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# SPECTRAL MULTIPLIERS FOR SCHRÖDINGER OPERATORS: I

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ABSTRACT. We prove a sharp Mihlin-Hörmander multiplier theorem for Schrödinger operators $H$ on $\mathbb{R}^n$. The method, which allows us to deal with general potentials, improves Hebisch's method relying on heat kernel estimates for positive potentials [22, 12]. Our result applies to, in particular, the *negative* Pöschl-Teller potential $V(x) = -\nu(\nu+1)\,\mathrm{sech}^2 x$, $\nu \in \mathbb{N}$, for which $H$ has a resonance at zero. Moreover, a modified heat kernel approach is used to obtain the multiplier result for unbounded electric and magnetic potentials arising in a relativistic quantum-mechanical background. Thus it helps the understanding of quantum scattering for wave and Schrödinger systems.

## 1. INTRODUCTION

Spectral multiplier theorem for differential operators plays a significant role in harmonic analysis and PDEs. It is closely related to the study of the associated function spaces and Littlewood-Paley theory. Let $H = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, where $V$ is real-valued. Spectral multipliers for $H$ have been considered in [22, 16, 14, 15, 3] and [12] for positive potentials. The case of negative potential is quite different and is not covered by the methods in these papers. Resonance and eigenvalue can occur that makes the analysis more involved. In this paper we are mainly concerned with proving a Mihlin-Hörmander type multiplier theorem on $L^p$ spaces for the Schrödinger operator with the negative P-T potential

$$V(x) = -\nu(\nu+1)\,\mathrm{sech}^2 x, \qquad \nu \in \mathbb{N}. \tag{1}$$

In [31, 47] we are able to extend the sharp spectral multiplier theorem on Triebel-Lizorkin spaces by modifying the argument in this note.

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Spectral multiplier problem requires both high and low energy analysis. In high energy the kernel of the multiplier operator $m(H)$ can be controlled by a weighted $L^2$ estimates. In low energy, roughly, it can be controlled pointwise by an approximation to the identity. This is the idea when dealing with positive selfadjoint operators [22, 12] where there is a rough kernel.

However for negative operator in the Schrödinger case, resonance and eigenvalue can occur even for smooth and rapidly decaying potentials, which lead to failure of the pointwise control of the kernel in lower energy. The purpose of this paper is to develop a general treatment to overcome this difficulty. We find that this pointwise estimate can be substituted with a weaker estimate (in integral form) that turns out still work. This is the approach we will apply to the Pöschl-Teller potentilal model. The P-T potential arises in standing wave problem for the cubic wave equation

$$-u_{tt} + u_{xx} + 2u - u^3 = 0\,. \tag{2}$$

In [47] we give a general treatment on spectral multiplier problem for Schrödinger operators satisfying for every $j \in \mathbb{Z}$

$$|\Phi_j(H)(x,y)| \le c_n \frac{2^{nj/2}}{(1+2^{j/2}|x-y|)^{n+\epsilon}}. \tag{3}$$

where $\Phi_j(x) = \Phi(2^{-j}x)$, $\Phi \in C_0^\infty(\mathbb{R})$.

The assumption is verified when $H$ is a Schrödinger operator $-\Delta+V$, $V \ge 0$ is in $L^1_{loc}(\mathbb{R}^n)$ [32, 22] or $H$ is a uniformly elliptic operator on $L^2(\mathbb{R}^n)$ [9, Theorem 3.4.10]. It is showed in [22, 46] that the decay (3) is satisfied whenever the heat kernel of $e^{-tH}$ satisfies the upper Gaussian bound

$$0 \le e^{-tH}(x,y) \le c_n t^{-n/2} e^{-c|x-y|^2/t}\,.$$

However, when $V$ is negative, eigenvalue and resonance may occur at the origin. the seemingly ubiquitous decay (3) for general selfadjoint operators is not valid for all $j$. Our approach shows that if (3) replaced by an integral version (1), the argument in [22, 12] can still work. This treatment will be further elaborated in the study of spectral calculus for rough potentials in the critical class in a following paper.

The basic ingredients we need to show are two weighted inequalities

(a) If $\Phi \in C_0^\infty(\mathbb{R})$ there exists a finite measure $\mu$ such that for each interval $I$ with length $2^{-j/2}$, $j \in \mathbb{Z}$,

$$\begin{aligned} |\Phi_j(H)(x,y)| \le& c \int_{u\in\mathbb{R}^n} \frac{2^{jn/2}}{(1+2^{j/2}|x-y-u|)^{n+\epsilon}} d\mu(u) \\ \approx& \rho_j * \mu\,(x-y) \end{aligned}$$

where $\rho_j(x) := 2^{jn/2}(1+2^{j/2}|x|)^{-n-\epsilon}$.

(b)

$$\sup_y \|\langle 2^j(x-y)\rangle^\alpha \phi_j(H)(x,y)\|_{L^2_x} \le C \sup_j \|\chi\phi(2^{-j}\xi^2)\|_{X^\alpha}$$

where $X^\alpha$ is $C^\alpha(\mathbb{R})$ or $W^\alpha$, $\alpha > n/2$. $\chi$ is a given smooth cut-off function with support off 0.

Observe that (a) is a weaker condition than the pointwise estimate for the decay of $\phi_j(H)$. Indeed, the ponitwise decay (3) is a special case when $\mu = \delta$.

1.1. **Weighted $L^2$ estimates for kernel of $m(H)$.** There are a few ways to prove $L^p$ boundedness for $m(H)$.

The usual condition is the Hörmander integral condition

$$\int_{|x-y|>2|x-\bar{y}|} |m(H)(x,y) - m(H)(x,\bar{y})| dx \le A$$

for $y \in I$, $\bar{y}$ the center of $I$, $I$ being any cube in $\mathbb{R}^n$. This is what is shown in [3]. For P-T potential similar estimate is not valid in low energy. This is the reason why we need to consider the weighted $L^2$ estimate in (b).

The other way is to use wave operator method [41, 42]. However wave operator method does not give the sharper weak $(1,1)$ result.

## 2. Main results

Let $H$ be a selfadjoint operator on $L^2(\mathbb{R}^n)$. Then if $\phi \in L^\infty$, we can define $\phi(H) = \int \phi(\lambda) dE_\lambda$ by functional calculus, where $H = \int \lambda dE_\lambda$ is the spectral resolution of $H$.

Our main result is the following

**Theorem 2.1.** *Suppose $H$ verifies the weighted decay (a) and weighted $L^2$ inequality (b). Then $m(H)$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ and of weak type $(1,1)$. Moreover,*

$$\|m(H)\|_{L^1 \to w\text{-}L^1} \le C(m)$$

$C(m) := \|m\|_\infty + \sup_{\lambda>0} \|\chi(\cdot)m(\lambda\cdot)\|_{X^\alpha}$ .

**Remark.** One may view condition (a) as a "pointwise" control of the kernel in *lower* energy while condition (b) as a norm control in *higher* energy.

The conditions of Theorem 2.1 applies to the case when the kernel of $\phi(H)$ is slowly decaying or lack of smoothness. It can also simplifies the proof of some known results on other potential of polynomial growth, e.g. the Hermite operator on $\mathbb{R}^n$ [47].

Applying the theorem to P-T potential we obtain

**Theorem 2.2.** *Let* $H = -d^2/dx^2 + V_\nu$*,* $\nu \in \mathbb{N}$*, where* $V_\nu$ *is the P-T potential in (1). Then* $H$ *satifies the weighted decay (a) and weighted* $L^2$ *inequality (b) with* $X^\alpha = C^\alpha$*,* $\alpha = 1$*. Therefore the conclusion of Theorem 2.1 holds for the one dimensional P-T model.*

From section 5 we know the derivative of the kernel fail to satisfy nice decay, making it difficult to control the difference of $m(H)$ and thus leading to the failing of low energy estimates for the Hörmander integral condition.

A recently developed approach [12] by Sikora et al extended Hebisch's method [22] that apply to positive operators efficiently but rely heavily on heat kernel estimates. However, in dimensions one and two when the potential is negative, such a heat kernel estimate is NOT available. Therefore we consider more direct approach and would rather state and prove the multiplier result in Theorem 2.1 for general dimensions.

We will assume $\Phi, \varphi \in C_0^\infty(\mathbb{R})$ satisfy the condition

$$\sum_{j=-\infty}^{\infty} \varphi_j(x) = 1 \quad x \neq 0 \tag{4}$$

$$\Phi(x) + \sum_{j=1}^{\infty} \varphi_j(x) = 1, \quad \forall x \tag{5}$$

where $\varphi_j(x) = \varphi(2^{-j}x)$.

## 3. Proof of Theorem 2.1

The technical lemmas we need in proving Theorem 2.1 are:

**Lemma 3.1.** *Let* $y \in I$*,* $I \subset \mathbb{R}^n$ *a cube with length* $t = \ell(I)$*. Let* $2^{-j_I/2} \sim t$*. Then (a) For* $t > 0, j \in \mathbb{Z}$*,*

$$\int_{|x-y|\geq 2t} |m_j(H)(1-\Phi_{j_I}(H))(x,y)|dx \leq C(2^{j/2}t)^{\frac{n}{2}-s}\|m(\xi^2)\|_{\dot{H}^s}$$

$s > n/2$*. (b)*

$$\int_{|x-y|\geq 2t} \sum_{j=-\infty}^{\infty} |m_j(H)(1-\Phi_{j_I}(H))(x,y)|dx \leq A\,.$$

*In particular,*

$$\int_{|x-y|\geq 2t} \sup_{j\in\mathbb{Z}} |m_j(H)(1-\Phi_{j_I}(H))(x,y)|dx \leq A\,.$$

**Lemma 3.2.** *Condition (b) implies*

$$\begin{aligned}\max_{y\in I}|\Phi_j(H)(x,y)| \leq & c\min_{y\in I}\int_{u\in\mathbb{R}^n}\frac{2^{jn/2}}{(1+2^{j/2}|x-y-u|)^{n+\epsilon}}d\mu(u)\\ \approx & \min_{y\in I}\rho_j*\mu\,(x-y)\\ \leq & \frac{1}{|I|}\int_{z\in I}\rho_j*\mu\,(x-z)dz\end{aligned}$$

*where* $\rho_j(x) := 2^{jn/2}(1+2^{j/2}|x|)^{-n-\epsilon}$.

The proof is based on the observation if $\ell(I) = r_I$, the side length of a cube $I$ in $\mathbb{R}^n$,

$$\sup_{y\in I}(1+|x-y|)^{-n-\epsilon} \leq C\min_{y\in I}(1+|x-y|)^{-n-\epsilon}$$

hence

$$\sup_{y\in I}(1+|x-y|/t)^{-n-\epsilon} \leq C\min_{y\in I}(1+|x-y|/t)^{-n-\epsilon} \leq \frac{C}{|I|}\int_I(1+|x-y|/t)^{-n-\epsilon}dy$$

where $t \sim \ell(I)$, $I$ is any cube, see [22].

**Proof of weak (1,1).** If applying the C-Z decomposition we know the main part is how to handle the "bad" function $b = \sum_k b_k$ where $b_k \subset I_k$, $I_k$ being disjoint intervals in $\mathbb{R}$.

*Proof.* Let $\Phi \subset [-1,1]$, $\Phi_j(x) = \Phi(2^{-j}x)$. Write

$$m(H)b(x) = \sum_k m(H)(1-\Phi_{j_k}(H))b_k(x) + \sum_k m(H)\Phi_{j_k}(H)b_k(x).$$

where $2^{-j_k} \sim \ell(I_k)^2$. We need to show

$$\begin{aligned}&|\{x\in\mathbb{R}\setminus\cup_k I_k^*: |m(H)b(x)|>\lambda/2\}|\\ \leq & |\{x\in\mathbb{R}\setminus I_k^*: \sum_k|m(H)(1-\Phi_{j_k}(H))b_k(x)|>\lambda/4\}\\ + & |\{x\in\mathbb{R}\setminus I_k^*: \sum_k|m(H)\Phi_{j_k}(H)b_k(x)|>\lambda/4\}\\ \leq & \lambda^{-1}\|f\|_1,\end{aligned}$$

where $b = \sum_k b_k$ (convergence in $L^1\cap L^q$ so $Tb(x) = \sum_k Tb_k$ in $L^q$)

*Higher* energy

Denote $I_k^*$ the cube having length three times the length of $I_k$ with the same center as $I_k$. If $x \notin \cup_k I_k^*$, $I_k \subset \{y : |y-x| > r_k\}$, $r_k$ being the length of $I_k$.

$$m(H)(1-\Phi_{j_k}(H)b_k(x)=\int_{|y-x|>r_k} m(H)(1-\Phi_{j_k}(H)(x,y)b_k(y)dy$$

Apply weighted condition Lemma 3.1 (c)

$$\begin{aligned}
&|\{x\notin \cup I_k^*: |\sum_k m(H)(1-\Phi_{j_k}(H)b_k(x)|>\alpha/4\}|\\
\le& C(\alpha/4)^{-1}\int_{\mathbb{R}^n\setminus \cup I_k^*}|\sum_k m(H)(1-\Phi_{j_k}(H))b_k(x)|dx\\
\le& C\alpha^{-1}\int\sum_k |b_k(y)|dy\int_{|y-x|>r_k}|m(H)(1-\Phi_{j_k}(H)(x,y)|dx\\
\le& C\sup_\lambda \|\chi m(\lambda\xi^2)\|_{C^s_{loc}}\alpha^{-1}\int|b(y)|dy\\
\le& C\sup_\lambda \|\chi m(\lambda\xi^2)\|_{C^s_{loc}}\alpha^{-1}\|f\|_1.
\end{aligned}$$

where we note

$$\begin{aligned}
&\int_{|x-y|>r_k}|m(H)(1-\Phi_{j_k}(H))(x,y)|dx\\
\le&\sum_{2^j>r_k^2}\int_{|x-y|>r_k}|m_j(H)(x,y)|dx\le C
\end{aligned}$$

because if $\Phi(x)+\sum_{j=1}^\infty \phi(2^{-j}x)=1$ then for any $j_0\in\mathbb{Z}$, $\Phi(2^{-j_0}x)=1-\sum_{j=j_0+1}^\infty\phi(2^{-j}x)$.

*Lower* energy

Since $m(H)$ is bounded on $L^2$. The proof is complete if we can show

$$\int|\sum_k \Phi_{j_k}(H)b_k(x)|^2dx\le C\alpha\|f\|_1 \tag{6}$$

To show this let $h \in L^2(\mathbb{R}^n)$, $2^{-j_k} \sim \ell(I_k)^2$. According to (b)

$$
\begin{aligned}
&\langle \sum_k \Phi_{j_k}(H) b_k, h\rangle \\
=&\sum_k \int_x h(x)dx \int_{y\in I_k} K_{j_k}(x,y) b_k(y)dy \\
\le&\sum_k |I_k|^{-1} \int_x |h(x)|dx \int_{z\in I_k}\int_u \rho_j(x-z-u)d\mu(u)dz \int_y |b_k(y)|dy \\
\le&\sum_k \|b_k\|_1 |I_k|^{-1} \int_{z\in I_k}\int_u (Mh)(z+u)d\mu dz \\
&(bec\ \rho_j = 2^{jn/2}(1+2^{j/2}(\cdot))^{-n-\epsilon} \text{ is} \sim \text{an approximation to the id}) \\
\le& C\alpha \int_z \sum_k \chi_{I_k}(z)(Mh)*d\mu(z)dz \\
\le& C\alpha \|\sum_k \chi_{I_k}\|_2 \|Mh*d\mu\|_2 \\
\le& C\alpha (\sum_k |I_k|)^{1/2}\|h\|_2 \\
\le& C\alpha(\alpha^{-1}\|f\|_1)^{1/2}\|h\|_2 = C\alpha^{1/2}\|f\|_1^{1/2}\|h\|_2
\end{aligned}
$$

which proves (11).We have used the fact that if $\rho_t = t^{-n}\rho(x/t)$ is any approximation kernel to the identity, so that $\rho \in L^1(\mathbb{R}^n)$ is positive and decreasing, then

$$\sup_{t>0} |\rho_t * f(x)| \le Mf(x)$$

where $M$ denotes the Hardy-Littlewood maximal function on $\mathbb{R}^n$. □

**Remark.** Note that in analyzing the kernel of $\Phi_j(H)$, oscillatory integral, if $j < 0$ we can prove the rapid decay from the spectrum side; if $j > 0$ it only gives a decay of $|x-y|^{-n}$; we will have to work in the space side and use the (average) integral version of rapid decay.

**Remark.** From the proof above we see the the weighted $L^2$ inequality in (b) somehow plays the role of Hörmander condition in classical case [37, 3]

$$\int_{|x-y|>2|x-\bar{y}|} |K(x,y) - K(x,\bar{y})|dx \le A \tag{7}$$

which requires the gradient estimate for $K_j(x,y)$.

**Remark.** A nontrivial vector-valued version of the proof of the $L^p$ result yields the multiplier result on the homogenous spaces $\dot{F}(H)$ and $\dot{B}(H)$ [31, 47].

In [30] when identifying the inhomogeneous space $F_p^{0,2}(H) = L^p$ we used decay for the derivative of the kernel in high energy, namely, (b') For $t > 0, j \geq 0$,

$$\int_{|x-y|\geq 2t} |m_j(H)(x,y) - m_j(H)(x,\bar{y})|dx \leq C(2^{j/2}t)^{\frac{1}{2}}$$

(c')

$$\int_{|x-y|\geq 2t} \sum_{2^j \geq t^2}^{\infty} |m_j(H)(x,y)|dx \leq C.$$

which is valid only for high energy estimate because we only have available for $j \geq 0$ the weighted estimate

$$\|(x-y)\partial_y K_j(x,y)\|_2 \leq 2^{j/4}\,.$$

To deal with the problem in low energy we avoid using the estimate for $\partial_y K_j(x,y)$ and follow the line of proof of Theorem 2.1 [31] for the homogeneous spaces $\dot{F}(H)$. Thus we obtain the identification of $\dot{F}(H)$ and $L^p$ spaces.

**Corollary 3.3.** *Let* $1 < p < \infty$. *Then*

$$\dot{F}_p^{0,2}(H) = \dot{F}_p^{0,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n),$$

*meaning that the Littlewood-Paley characterization holds*

$$\|\Big(\sum_{j\in\mathbb{Z}} |\phi_j(H)f|^2\Big)^{1/2}\|_p \approx \|f\|_p\,.$$

## 4. Proof of Corollary 3.3

Identification of $\dot{F}_p^{\alpha,q}(H) = L^p$. *homogeneous* spaces

Let $Q_j = \phi_j(H)$, $j \in \mathbb{Z}$. Define

$$Q : f \mapsto \{\phi_j(H)f\}$$

and

$$R : \{f_j\} \mapsto \sum_{j=-\infty}^{\infty} \psi_j(H)f_j$$

We show that the same method in showing spectral multiplier theorem for $F$ spaces yields

$$Q : L^1 \to w - L^1(\ell^2)$$

and

$$R : L^1(\ell^2) \to w - L^1$$

Hence, this, together with the boundedness $Q$: $L^2 \to L^2(\ell^2)$ and $R$ : $L^2(\ell^2) \to L^2$, proves that if $1 < p < \infty$

$$\|f\|_{L^p} \sim \|\phi_j(H)f\|_{L^p(\ell^2)} = \|f\|_{\dot{F}_p^{0,2}(H)}$$

**Lemma 4.1.** *$Q$: $L^2(\mathbb{R}^n) \to L^2(\ell^2)$. $R$ : $L^2(\ell^2) \to L^2(\mathbb{R}^n)$.*

Proof.

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}(\phi_j(H)f, \phi_j(H)f)\\ =&(\sum_{j\in\mathbb{Z}}(\overline{\phi_j}\phi_j)(H)f, f) \lesssim \|f\|_2^2\end{aligned}$$

because $\sum_{j\in\mathbb{Z}} |\phi_j(x)|^2 \approx 1$. The proof for R is similar. □

**Lemma 4.2.** *$Q$ is $L^1 \to w - L^1(\ell^2)$, i.e.,*

$$|\{\sum_{j\in\mathbb{Z}} |\phi_j(H)f(x)|^2)^{1/2} > \alpha\}| \le C\alpha^{-1}\|f\|_1$$

4.3. **Proof of $Q$: weak $(L^1, L^1(\ell^2))$.** Let $f \in L^1(\mathbb{R}^n)$. Given $\alpha > 0$, let $f = g + b$ be the Calderón-Zygmund decomposition, where $g \in L^2 \cap L^1$, $b = \sum_k b_k$, $b_k \subset I_k$, $I_k$ disjoint.

i) $|g(x)| \le C\alpha$

ii) $|I_k|^{-1} \int_{I_k} |f|dx \le C_n\alpha$.

Note that (i), (ii) imply

$$|I_k|^{-1} \int_{I_k} |b|dx \le C_n\alpha.$$

Since $\phi_j(H)$ is bounded $L^2 \to L^2(\ell^2)$.

$$\int \sum_{j\in\mathbb{Z}} |\phi_j(H)g(x)|^2 dx \le C\|g\|_2^2 \le C\alpha\|f\|_1$$

(i) gives

$$|\{x : (\sum_{j\in\mathbb{Z}} |\phi_j(H)g(x)|^2)^{1/2} > \alpha/2\}| \le C\|\phi\|_\infty\alpha^{-1}\|f\|_1.$$

We estimate $b$ in more detail. Let $2^{-j_k} \sim t_k =$ diameter of the cube $I_k$.

$$\begin{aligned}
&|\{x : (\sum_{j\in\mathbb{Z}} |\phi_j(H)b(x)|^2)^{1/2} > \alpha/2\}| \\
\le&|\{x : (\sum_j |\sum_k \phi_j(H)(1-\Phi_{j_k}(H)b_k(x)|^2)^{1/2} > \alpha/4\}| \\
+&|\{x : (\sum_j |\sum_k \phi_j(H)\Phi_{j_k}(H)b_k(x)|^2)^{1/2} > \alpha/4\}| \\
:=&B_1 + B_2.
\end{aligned}$$

For the first term since $|\cup I_k| \le \alpha^{-1}\|f\|_1$, it suffices to estimate:

Denote $I_k^*$ the cube having length three times the length of $I_k$ with the same center as $I_k$. If $x \notin \cup_k I_k^*$, $I_k \subset \{y : |y-x| > r_k\}$, $r_k$ being the length of $I_k$.

$$\phi_j(H)(1-\Phi_{j_k}(H)b_k(x) = \int_{|y-x|>r_k} \phi_j(H)(1-\Phi_{j_k}(H)(x,y)b_k(y)dy$$

Apply Hebisch-Zheng condition (b)

$$\begin{aligned}
&|\{x \notin \cup I_k^* : (\sum_j |\sum_k \phi_j(H)(1-\Phi_{j_k}(H)b_k(x)|^2)^{1/2} > \alpha/4\}| \\
\le& C(\alpha/4)^{-1}\int_{\mathbb{R}^n\setminus\cup I_k^*} (\sum_j |\sum_k \phi_j(H)(1-\Phi_{j_k}(H)b_k(x)|^2)^{1/2} dx \\
\le& C\alpha^{-1}\int_{\mathbb{R}^n\setminus\cup I_k^*} \sum_k (\sum_j |\phi_j(H)(1-\Phi_{j_k}(H)b_k(x)|^2)^{1/2} dx \quad (Minkowski) \\
\le& C\alpha^{-1}\sum_k \int |b_k(y)|dy \int_{\mathbb{R}^n\setminus\cup I_k^*} \sum_{j\in\mathbb{Z}} |\phi_j(H)(1-\Phi_{j_k}(H)(x,y)|dx \quad (Jensen) \\
\le& C\alpha^{-1}\sum_k \int |b_k(y)|dy \int_{\mathbb{R}^n\setminus I_k} \sum_{j\ge j_k} |\phi_j(H)(1-\Phi_{j_k}(H)(x,y)|dx \\
\le& C\sup_\lambda \|\phi_j(\lambda\xi^2)\|_{H^s_{loc}}\alpha^{-1}\|f\|_1.
\end{aligned}$$

It remains to deal with $B_2$. (Low energy) The proof is finished if we can show

$$\int \sum_j |\phi_j(H)\sum_k \Phi_{j_k}(H)b_k(x)|^2 dx \le C\alpha\|f\|_1 \tag{8}$$

Since $Q = \{\phi_j(H)\}_{j\in\mathbb{Z}}$ is bounded $L^2 \to L^2(\ell^2)$, we only need to prove

$$\int \sum_k |\Phi_{j_k}(H)b_k(x)|^2 dx \le C\alpha \|f\|_1 \, . \tag{9}$$

Similar to the proof of Theorem 2.1 we use duality argument. Let $h \in L^2(\mathbb{R}^n)$, $\|h\|_{L^2} \le 1$. Then by condition (b)

$$\begin{aligned} &< \sum_k \Phi_{j_k}(H)b_k(x), h > \\ \le& c\sum_k \|b_k\|_1 |I_k|^{-1} \int_{z\in I_k} dz \int_u d\mu(u) \int_x \rho_t(x-z-u)|h(x)|dx \\ =& c\sum_k |I_k|^{-1}\|b_k\|_1 \int_z \chi_{I_k}(z)dz \int_u d\mu(u) \int_x \rho_t(x-z-u)|h(x)|dx \\ \le& c\alpha \int_z \sum_k \chi_{I_k}(z)dz \int Mh(z+u)d\mu(u) \\ \le& c\alpha \int_z \|\sum_k \chi_{I_k}\|_2 \|(Mh)*\mu\|_2 \\ \le& c\alpha(\alpha^{-1}\|f\|_1^2)^{1/2}\|h\|_2 \le c(\alpha\|f\|_1)^{1/2} \end{aligned}$$

which proves (9) hence (8). This completes the proof of Lemma 4.2. □

Similarly we can show

**Lemma 4.4.** *$R$ is $L^1(\ell^2)) \to w - L^1$, i.e.,*

$$|\{x : |\sum_{j\in\mathbb{Z}} \psi_j(H)f_j(x)| > \alpha\}| \le C\alpha^{-1}\|(\sum_j |f_j|^2)^{1/2}\|_1$$

*Proof.* Let $\{f_j\} \in L^1(\ell^2)$, $\alpha > 0$. Let $F(x) = (\sum_{j=-\infty}^{\infty} |f_j(x)|^2)^{1/2}$. By the C-Z decomposition for $F \in L^1$ there exists a collection of disjoint open cubes $\{I_k\}$ such that

$$\begin{aligned} &i) \quad |F(x)| \le \alpha, \ \ \text{a.e. } x \in \mathbb{R}^n \setminus \cup_k I_k \\ &ii) \quad \lambda \le |I_k|^{-1} \int_{I_k} |F(x)|dx \le 2^n\alpha, \quad \forall k. \end{aligned}$$

Define

$$g_j(x) = \begin{cases} |I_k|^{-1}\int_{I_k} f_j dx & x \in I_k \\ f_j(x) & \mathit{otherwise.} \end{cases}$$

and $b_j(x) = f_j(x) - g_j(x)$. Then $|g_j(x)| \le C\alpha$.

Let $x \in I_k$. Minkowski's inequality gives

$$\begin{aligned}(\sum_{j=-\infty}^{\infty} |g_j(x)|^2)^{1/2} &= (\sum_{j=-\infty}^{\infty} (|I_k|^{-1} \int_{I_k} |f_j| dx)^2)^{1/2} \\ &\leq |I_k|^{-1} \int_{I_k} (\sum_{j=-\infty}^{\infty} |f_j|^2)^{1/2} dx \leq 2^n \alpha.\end{aligned}$$

Thus $\int \sum_j |g_j|^2 \leq c\alpha \|F\|_1$ and by Lemma 4.1

$$\begin{aligned}&|\{x : \sum_j |R_j g_j(x)|^2)^{1/2} > \alpha/2\}| \leq C\alpha^{-2} \|\{R_j g_j\}\|^2_{L^2(\ell^2)} \\ &\leq c\alpha^{-2} \|\{g_j\}\|^2_{L^2(\ell^2)} \leq c\alpha^{-1} \|\{f_j\}\|_{L^1(\ell^2)}.\end{aligned}$$

(use the system $\{\phi_j\}_{-\infty}^{\infty}$)

It remains to estimate for $\{b_j(x)\}$. Let $2^{j_k} \sim \ell(I_k)^2$.

$$\begin{aligned}&\int_{\mathbb{R}^n \setminus I_k^*} (\sum_{j \in \mathbb{Z}} |R_j (1 - \Phi_{j_k}(H)) b_{j,k}(x)|^2)^{1/2} dx \\ &= \int_{\mathbb{R}^n \setminus I_k^*} (\sum_{j \sim j_k} | \int_{I_k} \psi_j(H)(1 - \Phi_{j_k}(H)(x,y) b_{j,k}(y) dy|^2)^{1/2} dx \\ &\leq \int_{\mathbb{R}^n \setminus I_k^*} \int_{I_k} (\sum_j |\psi_j(H)(1 - \Phi_{j_k}(H)(x,y) b_{j,k}(y)|^2)^{1/2} dy dx \quad \text{Minkowski} \\ &\leq \int_{\mathbb{R}^n \setminus I_k^*} \int_{I_k} \sup_j |\psi_j(H)(x,y)(1 - \Phi_{j_k}(H)(x,y)| (\sum_j |b_{j,k}(y)|^2)^{1/2} dy dx \\ &\leq \int_{I_k} (\sum_j |b_{j,k}(y)|^2)^{1/2} dy \int_{\mathbb{R}^n \setminus I_k^*} \sum_{j \geq j_k} |\psi_j(H)(1 - \Phi_{j_k}(H)(x,y)| dx \\ &\leq C \int_{I_k} (\sum_j |b_{j,k}(y)|^2)^{1/2} dy \leq C' \int_{I_k} (\sum_j |f_j(y)|^2)^{1/2} dy.\end{aligned}$$

where we used

$$\int_{|x-y| > \ell(I_k)} |\psi_j(H)(x,y)| dx \leq (2^{j/2} \ell(I_k))^{-1} \qquad (N = n+1)$$

by condition (1).

Hence,

$$
\begin{aligned}
&|\{x \in \mathbb{R}^n \setminus \cup_k I_k^* : (\sum_j |R_j \sum_k (1-\Phi_{j_k}(H))b_{j,k}(x)|^2)^{1/2} > \alpha/2\}| \\
\leq & 2\alpha^{-1} \sum_k \int_{\mathbb{R}^n \setminus I_k^*} (\sum_j |R_j(1-\Phi_{j_k}(H))b_{j,k}(x)|^2)^{1/2} dx \\
\leq & C\alpha^{-1} \int (\sum_j |f_j(y)|^2)^{1/2} dy,
\end{aligned}
$$

where $b_j = \sum_k b_{j,k}$ (convergence in $L^1 \cap L^2$ so $T_j b_j(x) = \sum_k T_j b_{j,k}$ in $L^2$) and we used

$$
(\sum_j (\sum_k |T_j b_{j,k}(x)|)^2)^{1/2} \leq \sum_k (\sum_j |T_j b_{j,k}(x)|^2)^{1/2}
$$

by Minkowski inequality.

To estimate $|\{x \notin \cup_k I_k^* : (\sum_j |R_j \sum_k \Phi_{j_k}(H))b_{j,k}(x)|^2)^{1/2} > \alpha/2\}|$, enough

$$
\sum_j \int_{\mathbb{R}^n} |\psi_j(H) \sum_k \Phi_{j_k}(H) b_{j,k}(x)|^2 dx \leq C\alpha \|F\|_1. \tag{10}
$$

Since $R = \{\psi_j(H)\}_{j\in\mathbb{Z}}$ is uniformly bounded $L^2 \to L^2$ or equivalently, $L^2(\ell^2) \to L^2(\ell^2)$, we only need show

$$
\sum_j \int_{\mathbb{R}^n} |\sum_k \Phi_{j_k}(H) b_{j,k}(x)|^2 dx \leq C\alpha \|F\|_1. \tag{11}
$$

Let $h=\{h_j\}\in L^2(\ell^2)$. $\|h\|_{L^2(\ell^2)}\le 1$.

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}\langle\sum_k \Phi_{j_k}(H)b_{j,k}(x),h_j\rangle\\
\le& C\sum_j\sum_k \|b_{j,k}\|_1|I_k|^{-1}\int_{z\in I_k}dz\int_u d\mu(u)\int_x \rho_t(x-z-u)|h_j(x)|dx\\
\le& C\sum_j\sum_k |I_k|^{-1}\|b_{j,k}\|_1\int_z \chi_{I_k}(z)(\mu * Mh_j)(z)dz\\
\le& C(\sum_j\|\sum_k|I_k|^{-1}\|b_{j,k}\|_1\chi_{I_k}\|_2^2)^{1/2}\|Mh_j\|_{L^2(\ell^2)}\\
\le& C(\sum_j\sum_k|I_k|^{-1}\|b_{j,k}\|_1^2)^{1/2}\|h_j\|_{L^2(\ell^2)}\\
\le& C[\sum_k|I_k|^{-1}\sum_j(\int_{I_k}|b_j(y)|dy)^2]^{1/2}\\
\le& C[\sum_k|I_k|^{-1}\big(\int_{I_k}(\sum_j|b_j|^2)^{1/2}dy\big)^2]^{1/2}\le C(\alpha\|F\|_1)^{1/2}.
\end{aligned}
$$

□

where we have used the Fefferman-Stein inequality: if $1<p<\infty$

$$\|(\sum_j(Mf_j)^2)^{1/2}\|_p\le\|(\sum_j|f_j|^2)^{1/2}\|_p\,.$$

**Remark.** In the above proof of two lemmas we can replace $\{\phi_j\}_{j=-\infty}^{\infty}$ with $\{\Phi,\phi_j\}_{j=1}^{\infty}$ to obtain the inhomogeneous result. The homogeneous result is necessary and useful for obtaining Strichartz estimates for wave equation.

## 5. Pöschl-Teller potential

For $H=-d^2/dx^2+V_\nu$ solve the Helmholtz equation

$$He(x,z)=z^2e(x,z). \tag{12}$$

Under suitable asymptotic condition the solution also solves the Lippman-Schwinger equation

$$e(x,k)=e^{ikx}+\frac{1}{2ik}\int e^{ik\,|x-y|}V(y)e(y,k)\,dy\,, \tag{13}$$

that is

$$\begin{aligned} e(x,k) =& e^{ikx} - R_0(k^2+i0)Ve(\cdot,k)(x) \\ =& \sum_{n=0}^{\infty}(-R_0(k^2+i0)V)^n e^{ikx} \\ =& (I+R_0(k^2+i0)V)^{-1}e^{ikx}\,. \end{aligned}$$

Alternatively we can also using the above equations to write

$$e(x,k) = e^{ikx} - R_V(k^2)Ve^{ikx}\,.$$

where the free resolvent has the kernel

$$R_0(k^2+i0)(x,y) = -\frac{1}{2ik}e^{ik|x-y|} \qquad k\in\mathbb{R}\setminus\{0\}.$$

From [30] we know that the continuous spectrum is $\sigma_c = [0,\infty)$, and the point spectrum is $\sigma_p = -1,-4,\cdots,-n^2$. Bound states are Schwartz functions that are bounded by $c\,e^{-|x|}$.

We obtained in [30] the following formula for the continuum eigenfunctions.

**Proposition 5.1.** *Let $k\in\mathbb{R}\setminus\{0\}$. Then*

$$e_n(x,k) = (\operatorname{sign}(k))^n\left(\prod_{j=1}^{n}\frac{1}{j+i|k|}\right)P_n(x,k)e^{ikx},$$

*where $P_n(x,k) = p_n(\tanh x, ik)$ is defined by the recursion formula*

$$p_n(\tanh x, ik) = \frac{d}{dx}\big(p_{n-1}(\tanh x, ik)\big) + (ik - n\tanh x)p_{n-1}(\tanh x, ik)\,.$$

*In particular, the function*

$$\mathbb{R}\times(\mathbb{R}\setminus\{0\}) \ni (x,k)\mapsto e_n(x,k)\in\mathbb{C}$$

*is analytic with $e_n(x,-k) = e_n(-x,k)$. Moreover, the function*

$$(x,y,k)\mapsto e_n(x,k)\overline{e_n(y,k)} = \left(\prod_{j=1}^{n}\frac{1}{j^2+k^2}\right)P_n(x,k)P_n(y,-k)e^{ik(x-y)}$$

*is real analytic on $\mathbb{R}^3$.*

5.2. **Resonance.** The Wronskian can be computed by using Jost functions

$$W(z) = -2(-1)^n ik\prod_{\iota=1}^{n}\frac{\iota+ik}{\iota-ik}\,.$$

The remaining of this section is devoted to proving that conditions (a) and (b) in Theorem 2.2 are true for $H_\nu = H_0 + V_\nu$.

5.3. **Weighted $L^2$ for $m(H)(x,y)$.** Let $K_j(x,y)$ denote the kernel of $(m\phi_j)(H)E_{ac}$. Let $\lambda = 2^{-j/2}$.

**Lemma 5.4.**

$$\|(x-y)K_j(\cdot,y)\|_2 \le \lambda^{1/2} \quad \forall j \tag{14}$$

$$\|(x-y)\partial_y K_j(\cdot,y)\|_2 \le \lambda^{-1/2} \quad j \ge 0 \tag{15}$$

*Proof.* Proof of (14).

$$\begin{aligned}
&2\pi i(x-y)K_j(\cdot,y)\\
=&i(x-y)\int m_j(k^2)e(x,k)\overline{e(y,k)}dk\\
=&\int_{|k|\sim\lambda^{-1}} m_j(k^2)\left(\prod_{j=1}^n \frac{1}{j^2+k^2}\right) P_n(x,k)P_n(y,-k)\partial_k(e^{ik(x-y)})dk\\
=&-\int_{|k|\sim\lambda^{-1}} \partial_k[m_j(k^2)\left(\prod_{j=1}^n \frac{1}{j^2+k^2}\right) P_n(x,k)P_n(y,-k)]e^{ik(x-y)}dk,
\end{aligned}$$

which can be written as finite sums of

$$(\tanh x)^\ell(\tanh y)^k\big[(m_j(k^2))'(\prod_{j=1}^n(j^2+k^2))^{-1}r_{2n}(k)\big]^\vee(x-y)$$

$$(\tanh x)^\ell(\tanh y)^k\big[m_j(k^2)(\prod_{j=1}^n(j^2+k^2))^{-1}r_{2n-1}(k)\big]^\vee(x-y)$$

and

$$(\tanh x)^\ell(\tanh y)^k\big[m_j(k^2)\frac{2k}{\iota^2+k^2}(\prod_{j=1}^n(j^2+k^2))^{-1}q_{2n}(k)\big]^\vee(x-y)$$

$0 \le \ell, k, \iota \le n$, $0 \le i \le 2n$, $r_i, q_i$ are polynomials of degree $i$.

Plancherel formula for Fourier transform gives

$$\|(x-y)K_j(\cdot,y)\|_2 = O(\lambda^{1/2}) = O(2^{-j/4}) \qquad \forall j.$$

using

$$\begin{cases}(m_j(k^2))^{(i)} = O(\frac{1}{k^i}) \quad i=0,1\\ (\prod_{j=1}^n(j^2+k^2))^{-1}r_i(k) = O(1/\langle k\rangle)\\ \frac{2k}{\iota^2+k^2} = O(1/\langle k\rangle)\end{cases}$$

Proof of (15).

$$
\begin{aligned}
&2\pi i(x-y)\partial_y K_j(\cdot,y)\\
=&i(x-y)\int m_j(k^2)e(x,k)\overline{\partial_y e(y,k)}dk\\
=&\int_{|k|\sim\lambda^{-1}}(-ik)m_j(k^2)\left(\prod_{j=1}^{n}\frac{1}{j^2+k^2}\right)P_n(x,k)P_n(y,-k)\partial_k(e^{ik(x-y)})dk\\
=&\int_{|k|\sim\lambda^{-1}}m_j(k^2)\left(\prod_{j=1}^{n}\frac{1}{j^2+k^2}\right)P_n(x,k)\partial_y(P_n(y,-k))\partial_k(e^{ik(x-y)})dk
\end{aligned}
$$

$:= I_1+I_2$

$$
I_1=-\int_{|k|\sim\lambda^{-1}}\partial_k[-ikm_j(k^2)\left(\prod_{j=1}^{n}\frac{1}{j^2+k^2}\right)P_n(x,k)P_n(y,-k)]e^{ik(x-y)}dk,
$$

A similar argument as proving (14) yields

$$
\|(x-y)K'_{1,j}(\cdot,y)\|_2=O(\lambda^{-1/2})=O(2^{j/4})\qquad\forall j.
$$

using

$$
\begin{cases}(m_j(k^2))^{(i)}=O(\frac{1}{k^i}) & i=0,1\\ k=O(k)\\ (\prod_{j=1}^{n}(j^2+k^2))^{-1}r_i(k)=O(1/\langle k\rangle)\\ \frac{2k}{\iota^2+k^2}=O(1/\langle k\rangle)\end{cases}
$$

$$
\begin{aligned}
I_2=&-\int_{|k|\sim\lambda^{-1}}\partial_k[m_j(k^2)\left(\prod_{j=1}^{n}\frac{1}{j^2+k^2}\right)P_n(x,k)\partial_y(P_n(y,-k))]e^{ik(x-y)}dk\\
=&-\mathrm{sech}^2 y\int_{|k|\sim\lambda^{-1}}\partial_k[m_j(k^2)\left(\prod_{j=1}^{n}\frac{1}{j^2+k^2}\right)P_n(x,k)(\partial_y p_n)(\tanh y,-k)]e^{ik(x-y)}dk.
\end{aligned}
$$

where we find if $k\sim\lambda^{-1}$

$$
\partial_k[m_j(k^2)\left(\prod_{j=1}^{n}\frac{1}{j^2+k^2}\right)P_n(x,k)(\partial_y p_n)(\tanh y,-k)]=\begin{cases}O(k^{-2})=O(1) & |k|\to\infty\\ O(k^{-1}) & |k|\to 0\end{cases}
$$

Plancherel formula for Fourier transform gives

$$
\|(x-y)K'_{2,j}(\cdot,y)\|_2=\mathrm{sech}^2 y\begin{cases}O(\lambda^{3/2})=O(\lambda^{-1/2})=O(2^{j/4}) & j\in\mathbb{N}_0\\ O(\lambda^{1/2})=O(2^{-j/4}) & j<0\end{cases}
$$

□

**Lemma 5.5.** *Let* $j \in \mathbb{Z}$.

$$\|K_j(\cdot, y)\|_2 \le \lambda^{-1/2} = 2^{j/4} \quad \forall y$$
$$\|K_j(\cdot, y)\|_\infty \lesssim 2^{j/2}.$$

*Proof.* $K_j(x, y) = (m\phi_j)(H)E_{ac}(x, y)$.

$$2\pi K_j(\cdot, y) = \int m_j(k^2)e(x,k)\overline{e(y,k)}dk$$
$$= \int_{|k|\sim\lambda^{-1}} m_j(k^2)\left(\prod_{j=1}^{n} \frac{1}{j^2+k^2}\right) P_n(x,k)P_n(y,-k)(e^{ik(x-y)})dk$$

which can be written as finite sums of

$$(\tanh x)^\ell(\tanh y)^k \left[(m_j(k^2))(\prod_{j=1}^{n}(j^2+k^2))^{-1}r_{2n}(k)\right]^\vee(x-y)$$

$\therefore$

$$\|m_j(H)(x,y)\|_2 \le \|m_j(k^2)(\prod_{j=1}^{n}(j^2+k^2))^{-1}r_{2n}(k)\|_{L^2_k}$$
$$\sim \|m_j(k^2)\|_{L^2_k} \sim 2^{j/4}(\int_{|k|\le 1} |\phi(k^2)|^2 dk)^{1/2}, \quad \forall j$$

(if $m = 1$, $\phi_j \subset [-2^j, 2^j]$), using

$$\begin{cases} m_j(k^2) = O(1) \\ r_{2n}(k)\prod_{j=1}^{n}(j^2+k^2)^{-1} = O(1) \end{cases}$$

□

**Lemma 5.6.**

$$\|(x-y)\partial_y K_j(\cdot,y)\|_2 \lesssim \begin{cases} 2^{j/4} + \operatorname{sech}^2 y 2^{-j/4} & j \to -\infty \\ 2^{j/4} & j \to \infty \end{cases}$$
$$\|(x-y)\partial_y K_j(\cdot,y)\|_\infty \lesssim 2^{j/2} + \operatorname{sech}^2 y \begin{cases} O(1) & j \to -\infty \\ 2^{-j/2} & j \to \infty \end{cases}$$

**Remark.** This means for $j \to -\infty$, $\|(x-y)\partial_y K_j(\cdot,y)\|_r \sim \operatorname{sech}^2 y 2^{-j/(2r)}$, $r \in [2, \infty]$, which does not seem to help establish the Hörmander integral condition even if using $r$-norm instead of 2-norm.

*Proof.* For 2-norm, it is proved before. For $r = \infty$,

$$
\begin{aligned}
& i2\pi(x-y)\partial_y K_j(\cdot, y) \\
&= -\int \partial_k[km_j(k^2)\left(\prod_{j=1}^{n} \frac{1}{j^2+k^2}\right) p_n(\tanh x, k) p_n(\tanh y, -k)] e^{ik(x-y)} dk \\
&+(-i)\mathrm{sech}^2 y \int_{|k|\sim\lambda^{-1}} \partial_k[m_j(k^2)\left(\prod_{j=1}^{n} \frac{1}{j^2+k^2}\right) p_n(\tanh x, k)(\partial_y p_n)(\tanh y, -k)] e^{ik(x-y)} dk \\
&\sim \int_{|k|\sim\lambda^{-1}} O(1) dk + \mathrm{sech}^2 y \int_{|k|\sim\lambda^{-1}} \begin{cases} O(1/k) & k \to 0 \\ O(1/k^2) = O(1) & k \to \infty \end{cases} dk
\end{aligned}
$$

$(\lambda = 2^{-j/2})$
$\therefore$

$$\|(x-y)\partial_y K_j(\cdot, y)\|_\infty \sim 2^{j/2} + \mathrm{sech}^2 y \begin{cases} O(1) & j \to -\infty \\ 2^{-j/2} & j \to \infty. \end{cases}$$

□

5.7. **Weighted pointwise decay of the kernel.** The problem for pointwise decay of $\Phi_j(H)(x, y)$ in higher energy can be overcome by using an integral version of (3) with a finite measure [1].

**Lemma 5.8.** *(Hebisch-Zheng) Let $\Psi \in C_0^\infty$ be supported in $[-1, 1]$ and let $I$ be any cube in $\mathbb{R}^n$ with length $\ell(I)$. Then for all $x \in \mathbb{R}^n$ and $y \in I$ with $\ell(I) = 2^{-j/2}$ we have*

*a)*

$$|\Phi_j(H)(x, y)| \leq c \int_{u\in\mathbb{R}^n} \frac{2^{jn/2}}{(1 + 2^{j/2}|x - y - u|)^{n+\epsilon}} d\mu(u)$$

*$d\mu(u) = \delta(u) + \langle u\rangle^m e^{-c|u|} du$, some $m \geq 0$.*

*Hence b)*

$$\sup_{y\in I} |\Phi_j(H)(x, y)| \leq \frac{c}{|I|} \int_{z\in I} \int_{u\in\mathbb{R}^n} \frac{2^{jn/2}}{(1 + 2^{j/2}|x - z - u|)^{n+\epsilon}} d\mu(u) dz\,.$$

[1] We modify Hebisch method when the kernel is rough (and slowly decaying), not having Lipschitz smoothness as needed in the Hörmander method

*Proof.* Let $\Psi(x)=\Phi(x^2)$, $\lambda=2^{-j/2}$. According to the formula for the kernel in preceding subsection

$$\begin{aligned}
&\Phi_j(H)(x,y)\\
=&\big[\Psi(\lambda k)\big]^{\vee}(x-y)+\big[\Psi(\lambda k)\sum_{\substack{\iota=1,\dots,n\\ \mu_\iota=0,\dots,N}}\frac{a_{\mu_\iota,\iota}+b_{\mu_\iota,\iota}k}{(\iota^2+k^2)^{\mu_\iota+1}}\big]^{\vee}(x-y)\\
=&\lambda^{-1}\Psi^{\vee}(\lambda^{-1}(x-y))\\
&+c\lambda^{-1}\int_{\mathbb{R}}\Psi^{\vee}(\lambda^{-1}(x-y-u))|u|^m e^{-c|u|}du\,,
\end{aligned}$$

$m\in\mathbb{N}_0$ is an integral constant. Thus

$$\begin{aligned}
&|\Phi_j(H)(x,y)|=|\int_{|k|\lesssim 2^{j/2}}\Psi(2^{-j/2}k)e^{i(x-y)k}(1+\hat{a}(k))dk|\\
=&\big(\lambda^{-n}\Psi^{\vee}(\lambda^{-1}\cdot)*(1+\hat{a})^{\vee}\big)(x-y)\\
\lesssim&\lambda^{-n}\int_{\mathbb{R}^n}(1+\lambda^{-1}|x-y-u|)^{-n-\epsilon}d\mu(u)\ ,
\end{aligned}$$

where $y\in I$, $d\mu(u)=\delta(u)+a(u)du$, $a(u)=|u|^m e^{-c|u|}$ . □

**Remark.** Observe that for $j>0$, $\Phi_j$ actually contains both high+low energy information (if $0\in\operatorname{supp}\Phi$). This is the most technically difficult part. Fortunately with Lemma 5.8 we can control it by maximal function.

5.9. **Kernel decay for a positive potential.** The case is simpler when $V$ is nonnegative. It can be shown [46, 9] that (3) is a weaker assumption than the heat kernel estimate

$$0\le e^{-tH}(x,y)\le c_n t^{-n/2}e^{-cd(x,y)^2/t}\qquad\forall t>0\,.$$

Examples include $H$ being a uniform elliptic operator or its perturbation of order 0. i.e., a Schrödinger operator with $V=V_+-V_-$, $V_-$ is small in Kato norm, cf. [8].

In [32] it is shown if $\sigma(H)\subset[0,\infty)$ then

$$e^{-tH}(x,y)\le Ct^{-n/2}e^{-|x-y|^2/4t}(1+\delta t+|x-y|^2/t)^{n/2}\quad\forall t>0$$

$\delta=\delta(V_-)$ and $\delta=0$ if $V_-=0$.

**Proposition 5.10.** *Let $H$ denote a selfadjoint operator on $(M,g)$ with dimension $n$. Suppose $e^{-tH}$ verifies the upper Gaussian bound*

$$e^{-tH}(x,y)\le C_n t^{-n/2}e^{-cd(x,y)^2/t}\quad\forall t>0.$$

*Then for each $\ell$*

$$|\phi_j(H)(x,y)| \le C_\ell 2^{jn/2}(1+2^{j/2}d(x,y))^{-\ell} \quad \forall j \in \mathbb{Z}\,.$$

5.11. **Unbounded electric and magnetic potentials.** From the recent result of Zoltan Szabo and Tie 2006 on Zeeman operators and twisted Laplacian, which were constructed to represent the physical kinetic energy and potential energy together with a spin phenomenon in a Coulomb model.

† Perturbation of harmonic osillator
$-\Delta + |x|^2 + V(x)$, $\|V\|_\infty < 1 \quad (n = 1, 2, 3)$

† Non-isotropic magnetic potential

$$-\frac{1}{2}\sum_{j=1}^{n}(\partial_{x_j} + ia_j y_j)^2 + (\partial_{y_j} - ia_j x_j)^2$$

$$a_j > 0$$

A detailed study of the heat and Schrödinger kernel can be found in [Sz07], [Tie06]. It is further elaborated in [Z06] on the use of a modified heat kernel estimate for the study of spectral multipliers, dispersive and Strichartz estimates in measuring the regularity of quantum waves in spacetime.

**Remark.** It is worth mentioning that Koch-Tataru [KT06] introduced a semiclassical analysis that deal with general Laplacians for Hermite operator and Laplacian opeator with $C^\infty$ potential on compact manifolds. It provides a general treatment on problems involving the spectral projection, Strichartz estimates and microlocal analysis. However it remains a very interesting and challenging question on how to develop such a treatment for magnetic potentials and noncompact Riemannian manifolds.

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