 5.3.4]) that if all $\kappa_i > 0$ the kernel $P_n(w_\mu; x, y)$ of the orthogonal projector onto $\mathcal{V}_n(w_\kappa)$ in $L^2(\mathbb{T}^d, w_\kappa)$ has the following representation

$$(3.3) \quad P_n(w_\mu; x, y) = c_\kappa P_n^{(\lambda - \frac{1}{2}, -\frac{1}{2})}(1) \\ \times \int_{[-1, 1]^{d+1}} P_n^{(\lambda - \frac{1}{2}, -\frac{1}{2})}(2z(u; x, y)^2 - 1) \prod_{i=1}^{d+1} (1 - u_i^2)^{\kappa_i} du,$$

where

$$z(u; x, y) := \sum_{k=1}^{d+1} u_k \sqrt{x_k y_k}, \quad x_{d+1} := 1 - |x|, \quad y_{d+1} := 1 - |y|, \quad \lambda := |\kappa| + (d - 1)/2,$$

in which $|x| = x_1 + \dots + x_d$. In the case when some or all $\kappa_i = 0$, this identity holds under the limit $\kappa_i \rightarrow 0$, using that

$$\lim_{\kappa \rightarrow 0^+} \frac{\int_{-1}^1 f(x) (1 - x^2)^{\kappa-1} dx}{\int_{-1}^1 (1 - x^2)^{\kappa-1} dx} = \frac{1}{2} [f(1) + f(-1)].$$

Assume $\kappa_i > 0$, $i = 1, \dots, n + 1$. Combining (1.15), (1.23), and (3.3) we obtain the representation

$$(3.4) \quad e^{t\mathcal{D}_\kappa}(x, y) = c_\kappa \int_{[-1, 1]^{d+1}} e^{tL_{\lambda - \frac{1}{2}, -\frac{1}{2}}}(1, 2z(u; x, y)^2 - 1) \prod_{i=1}^{d+1} (1 - u_i^2)^{\kappa_i} du.$$

Note that from (1.11) we have $\sum_{k=1}^{d+1} \sqrt{x_k y_k} = \cos d_{\mathbb{T}}(x, y)$ and hence $|z(u; x, y)| \leq 1$. Just as in (2.11) we obtain

$$(3.5) \quad \rho(1, 2z^2 - 1) := |\arccos 1 - \arccos(2z^2 - 1)| \sim \sqrt{1 - (2z^2 - 1)} \sim \sqrt{1 - z^2}.$$

On the other hand, with $\alpha = \lambda - 1/2$ and $\beta = -1/2$ we infer from (1.21) that $V(x, r) \sim r(1 - x + r^2)^\lambda$ and hence

$$V(1, \sqrt{t}) \sim t^{\lambda+1/2} \quad \text{and} \quad V(2z^2 - 1, \sqrt{t}) \sim t^{1/2} (t + 2(1 - z^2))^\lambda \sim t^{\lambda+1/2} \left(1 + \frac{1 - z^2}{t}\right)^\lambda.$$

We use these equivalences, (3.4), (1.24), and (3.5) to obtain

$$(3.6) \quad e^{t\mathcal{D}^\kappa}(x, y) \leq c_1 \int_{[-1,1]^{d+1}} \frac{\exp\left\{-\frac{1-z(u;x,y)^2}{c_2 t}\right\}}{t^{\lambda+1/2} \left(1 + \frac{1-z(u;x,y)^2}{t}\right)^\lambda} \prod_{i=1}^{d+1} (1-u_i^2)^{\kappa_i-1} du$$

and

$$(3.7) \quad e^{t\mathcal{D}^\kappa}(x, y) \geq c_3 \int_{[-1,1]^{d+1}} \frac{\exp\left\{-\frac{1-z(u;x,y)^2}{c_4 t}\right\}}{t^{\lambda+1/2} \left(1 + \frac{1-z(u;x,y)^2}{t}\right)^\lambda} \prod_{i=1}^{d+1} (1-u_i^2)^{\kappa_i-1} du.$$

Just as in the proof of Theorem 1.1 by replacing the constant c_4 in (3.7) by a smaller constant c'_4 we can eliminate the term $\left(1 + \frac{1-z(u;x,y)^2}{t}\right)^\lambda$ in the denominator. Thus, it follows that

$$(3.8) \quad e^{t\mathcal{D}^\kappa}(x, y) \geq \frac{c'_3}{t^{\lambda+1/2}} \int_{[-1,1]^{d+1}} \exp\left\{-\frac{1-z(u;x,y)^2}{c'_4 t}\right\} \prod_{i=1}^{d+1} (1-u_i^2)^{\kappa_i-1} du.$$

By simply deleting that term in (3.6) we get

$$e^{t\mathcal{D}^\kappa}(x, y) \leq \frac{c_1}{t^{\lambda+1/2}} \int_{[-1,1]^{d+1}} \exp\left\{-\frac{1-z(u;x,y)^2}{c_2 t}\right\} \prod_{i=1}^{d+1} (1-u_i^2)^{\kappa_i-1} du.$$

Evidently,

$$1-z(u;x,y)^2 = (1+|z(u;x,y)|)(1-|z(u;x,y)|) \geq 1-|z(u;x,y)| \geq 1 - \sum_{i=1}^{d+1} |u_i| \sqrt{x_i y_i}.$$

Using the symmetry of the last term above with respect to sign changes of u_i , and that $1-u_i^2 \sim 1-u_i$ when $0 \leq u_i \leq 1$, we conclude that

$$(3.9) \quad e^{t\mathcal{D}^\kappa}(x, y) \leq \frac{c'_1}{t^{\lambda+1/2}} \int_{[0,1]^{d+1}} \exp\left\{-\frac{1-z(u;x,y)}{c_2 t}\right\} \prod_{i=1}^{d+1} (1-u_i)^{\kappa_i-1} du.$$

Similarly, using that $1-z(u;x,y)^2 \leq 2(1-z(u;x,y))$ we infer from (3.8) that

$$(3.10) \quad e^{t\mathcal{D}^\kappa}(x, y) \geq \frac{c''_3}{t^{\lambda+1/2}} \int_{[0,1]^{d+1}} \exp\left\{-\frac{1-z(u;x,y)}{c''_4 t}\right\} \prod_{i=1}^{d+1} (1-u_i)^{\kappa_i-1} du.$$

By the definition of $d_{\mathbb{T}}(x, y)$ in (1.11) we have

$$1 - \sum_{i=1}^{d+1} \sqrt{x_i y_i} = 1 - \cos d_{\mathbb{T}}(x, y) = 2 \sin^2 \frac{d_{\mathbb{T}}(x, y)}{2} \sim d_{\mathbb{T}}(x, y)^2$$

and hence

$$(3.11) \quad 1 - z(u;x,y) = 1 - \sum_{i=1}^{d+1} \sqrt{x_i y_i} + \sum_{i=1}^{d+1} (1-u_i) \sqrt{x_i y_i} \sim d_{\mathbb{T}}(x, y)^2 + \sum_{i=1}^{d+1} (1-u_i) \sqrt{x_i y_i}.$$

Consequently,

$$(3.12) \quad \exp\left\{-\frac{1-z(u;x,y)}{c_2 t}\right\} \leq \exp\left\{-\frac{d_{\mathbb{T}}(x, y)^2}{c' t}\right\} \prod_{i=1}^{d+1} \exp\left\{-\frac{(1-u_i) \sqrt{x_i y_i}}{c' t}\right\}$$

and

$$(3.13) \quad \exp \left\{ -\frac{1-z(u; x, y)}{c''t} \right\} \geq \exp \left\{ -\frac{d_{\mathbb{T}}(x, y)^2}{c''t} \right\} \prod_{i=1}^{d+1} \exp \left\{ -\frac{(1-u_i)\sqrt{x_i y_i}}{c''t} \right\}.$$

For $x, y \in [0, 1]$ and $\kappa > 0$, denote

$$(3.14) \quad A_t(\kappa; x, y) := \kappa \int_0^1 \exp \left\{ -\frac{(1-u)\sqrt{xy}}{ct} \right\} (1-u)^{\kappa-1} du,$$

where $c > 0$ is a constant. We claim that for any $0 < \varepsilon \leq 1$

$$(3.15) \quad \frac{c^\diamond t^{|\kappa|}}{\prod_{i=1}^{d+1} (x_i + t)^{\kappa_i/2} (y_i + t)^{\kappa_i/2}} \leq \prod_{i=1}^{d+1} A_t(\kappa_i; x_i, y_i) \leq \frac{c^\diamond t^{|\kappa|} \exp \left\{ \varepsilon \frac{d_{\mathbb{T}}(x, y)^2}{t} \right\}}{\prod_{i=1}^{d+1} (x_i + t)^{\kappa_i/2} (y_i + t)^{\kappa_i/2}},$$

where $c^\diamond > 0$ depends on ε .

Assume for a moment that the inequalities (3.15) are valid. Then by (3.9), (3.12), and the right-hand side inequality in (3.15) we obtain

$$\begin{aligned} e^{tD_\kappa}(x, y) &\leq \frac{c}{t^{|\kappa|+d/2}} \exp \left\{ -\frac{d_{\mathbb{T}}(x, y)^2}{c't} \right\} \frac{t^{|\kappa|} \exp \left\{ \varepsilon \frac{d_{\mathbb{T}}(x, y)^2}{t} \right\}}{\prod_{i=1}^{d+1} (x_i + t)^{\kappa_i/2} \prod_{i=1}^{d+1} (y_i + t)^{\kappa_i/2}} \\ &\leq \frac{c \exp \left\{ -\frac{d_{\mathbb{T}}(x, y)^2}{2c't} \right\}}{[V_{\mathbb{T}}(x, \sqrt{t}) V_{\mathbb{T}}(y, \sqrt{t})]^{1/2}}. \end{aligned}$$

Here we used that $\lambda = |\kappa| + (d-1)/2$, and $V_{\mathbb{T}}(x, \sqrt{t}) = t^{d/2} \prod_{i=1}^{d+1} (x_i + t)^{\kappa_i}$ and the similar expression for $V_{\mathbb{T}}(y, \sqrt{t})$, which follow by (1.13). We also used the right-hand side estimate in (3.15) with $\varepsilon = (2c')^{-1}$. The above inequalities yields the upper bound estimate in (1.16). One similarly shows that (3.10), (3.13), and the left-hand side inequality in (3.15) imply the lower bound estimate in (1.16).

It remains to prove the estimates in (3.15). We first focus on the lower bound estimate in (3.15). If $\sqrt{xy}/t \leq 1$, then $\exp \left\{ -\frac{(1-u)\sqrt{xy}}{ct} \right\} \geq c' > 0$ and hence

$$A_t(\kappa; x, y) \geq c' \geq c'(t/\sqrt{xy})^\kappa.$$

Assume $\sqrt{xy}/t > 1$. Then applying the substitution $v = \frac{(1-u)\sqrt{xy}}{t}$ we obtain

$$A_t(\kappa; x, y) = \frac{t^\kappa}{(\sqrt{xy})^\kappa} \int_0^{\sqrt{xy}/t} e^{-v/c} v^{\kappa-1} dv \geq \frac{t^\kappa}{(\sqrt{xy})^\kappa} \int_0^1 e^{-v/c} v^{\kappa-1} dv = \frac{c't^\kappa}{(\sqrt{xy})^\kappa}.$$

Thus in both cases

$$A_t(\kappa; x, y) \geq \frac{c't^\kappa}{(\sqrt{xy})^\kappa} \geq \frac{c't^\kappa}{(x+t)^{\kappa/2}(y+t)^{\kappa/2}},$$

which yields the lower bound estimate in (3.15).

We now prove the upper bound estimate in (3.15). Clearly $\exp \left\{ -\frac{(1-u)\sqrt{xy}}{ct} \right\} \leq 1$ and hence $A_t(\kappa; x, y) \leq c'$. On the other hand, from above it follows that

$$A_t(\kappa; x, y) \leq \frac{t^\kappa}{(\sqrt{xy})^\kappa} \int_0^\infty e^{-v/c} v^{\kappa-1} dv = \frac{c''t^\kappa}{(\sqrt{xy})^\kappa}.$$

Together, these two estimates yield

$$A_t(\kappa; x, y) \leq \frac{c^*t^\kappa}{(\sqrt{xy} + t)^\kappa},$$

implying

$$(3.16) \quad \prod_{i=1}^{d+1} A_t(\kappa_i; x_i, y_i) \leq \frac{c^* t^{|\kappa|}}{\prod_{i=1}^{d+1} (\sqrt{x_i y_i} + t)^{\kappa_i}}.$$

To show that this leads to the desired upper bound estimate, we need the following simple inequality (see [6, (2.50)])

$$|\sqrt{x_i} - \sqrt{y_i}| \leq d_{\mathbb{T}}(x, y), \quad i = 1, \dots, d+1, \quad x, y \in \mathbb{T}^d.$$

This along with (2.22) implies

$$(\sqrt{x_i} + \sqrt{t})(\sqrt{y_i} + \sqrt{t}) \leq c(\sqrt{x_i y_i} + t) \left(1 + \frac{d_{\mathbb{T}}(x, y)}{\sqrt{t}}\right),$$

which leads to

$$\begin{aligned} \prod_{i=1}^{d+1} (x_i + t)^{\kappa_i/2} (y_i + t)^{\kappa_i/2} &\sim \prod_{i=1}^{d+1} (\sqrt{t} + \sqrt{x_i})^{\kappa_i} (\sqrt{t} + \sqrt{y_i})^{\kappa_i} \\ &\leq c \prod_{i=1}^{d+1} (\sqrt{x_i y_i} + t)^{\kappa_i} \left(1 + \frac{d_{\mathbb{T}}(x, y)}{\sqrt{t}}\right)^{|\kappa|} \\ &\leq c(\varepsilon) \prod_{i=1}^{d+1} (\sqrt{x_i y_i} + t)^{\kappa_i} \exp \left\{ \varepsilon |\kappa| \frac{d_{\mathbb{T}}(x, y)^2}{t} \right\}. \end{aligned}$$

Here for the last inequality we used (2.24) with $\mu = |\kappa|$. Together, the above and (3.16) yield the upper bound estimate in (3.15).

We now consider the case when one or more $\kappa_i = 0$, $1 \leq i \leq n+1$. In this case, the kernel representation (3.3) holds under the limit. If $\kappa_i = 0$, then the integral over u_i in (3.4) is replaced by the average of point evaluations at $u_i = 1$ and $u_i = -1$. It is easy to see that all deductions that lead to (3.15) are still valid with the realization that (3.14) holds under the limit

$$\lim_{\kappa \rightarrow 0^+} A_t(\kappa; x, y) = \lim_{\kappa \rightarrow 0^+} \kappa \int_0^1 \exp \left\{ -\frac{(1-u)\sqrt{xy}}{ct} \right\} (1-u)^{\kappa-1} du = 1.$$

This completes the proof. \square

REFERENCES

- [1] H. Berens, H.J. Schmid, and Y. Xu, Bernstein-Durrmeyer polynomials on a simplex, *J. Approx. Theory*, 68 (1992), 247–261.
- [2] T. Coulhon, G. Kerkyacharian, and P. Petrushev, Heat kernel generated frames in the setting of Dirichlet spaces, *J. Fourier Anal. Appl.* 18 (2012), 995–1066.
- [3] F. Dai, Y. Xu, Moduli of smoothness and approximation on the unit sphere and the unit ball, *Adv. in Math.* 224 (2010), 1233–1310.
- [4] Z. Ditzian, Multidimensional Jacobi-type Bernstein-Durrmeyer operators, *Acta Sci. Math. (Szeged)*, 60 (1995), 225–243.
- [5] C. F. Dunkl, Y. Xu, *Orthogonal polynomials of several variables*, Cambridge Univ. Press, 2001.
- [6] K. Ivanov, P. Petrushev, and Y. Xu, Sub-exponentially localized kernels and frames induced by orthogonal expansions, *Math. Z.* 264 (2010), 361–397.
- [7] G. Kerkyacharian, P. Petrushev, Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces. *Trans. Amer. Math. Soc.* 367 (2015), 121–189.
- [8] G. Kerkyacharian, P. Petrushev, and Y. Xu, Gaussian bounds for the weighted heat kernels on the interval, ball and simplex, preprint.

- [9] A. Nowak and P. Sjögren, Sharp estimates of the Jacobi heat kernel, *Studia Math.* 218 (2013), no. 3, 219–244.
- [10] P. Petrushev, Y. Xu, Localized polynomial frames on the interval with Jacobi weights, *J. Fourier Anal. and Appl.* 11 (2005), 557–575.
- [11] P. Petrushev, Y. Xu, Localized polynomial frames on the ball, *Const. Approx.* 27 (2008), 121–148.
- [12] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Vol. 23, Providence, 4th edition, 1975.

LPSM, CNRS-UMR 7599, AND CREST
E-mail address: `kerk@math.univ-paris-diderot.fr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208
E-mail address: `pencho@math.sc.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403-1222
E-mail address: `yuan@math.uoregon.edu`