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# JACOBI DECOMPOSITION OF WEIGHTED TRIEBEL-LIZORKIN AND BESOV SPACES

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ABSTRACT. The Littlewood-Paley theory is extended to weighted spaces of distributions on  $[-1, 1]$  with Jacobi weights  $w(t) = (1-t)^\alpha(1+t)^\beta$ . Almost exponentially localized polynomial elements (needlets)  $\{\varphi_\xi\}$ ,  $\{\psi_\xi\}$  are constructed and, in complete analogy with the classical case on  $\mathbb{R}^n$ , it is shown that weighted Triebel-Lizorkin and Besov spaces can be characterized by the size of the needlet coefficients  $\{\langle f, \varphi_\xi \rangle\}$  in respective sequence spaces.

## 1. INTRODUCTION

The  $\varphi$ -transform of Frazier and Jawerth [6, 7, 8] is a powerful tool for decomposition of spaces of functions or distributions on  $\mathbb{R}^n$ . Our goal in this paper is to develop similar tools for decomposition of weighted spaces of distributions on  $[-1, 1]$  with Jacobi weights

$$(1.1) \quad w(x) := w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta, \quad \alpha, \beta > -1/2.$$

We will build upon the elements constructed in [12] and termed *needlets*. The targeted spaces are weighted Triebel-Lizorkin and Besov spaces on  $[-1, 1]$ .

The main vehicle in constructing our building blocks will be the classical Jacobi polynomials  $\{P_n^{(\alpha,\beta)}\}_{n=0}^\infty$ , which form an orthogonal basis for  $L^2(w) := L^2([-1, 1], w)$  and are normalized by  $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$  [17]. In particular,

$$(1.2) \quad \int_{-1}^1 P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w(x) dx = \delta_{n,m} h_n^{(\alpha,\beta)},$$

where  $h_n^{(\alpha,\beta)} \sim n^{-1}$  with constants of equivalence depending only on  $\alpha$  and  $\beta$ . Then the normalized Jacobi polynomials  $P_n(x) = P_n^{(\alpha,\beta)}(x)$ , defined by

$$(1.3) \quad P_n(x) := (h_n^{(\alpha,\beta)})^{-1/2} P_n^{(\alpha,\beta)}(x), \quad n = 0, 1, \dots,$$

form an orthonormal basis for  $L^2(w)$ , where the inner product is defined by

$$(1.4) \quad \langle f, g \rangle := \int_{-1}^1 f(x) \overline{g(x)} w(x) dx.$$

Consequently, for every  $f \in L^2(w)$

$$(1.5) \quad f = \sum_{n=0}^{\infty} a_n(f) P_n \quad \text{with} \quad a_n(f) := \langle f, P_n \rangle.$$

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Then the kernel of the  $n$ th partial sum operator is

$$(1.6) \quad K_n(x, y) := \sum_{\nu=0}^n P_\nu(x)P_\nu(y).$$

Our construction of needlets relies on the fundamental fact [12] that if the coefficients on the right in (1.6) are “smoothed out” by sampling a compactly supported  $C^\infty$  function, then the resulting kernel has nearly exponential localization around the main diagonal  $y = x$  in  $[-1, 1]^2$ . To be more specific, let

$$(1.7) \quad L_n(x, y) := \sum_{j=0}^{\infty} \widehat{a}\left(\frac{j}{n}\right) P_j(x)P_j(y)$$

with  $\widehat{a}$  admissible in the sense of the following definition:

**Definition 1.1.** *A function  $\widehat{a} \in C^\infty[0, \infty)$  is said to be admissible of type*

- (a) *if  $\text{supp } \widehat{a} \subset [0, 2]$  and  $\widehat{a}(t) = 1$  on  $[0, 1]$ , and of type*
- (b) *if  $\text{supp } \widehat{a} \subset [1/2, 2]$ .*

As a companion to the weight  $w(x)$  we introduce the quantity

$$(1.8) \quad \mathcal{W}(n; x) = \mathcal{W}_{\alpha, \beta}(n; x) := (1 - x + n^{-2})^{\alpha+1/2}(1 + x + n^{-2})^{\beta+1/2}.$$

We will also need the distance on  $[-1, 1]$  defined by

$$(1.9) \quad d(x, y) := |\arccos x - \arccos y|.$$

Now one of the main results from [12] can be stated as follows: Let  $\widehat{a}$  be admissible. Then for any  $\sigma > 0$  there is a constant  $c_\sigma > 0$  depending only on  $\sigma$ ,  $\alpha$ ,  $\beta$ , and  $\widehat{a}$  such that

$$(1.10) \quad |L_n(x, y)| \leq c_\sigma \frac{n}{\sqrt{\mathcal{W}(n; x)}\sqrt{\mathcal{W}(n; y)}(1 + nd(x, y))^\sigma}, \quad x, y \in [-1, 1].$$

The kernels  $L_n(x, y)$  are the main ingredient in constructing needlet systems here. Our construction utilizes a semi-discrete Calderón type decomposition combined with discretization using the Gaussian quadrature formula (see §3). Earlier in [10] a similar scheme has been used for the construction of frames on the sphere.

Denoting by  $\{\varphi_\xi\}_{\xi \in \mathcal{X}}$  and  $\{\psi_\xi\}_{\xi \in \mathcal{X}}$  the constructed analysis and synthesis needlet systems, indexed by a multilevel set  $\mathcal{X} = \cup_{j=0}^{\infty} \mathcal{X}_j$ , we show that every distribution  $f$  on  $[-1, 1]$  ( $f \in \mathcal{D}'$ ) has the representation

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \varphi_\xi \rangle \psi_\xi.$$

In this article we use the needlets to characterize two scales of weighted Triebel-Lizorkin (F-space) and Besov spaces (B-spaces) on  $[-1, 1]$  defined via Jacobi expansions. The idea of using orthogonal or spectral decompositions for introducing Triebel-Lizorkin and Besov spaces is natural and well known, see [14, 18]. To be more precise, let

$$\Phi_0(x, y) := P_0(x)P_0(y) \quad \text{and} \quad \Phi_j(x, y) := \sum_{\nu=0}^{\infty} \widehat{a}\left(\frac{\nu}{2^{j-1}}\right) P_\nu(x)P_\nu(y), \quad j \geq 1,$$

where  $\widehat{a}$  is admissible of type (b) (see Definition 1.1) and  $\widehat{a} > 0$  on  $[3/5, 5/3]$ .

The first scale of  $F$ -spaces  $F_p^{sq}$  with  $s \in \mathbb{R}$ ,  $0 < p < \infty$ ,  $0 < q \leq \infty$ , is defined (§4) as the space of all distributions  $f$  on  $[-1, 1]$  such that

$$\|f\|_{F_p^{sq}} := \left\| \left( \sum_{j=0}^{\infty} (2^{sj} |\Phi_j * f(\cdot)|)^q \right)^{1/q} \right\|_{L^p(w)} < \infty.$$

We define a second scale of  $F$ -spaces  $\mathcal{F}_p^{sq}$  (§5) as the space of all  $f \in \mathcal{D}'$  such that

$$\|f\|_{\mathcal{F}_p^{sq}} := \left\| \left( \sum_{j=0}^{\infty} \left[ 2^{sj} \mathcal{W}(2^j; \cdot)^{-s} |\Phi_j * f(\cdot)| \right]^q \right)^{1/q} \right\|_{L^p(w)} < \infty.$$

(For the definition of  $\Phi_j * f$ , see (2.32).) The corresponding scales of weighted Besov spaces  $B_p^{sq}$  (see [15, 18]) and  $\mathcal{B}_p^{sq}$  with  $s \in \mathbb{R}$ ,  $0 < p, q \leq \infty$ , are defined (§6-7) via the (quasi-)norms

$$\|f\|_{B_p^{sq}} := \left( \sum_{j=0}^{\infty} \left( 2^{sj} \|\Phi_j * f\|_{L^p(w)} \right)^q \right)^{1/q}$$

and

$$\|f\|_{\mathcal{B}_p^{sq}} := \left( \sum_{j=0}^{\infty} \left[ 2^{sj} \|\mathcal{W}(2^j; \cdot)^{-s} \Phi_j * f(\cdot)\|_{L^p(w)} \right]^q \right)^{1/q}.$$

To some extent the second scales of F- and B-spaces are more natural than the first scales since they scale (embed) correctly with respect to the smoothness parameter  $s$  (see §5, §7 for details). Also, the second scale of B-spaces provides the smoothness spaces of nonlinear n-term approximation from needlets (§8).

One of our main results (§4) shows that for all indices the weighted Triebel-Lizorkin spaces  $F_p^{sq}$  can be characterized in terms of the size of the needlet coefficients, namely,

$$\|f\|_{F_p^{sq}} \sim \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} \sum_{\xi \in \mathcal{X}_j} |\langle f, \varphi_\xi \rangle \psi_\xi(\cdot)|^q \right)^{1/q} \right\|_{L^p(w)}.$$

The needlet characterization of the Besov spaces  $B_p^{sq}$  (§6) takes the form

$$\|f\|_{B_p^{sq}} \sim \left( \sum_{j=0}^{\infty} 2^{sjq} \left[ \sum_{\xi \in \mathcal{X}_j} \|\langle f, \varphi_\xi \rangle \psi_\xi\|_p^p \right]^{q/p} \right)^{1/q}.$$

Characterizations of similar nature are obtained for the second scales of weighted Triebel-Lizorkin and Besov spaces  $\mathcal{F}_p^{sq}$  and  $\mathcal{B}_p^{sq}$  (see §5, §7). Using  $L^p(w)$  multipliers we show that the space  $F_p^{02} = \mathcal{F}_p^{02}$  can be identified as  $L^p(w)$  for  $1 < p < \infty$ .

Atomic and molecular decomposition of weighted Triebel-Lizorkin and Besov spaces can be developed using the approach of Frazier and Jawerth [6, 7]. This enables one to make the connection between the weighted F-spaces and the weighted Hardy spaces on  $[-1, 1]$  (see [2, 9]). To prevent this paper from exceeding some reasonable size we leave the atomic and molecular decompositions for elsewhere.

It is an open problem to extend the results from this article to dimensions  $d > 1$ . The missing key element is the nearly exponential localization of kernels of type (1.7) in the multivariate case.

The rest of the paper is organized as follows. In §2, some auxiliary facts are given, including localized and reproducing polynomial kernels, Gaussian quadrature, the maximal inequality, and basics of distributions on  $[-1, 1]$ . In §3, we construct

the needlets and show some of their properties. The first and second scales of weighted Triebel-Lizorkin spaces are defined and characterized via needlets in §4 and §5, respectively, while the first and second scales of Besov spaces are defined and characterized via needlets in §6 and §7. In §8, Besov spaces are applied to weighted nonlinear approximation from needlets; a Jackson theorem is proved. Section 9 is an appendix, where the proofs of some statements are given.

Throughout the paper we use the following notation:

$$\|f\|_p := \left( \int_{-1}^1 |f(x)|^p w(x) dx \right)^{1/p}, \quad 0 < p < \infty, \quad \text{and} \quad \|f\|_\infty := \sup_{x \in [-1,1]} |f(x)|.$$

For a measurable set  $E \subset [-1, 1]$ , we set  $\mu(E) := \int_E w(y) dy$ ;  $\mathbb{1}_E$  is the characteristic function of  $E$  and  $\tilde{\mathbb{1}}_E := |\mu(E)|^{-1/2} \mathbb{1}_E$  is the  $L^2(w)$  normalized characteristic function of  $E$ . Also,  $\Pi_n$  denotes the set of all univariate algebraic polynomials of degree  $\leq n$ . Positive constants are denoted by  $c, c_1, c_*, \dots$  and they may vary at every occurrence. The notation  $A \sim B$  means  $c_1 A \leq B \leq c_2 A$ .

## 2. PRELIMINARIES

**2.1. Localized kernels induced by Jacobi polynomials.** To a large extent our development in this paper relies on the nearly exponential localization (1.10) of kernels  $L_n(x, y)$  of the form (1.7) with admissible  $\hat{a}$ , established in [12]. To avoid some potential confusion, we note that the inner product in [12] is defined by  $\langle f, g \rangle := c_{\alpha, \beta} \int_{-1}^1 f(x) \overline{g(x)} w(x) dx$  with  $c_{\alpha, \beta}^{-1} := \int_{-1}^1 w(x) dx$  and as a result  $L_n(x, y)$  from (1.7) is a constant multiple of  $L^{\alpha, \beta}(x, y)$  from [12]. A similar remark applies to the constants  $h_n^{(\alpha, \beta)}$  from (1.2) and [12].

The proof of estimate (1.10) (see [12]) is based on the almost exponential localization of the univariate polynomial:

$$(2.1) \quad L_n^{\alpha, \beta}(x) := \sum_{j=0}^{\infty} \hat{a} \left( \frac{j}{n} \right) \left( h_j^{(\alpha, \beta)} \right)^{-1} P_j^{(\alpha, \beta)}(1) P_j^{(\alpha, \beta)}(x).$$

**Theorem 2.1.** [1, 12] *Assume that  $\alpha \geq \beta > -1/2$  and let  $\hat{a}$  be admissible. Then for every  $k \geq 1$  there exists a constant  $c_k > 0$  depending only on  $k, \alpha, \beta$ , and  $\hat{a}$  such that*

$$(2.2) \quad |L_n^{\alpha, \beta}(\cos \theta)| \leq c_k \frac{n^{2\alpha+2}}{(1+n\theta)^{k+\alpha-\beta}}, \quad 0 \leq \theta \leq \pi.$$

*The dependence of  $c_k$  on  $\hat{a}$  is of the form  $c_k = c(\alpha, \beta, k) \max_{1 \leq \nu \leq k} \|\hat{a}^{(\nu)}\|_{L^1}$ .*

This estimate was proved in [12] with  $\hat{a}$  admissible of type (b) and in [1] with  $\hat{a}$  admissible of type (a) (for a proof, see also [13]).

In [12, Proposition 1] it is shown that (1.10) yields the following upper bound for the weighted  $L^p$  integrals of  $|L_n(x, y)|$ :

$$(2.3) \quad \int_{-1}^1 |L_n(x, y)|^p w(y) dy \leq c \left( \frac{n}{\mathcal{W}(n; x)} \right)^{p-1}, \quad -1 \leq x \leq 1, \quad 0 < p < \infty.$$

The next theorem shows that in a sense the kernel  $L_n(x, y)$  from (1.7) is Lip 1 in  $x$  (and  $y$ ).

**Theorem 2.2.** *Let  $\alpha, \beta > -1/2$ . Suppose  $\widehat{a}$  is admissible and  $\sigma > 0$  is an arbitrary constant. If  $x, y, z, \xi \in [-1, 1]$ ,  $d(x, \xi) \leq c_* n^{-1}$  and  $d(z, \xi) \leq c_* n^{-1}$  with  $n \geq 1$ ,  $c_* > 0$ , then*

$$(2.4) \quad |L_n(x, y) - L_n(\xi, y)| \leq c_\sigma \frac{n^2 d(x, \xi)}{\sqrt{\mathcal{W}(n; y)} \sqrt{\mathcal{W}(n; z)} (1 + nd(y, z))^\sigma},$$

where  $c_\sigma > 0$  depends only on  $\sigma, \alpha, \beta, c_*$ , and  $\widehat{a}$ .

The proof of this theorem is given in the appendix.

Lower bound estimates for the integrals of  $|L_n(x, y)|$  are nontrivial and will be vital for our further development.

**Proposition 2.3.** *Let  $\widehat{a}$  be admissible and  $|\widehat{a}(t)| \geq c > 0$  for  $t \in [3/5, 5/3]$ . Then*

$$(2.5) \quad \int_{-1}^1 |L_n(x, y)|^2 w(y) dy \geq cn \mathcal{W}(n; x)^{-1}, \quad -1 \leq x \leq 1.$$

**Proof.** By the definition of  $L_n(x, y)$  in (1.7) and the orthogonality of the Jacobi polynomials, it follows that

$$\int_{-1}^1 |L_n(x, y)|^2 w(y) dy \geq c \sum_{k=[n/2]}^{2n} |\widehat{a}(k/n)|^2 [P_k^{(\alpha, \beta)}(x)]^2.$$

Since  $|\widehat{a}(t)| \geq c > 0$  for  $t \in [3/5, 5/3]$  and  $P_k(x) = (h_k^{(\alpha, \beta)})^{-1/2} P_k^{(\alpha, \beta)}(x) \sim k^{1/2} P_k^{(\alpha, \beta)}(x)$ , it suffices to prove that

$$\sum_{k=[3n/5]}^{[5n/3]} [P_k^{(\alpha, \beta)}(x)]^2 \geq c \mathcal{W}(n; x)^{-1}, \quad c > 0,$$

which is established in the following proposition.  $\square$

**Proposition 2.4.** *If  $\alpha, \beta > -1$  and  $\varepsilon > 0$ , then*

$$(2.6) \quad \Lambda_n(x) := \sum_{k=n}^{n+[\varepsilon n]} [P_k^{(\alpha, \beta)}(x)]^2 \geq c \mathcal{W}(n; x)^{-1}, \quad x \in [-1, 1], \quad n \geq 1/\varepsilon,$$

where  $c > 0$  depending only on  $\alpha, \beta$ , and  $\varepsilon$ .

This proposition is nontrivial and its proof is given in the appendix.

**2.2. Reproducing kernels and best polynomial approximation.** We let  $E_n(f)_p$  denote the best approximation of  $f \in L^p(w)$  from  $\Pi_n$ , i.e.

$$(2.7) \quad E_n(f)_p := \inf_{g \in \Pi_n} \|f - g\|_p.$$

To simplify our notation we introduce the following ‘‘convolution’’: For functions  $\Phi : [-1, 1]^2 \rightarrow \mathbb{C}$  and  $f : [-1, 1] \rightarrow \mathbb{C}$ , we write

$$(2.8) \quad \Phi * f(x) := \int_{-1}^1 \Phi(x, y) f(y) w(y) dy.$$

**Lemma 2.5.** *Suppose  $\widehat{a}$  is admissible of type (a) and let  $L_n(x, y)$  be the kernel defined in (1.7).*

(i)  $L_n(x, y)$  is a symmetric reproducing kernel for  $\Pi_n$ , i.e.  $L_n * g = g$  for  $g \in \Pi_n$ .

(ii) For any  $f \in L^p(w)$ ,  $1 \leq p \leq \infty$ , we have  $L_n * f \in \Pi_{2n}$ ,

$$(2.9) \quad \|L_n * f\|_p \leq c \|f\|_p, \quad \text{and} \quad \|f - L_n * f\|_p \leq c E_n(f)_p.$$

**Proof.** Part (i) is immediate since  $\widehat{a}(\frac{\nu}{n}) = 1$  for  $0 \leq \nu \leq n$ . The left-hand side estimate in (2.9) follows from (2.3) when  $p = 1$  and  $p = \infty$ ; the general case follows by interpolation. The right-hand side estimate in (2.9) follows by the left-hand side estimate and (i).  $\square$

Lemma 2.5 (i) and (2.3) are instrumental in proving Nikolski type inequalities.

**Proposition 2.6.** *For  $0 < q \leq p \leq \infty$  and  $g \in \Pi_n$ ,*

$$(2.10) \quad \|g\|_p \leq cn^{(2+2\min\{0, \max\{\alpha, \beta\}\})(1/q-1/p)} \|g\|_q,$$

furthermore, for any  $s \in \mathbb{R}$ ,

$$(2.11) \quad \|\mathcal{W}(n; \cdot)^s g(\cdot)\|_p \leq cn^{1/q-1/p} \|\mathcal{W}(n; \cdot)^{s+1/p-1/q} g(\cdot)\|_q.$$

The proof of this proposition is given in the appendix.

**2.3. Quadrature formula and subdivision of  $[-1, 1]$ .** For the construction of our building blocks (needlets) we will utilize an appropriate Gaussian quadrature formula. Let  $\xi_{j,\nu} =: \cos \theta_\nu$ ,  $\nu = 1, 2, \dots, 2^{j+1}$ , be the zeros of the Jacobi polynomials  $P_{2^{j+1}}^{(\alpha, \beta)}$  ordered so that

$$0 < \theta_1 < \dots < \theta_{2^{j+1}} < \pi.$$

It is well known that uniformly (see [5] and also (9.9)-(9.10) below)

$$(2.12) \quad \theta_{\nu+1} - \theta_\nu \sim 2^{-j} \quad \text{and hence} \quad \theta_\nu \sim \nu 2^{-j}.$$

Define now

$$(2.13) \quad \mathcal{X}_j := \{\xi_{j,\nu} : \nu = 1, 2, \dots, 2^{j+1}\}, \quad j \geq 0, \quad \text{and set} \quad \mathcal{X} := \cup_{j=0}^{\infty} \mathcal{X}_j.$$

As is well known [17] the zeros of the Jacobi polynomial  $P_{2^{j+1}}^{(\alpha, \beta)}$  serve as knots of the Gaussian quadrature

$$(2.14) \quad \int_{-1}^1 f(x)w(x)dx \sim \sum_{\xi \in \mathcal{X}_j} c_\xi f(\xi),$$

which is exact for all polynomials of degree at most  $2^{j+2} - 1$ . Furthermore, the coefficients  $c_\xi$  are positive and have the asymptotic

$$(2.15) \quad c_\xi \sim \lambda_{2^{j+1}}(\xi) \sim 2^{-j} w(\xi) (1 - \xi^2)^{1/2} \sim 2^{-j} \mathcal{W}(2^j; \xi),$$

where  $\lambda_{2^{j+1}}(t)$  is the Christoffel function and the constants of equivalence depend only on  $\alpha, \beta$  (cf. e.g. [11]).

We next introduce the  $j$ th level weighted dyadic intervals. Set as above  $\xi_{j,\nu} =: \cos \theta_\nu$  and define

$$(2.16) \quad I_{\xi_{j,\nu}} := [(\xi_{j,\nu+1} + \xi_{j,\nu})/2, (\xi_{j,\nu-1} + \xi_{j,\nu})/2], \quad \nu = 2, 3, \dots, 2^{j+1} - 1,$$

and

$$(2.17) \quad I_{\xi_{j,1}} := [(\xi_{j,2} + \xi_{j,1})/2, 1], \quad I_{\xi_{j,2^{j+1}}} := [-1, (\xi_{j,2^{j+1}} + \xi_{j,2^{j+1}-1})/2].$$

For  $\xi \in \mathcal{X}_j$  we will briefly denote  $I_\xi := I_{\xi_{j,\nu}}$  if  $\xi = \xi_{j,\nu}$ .

It follows by (2.12) that there exist constants  $c_1, c_2 > 0$  such that

$$(2.18) \quad B_\xi(c_1 2^{-j}) \subset I_\xi \subset B_\xi(c_2 2^{-j}),$$

where  $B_y(r) := \{x \in [-1, 1] : d(x, y) \leq r\}$  with  $d(\cdot, \cdot)$  being the distance from (1.9). Also, it is straightforward to show that

$$(2.19) \quad \mu(I_\xi) := \int_{I_\xi} w(x) dx \sim 2^{-j} \mathcal{W}(2^j; \xi) \sim c_\xi, \quad \xi \in \mathcal{X}_j, \quad j \geq 0.$$

It will be useful to note that

$$(2.20) \quad \mathcal{W}(n; \cos \theta) \sim (\sin \theta + n^{-1})^{2\alpha+1}, \quad 0 \leq \theta \leq 2\pi/3,$$

$$(2.21) \quad \mathcal{W}(n; \cos \theta) \sim (\sin \theta + n^{-1})^{2\beta+1}, \quad \pi/3 \leq \theta \leq \pi.$$

The following simple inequality will be instrumental in various proofs

$$(2.22) \quad \mathcal{W}(n; x) \leq c \mathcal{W}(n; y) (1 + nd(x, y))^{2\max\{\alpha, \beta\}+1}, \quad x, y \in [-1, 1], \quad n \geq 1.$$

For the proof see the appendix.

**2.4. The maximal inequality.** For every  $0 < t < \infty$  and  $x \in [-1, 1]$ , we define

$$(2.23) \quad \mathcal{M}_t f(x) := \sup_{I \ni x} \left( \frac{1}{\mu(I)} \int_I |f(y)|^t w(y) dy \right)^{1/t},$$

where the sup is over all intervals  $I \subset [-1, 1]$  containing  $x$ . It is not hard to see that  $\mu$  is a doubling measure on  $[-1, 1]$  and hence the general theory of maximal inequalities applies. In particular the Fefferman-Stein vector-valued maximal inequality holds (see [16]): If  $0 < p < \infty, 0 < q \leq \infty$  and  $0 < t < \min\{p, q\}$  then for any sequence of functions  $\{f_\nu\}_{\nu=1}^\infty$  on  $[-1, 1]$ ,

$$(2.24) \quad \left\| \left( \sum_{\nu=1}^\infty |\mathcal{M}_t f_\nu(\cdot)|^q \right)^{1/q} \right\|_p \leq c \left\| \left( \sum_{\nu=1}^\infty |f_\nu(\cdot)|^q \right)^{1/q} \right\|_p.$$

We need to estimate  $(\mathcal{M}_t \mathbb{1}_{I_\xi})(x)$  for the intervals  $I_\xi$  from (2.16)-(2.17) and other intervals.

**Lemma 2.7.** *Let  $\eta \in [0, 1]$  and  $0 < \varepsilon \leq \pi$ . Then for  $x \in [-1, 1]$*

$$(2.25) \quad (\mathcal{M}_t \mathbb{1}_{B_\eta(\varepsilon)})(x) \sim \left(1 + \frac{d(\eta, x)}{\varepsilon}\right)^{-1/t} \left(1 + \frac{d(\eta, x)}{\varepsilon + d(\eta, 1)}\right)^{-(2\alpha+1)/t}$$

and hence

$$(2.26) \quad c' \left(1 + \frac{d(\eta, x)}{\varepsilon}\right)^{-(2\alpha+2)/t} \leq (\mathcal{M}_t \mathbb{1}_{B_\eta(\varepsilon)})(x) \leq c \left(1 + \frac{d(\eta, x)}{\varepsilon}\right)^{-1/t}.$$

Here the constants depend only on  $\alpha, \beta$ , and  $t$ .

A similar lemma holds for  $\eta \in [-1, 0)$ . We relegate the proof of this lemma to the appendix.

**2.5. Distributions on  $[-1, 1]$ .** Here we give some basic and well known facts about distributions on  $[-1, 1]$ . We will use as test functions the set  $\mathcal{D} := C^\infty[-1, 1]$  of all infinitely differentiable complex valued functions on  $[-1, 1]$ , where the topology is induced by the semi-norms

$$(2.27) \quad \|\phi\|_k := \sup_{-1 \leq t \leq 1} |\phi^{(k)}(t)|, \quad k = 0, 1, \dots$$



Note that the Jacobi polynomials  $\{P_n\}$  belong to  $\mathcal{D}$ . More importantly, the space  $\mathcal{D}$  of test functions  $\phi$  can be completely characterized by the coefficients of their Jacobi expansions:  $a_n(\phi) := \langle \phi, P_n \rangle := \int_{-1}^1 \phi(x) P_n(x) w(x) dx$ . Denote

$$(2.28) \quad \mathcal{N}_k(\phi) := \sup_{n \geq 0} (n+1)^k |a_n(\phi)|.$$

**Lemma 2.8.** (a)  $\phi \in \mathcal{D}$  if and only if  $a_n(\phi) = \mathcal{O}(n^{-k})$  for all  $k$ .

(b) For every  $\phi \in \mathcal{D}$  we have  $\phi = \sum_{n=0}^{\infty} a_n(\phi) P_n$ , where the convergence is in the topology of  $\mathcal{D}$ .

(c) The topology in  $\mathcal{D}$  can be equivalently defined by the norms  $\mathcal{N}_k(\cdot)$ ,  $k = 0, 1, \dots$ .

**Proof.** If  $\phi \in \mathcal{D}$ , then due to the orthogonality of  $P_n$  to  $\Pi_{n-1}$ , we have for  $n = 1, 2, \dots$

$$|a_n(\phi)| = |\langle \phi, P_n \rangle| = |\langle \phi - Q_{n-1}, P_n \rangle| \leq E_{n-1}(\phi)_2 \leq c_k n^{-k} \|\phi^{(k)}\|_{\infty},$$

where  $Q_{n-1} \in \Pi_{n-1}$  is the polynomial of best  $L^2(w)$  approximation to  $\phi$ . Here we used a simple Jackson estimate for approximation from algebraic polynomials ( $E_n(\phi)_{\infty} \leq c_k n^{-k} \|\phi^{(k)}\|_{\infty}$ ). Therefore,  $a_n(\phi) = \mathcal{O}(n^{-k})$  and  $\mathcal{N}_k(\phi) \leq c_k \|\phi\|_k$  for  $k = 0, 1, \dots$ .

On the other hand, by Markov's inequality it follows that

$$\|P_n^{(k)}\|_{L^{\infty}[-1,1]} \leq n^{2k} \|P_n\|_{L^{\infty}[-1,1]} \leq c n^{2k} h_n^{-1/2} P_n(1) \leq c n^{2k+\alpha+1/2}.$$

Hence, if  $a_n(\phi) = \mathcal{O}(n^{-k})$  for all  $k$ , then  $\phi^{(k)} = \sum_{n=0}^{\infty} a_n(\phi) P_n^{(k)}$  with the series converging uniformly and

$$\|\phi\|_k \leq c \sum_{n=0}^{\infty} |a_n(\phi)| (n+1)^{2k+\alpha+1/2} \leq c \mathcal{N}_{2k+[\alpha+1/2]+1}(\phi), \quad k = 0, 1, \dots,$$

which completes the proof of the lemma.  $\square$

The space  $\mathcal{D}' := \mathcal{D}'[-1, 1]$  of distributions on  $[-1, 1]$  is defined as the set of all continuous linear functionals on  $\mathcal{D}$ . The pairing of  $f \in \mathcal{D}'$  and  $\phi \in \mathcal{D}$  will be denoted by  $\langle f, \phi \rangle := f(\bar{\phi})$ , which will be shown to be consistent with the inner product  $\langle f, g \rangle := \int_{-1}^1 f(x) \overline{g(x)} w(x) dx$  in  $L^2(w)$ . We will need the representation of distributions from  $\mathcal{D}'$  in terms of Jacobi polynomials.

**Lemma 2.9.** (a) A linear functional  $f$  on  $\mathcal{D}$  is a distribution ( $f \in \mathcal{D}'$ ) if and only if there exists  $k \geq 0$  such that

$$(2.29) \quad |\langle f, \phi \rangle| \leq c_k \mathcal{N}_k(\phi) \quad \text{for all } \phi \in \mathcal{D},$$

For  $f \in \mathcal{D}'$ , denote  $a_n(f) := \langle f, P_n \rangle$ . Then for some  $k \geq 0$

$$(2.30) \quad |\langle f, P_n \rangle| \leq c_k (n+1)^k, \quad n = 0, 1, \dots$$

(b) Every  $f \in \mathcal{D}'$  has the representation  $f = \sum_{n=0}^{\infty} a_n(f) P_n$  in distributional sense, i.e.

$$(2.31) \quad \langle f, \phi \rangle = \sum_{n=0}^{\infty} a_n(f) \langle P_n, \phi \rangle = \sum_{n=0}^{\infty} a_n(f) a_n(\bar{\phi}) \quad \text{for all } \phi \in \mathcal{D},$$

where the series converges absolutely.

**Proof.** (a) Part (a) follows immediately by the fact that the topology in  $\mathcal{D}$  can be defined by the norms  $\mathcal{N}_k(\cdot)$  defined in (2.28).

(b) Using Lemma 2.8 (b) we get for  $\phi \in \mathcal{D}$ ,

$$\langle f, \phi \rangle = \lim_{N \rightarrow \infty} \left\langle f, \sum_{n=0}^N a_n(\phi) P_n \right\rangle = \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n(\bar{\phi}) \langle f, P_n \rangle = \sum_{n=0}^{\infty} a_n(f) a_n(\bar{\phi}),$$

where for the last equality we used (2.30) and the fact that  $a_n(\bar{\phi})$  are rapidly decaying.  $\square$

It is convenient to us to extend the ‘‘convolution’’ from (2.8) to the case of distributions.

**Definition 2.10.** *Assuming that  $f \in \mathcal{D}'$  and  $\Phi : [-1, 1]^2 \rightarrow \mathbb{C}$  is such that  $\Phi(x, y)$  belongs to  $\mathcal{D}$  as a function of  $y$  ( $\Phi(x, \cdot) \in \mathcal{D}$ ), we define  $\Phi * f$  by*

$$(2.32) \quad \Phi * f(x) := \langle f, \overline{\Phi(x, \cdot)} \rangle,$$

where on the right  $f$  acts on  $\overline{\Phi(x, y)}$  as a function of  $y$ .

### 3. CONSTRUCTION OF BUILDING BLOCKS (NEEDLETS)

Following the ideas from [12] we next construct two sequences of companion ‘‘analysis’’ and ‘‘synthesis’’ needlets. Our construction is based on a Calderon type reproducing formula. Let  $\widehat{a}, \widehat{b}$  satisfy the conditions

$$(3.1) \quad \widehat{a}, \widehat{b} \in C^\infty[0, \infty), \quad \text{supp } \widehat{a}, \widehat{b} \subset [1/2, 2],$$

$$(3.2) \quad |\widehat{a}(t)|, |\widehat{b}(t)| > c > 0, \quad \text{if } t \in [3/5, 5/3],$$

$$(3.3) \quad \widehat{a}(t)\widehat{b}(t) + \widehat{a}(2t)\widehat{b}(2t) = 1, \quad \text{if } t \in [1/2, 1].$$

Hence,

$$(3.4) \quad \sum_{\nu=0}^{\infty} \widehat{a}(2^{-\nu}t)\widehat{b}(2^{-\nu}t) = 1, \quad t \in [1, \infty).$$

It is easy to show that if  $\widehat{a}$  satisfies (3.1)-(3.2), then there exists  $\widehat{b}$  satisfying (3.1)-(3.2) such that (3.3) holds true (see e.g. [7]).

Assuming that  $\widehat{a}, \widehat{b}$  satisfy (3.1)-(3.3), we define  $\Phi_0(x, y) = \Psi_0(x, y) := P_0(x)P_0(y)$ ,

$$(3.5) \quad \Phi_j(x, y) := \sum_{\nu=0}^{\infty} \widehat{a}\left(\frac{\nu}{2^{j-1}}\right) P_\nu(x)P_\nu(y), \quad j \geq 1,$$

$$(3.6) \quad \Psi_j(x, y) := \sum_{\nu=0}^{\infty} \widehat{b}\left(\frac{\nu}{2^{j-1}}\right) P_\nu(x)P_\nu(y), \quad j \geq 1.$$

Let  $\mathcal{X}_j$  be the set of knots of quadrature formula (2.14), defined in (2.13), and let  $c_\xi$  be the coefficients of the same quadrature. We define the  $j$ th level *needlets* by

$$(3.7) \quad \varphi_\xi(x) := c_\xi^{1/2} \Phi_j(x, \xi) \quad \text{and} \quad \psi_\xi(x) := c_\xi^{1/2} \Psi_j(x, \xi), \quad \xi \in \mathcal{X}_j.$$

As in (2.13) we write  $\mathcal{X} := \cup_{j=0}^{\infty} \mathcal{X}_j$ , where equal points from different levels  $\mathcal{X}_j$  are considered as distinct elements of  $\mathcal{X}$ , so that  $\mathcal{X}$  can be used as an index set. We define the *analysis* and *synthesis* needlet systems  $\Phi$  and  $\Psi$  by

$$(3.8) \quad \Phi := \{\varphi_\xi\}_{\xi \in \mathcal{X}}, \quad \Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}.$$

By estimate (1.10) it follows that the needlets have nearly exponential localization, namely, for  $x \in [-1, 1]$ ,

$$(3.9) \quad |\Phi_j(\xi, x)|, |\Psi_j(\xi, x)| \leq \frac{c_\sigma 2^j}{\sqrt{\mathcal{W}(2^j; \xi)} \sqrt{\mathcal{W}(2^j; x)} (1 + 2^j d(\xi, x))^\sigma} \quad \forall \sigma,$$

and hence

$$(3.10) \quad |\varphi_\xi(x)|, |\psi_\xi(x)| \leq \frac{c_\sigma 2^{j/2}}{\sqrt{\mathcal{W}(2^j; x)} (1 + 2^j d(\xi, x))^\sigma} \quad \forall \sigma.$$

Note that  $x$  in the term  $\sqrt{\mathcal{W}(2^j; x)}$  above can be replaced by  $\xi$  (replacing  $c_\sigma$  by a larger constant), namely,

$$(3.11) \quad |\varphi_\xi(x)|, |\psi_\xi(x)| \leq \frac{c_\sigma 2^{j/2}}{\sqrt{\mathcal{W}(2^j; \xi)} (1 + 2^j d(\xi, x))^\sigma} \quad \forall \sigma.$$

This estimate follows by (3.10) and (2.22).

We will need to estimate the norms of the needlets. We have for  $0 < p \leq \infty$ ,

$$(3.12) \quad \|\varphi_\xi\|_p \sim \|\psi_\xi\|_p \sim \|\mathbb{1}_{I_\xi}\|_p \sim \left( \frac{2^j}{\mathcal{W}(2^j; \xi)} \right)^{1/2-1/p}, \quad \xi \in \mathcal{X}_j.$$

Moreover, there exist constants  $c^*, c^\diamond > 0$  such that

$$(3.13) \quad \|\varphi_\xi\|_{L^\infty(B_\xi(c^* 2^{-j}))}, \|\psi_\xi\|_{L^\infty(B_\xi(c^* 2^{-j}))} \geq c^\diamond \left( \frac{2^j}{\mathcal{W}(2^j; \xi)} \right)^{1/2},$$

where  $B_\xi(c^* 2^{-j}) := \{x \in [-1, 1] : d(\xi, x) \leq c^* 2^{-j}\}$  which is an interval. Notice that if  $\widehat{a}, \widehat{b}$  in (3.1)-(3.3) are real valued then by Proposition 2.4

$$(3.14) \quad \varphi_\xi(\xi), \psi_\xi(\xi) \geq c \left( \frac{2^j}{\mathcal{W}(2^j; \xi)} \right)^{1/2}, \quad c > 0.$$

For the proofs of (3.12)-(3.13), see the appendix.

Our next goal is to establish needlet decompositions of  $\mathcal{D}'$  and  $L^p(w)$ .

**Proposition 3.1.** (a) For  $f \in \mathcal{D}'$ , we have

$$(3.15) \quad f = \sum_{j=0}^{\infty} \Psi_j * \overline{\Phi}_j * f \quad \text{in } \mathcal{D}'$$

and

$$(3.16) \quad f = \sum_{\xi \in \mathcal{X}} \langle f, \varphi_\xi \rangle \psi_\xi \quad \text{in } \mathcal{D}'.$$

(b) If  $f \in L^p(w)$ ,  $1 \leq p \leq \infty$ , then (3.15) – (3.16) hold in  $L^p(w)$ . Moreover, if  $1 < p < \infty$ , then the convergence in (3.15) – (3.16) is unconditional.

**Proof.** (a) Let  $f \in \mathcal{D}'$ . By (2.32) and Lemma 2.9, we have

$$(3.17) \quad \overline{\Phi}_j * f = \sum_{\nu=0}^{2^j} \widehat{a} \left( \frac{\nu}{2^{j-1}} \right) a_\nu(f) P_\nu$$

and further

$$(3.18) \quad \Psi_j * \overline{\Phi}_j * f = \sum_{\nu=0}^{2^j} \widehat{a} \left( \frac{\nu}{2^{j-1}} \right) \widehat{b} \left( \frac{\nu}{2^{j-1}} \right) a_\nu(f) P_\nu.$$

Now (3.15) follows by (3.4) and Lemmas 2.8-2.9.

Note that  $\Psi_j(x, y)$  and  $\Phi_j(x, y)$  are symmetric functions (e.g.  $\Psi_j(y, x) = \Psi_j(x, y)$ ) and hence  $\Psi_j * \overline{\Phi_j}(x, y)$  is well defined. Also,  $\Psi_j * (\overline{\Phi_j} * f) = (\Psi_j * \overline{\Phi_j}) * f$ . We observe that  $\Psi_j(x, u)\overline{\Phi_j}(y, u)$  belongs to  $\Pi_{2^{j+1}-1}$  as a function of  $u$  and apply the quadrature formula from (2.14) to obtain

$$\begin{aligned} \Psi_j * \overline{\Phi_j}(x, y) &= \int_{-1}^1 \Psi_j(x, u)\overline{\Phi_j}(y, u)w(u)du \\ &= \sum_{\xi \in \mathcal{X}_j} c_\xi \Psi_j(x, \xi)\overline{\Phi_j}(y, \xi) = \sum_{\xi \in \mathcal{X}_j} \psi_\xi(x)\overline{\varphi_\xi(y)}. \end{aligned}$$

Hence,

$$\Psi_j * \overline{\Phi_j} * f = \sum_{\xi \in \mathcal{X}_j} \langle f, \varphi_\xi \rangle \psi_\xi.$$

Substituting this in (3.15) yields (3.16).

(b) To prove (3.15) in  $L^p(w)$  we observe that  $\sum_{j=0}^\ell \Psi_j * \overline{\Phi_j} * f = L_\ell * f$  with  $L_\ell := \sum_{j=0}^\ell \Psi_j * \overline{\Phi_j}$ . Because of (3.4),  $L_\ell(x, y)$  is a reproducing kernel for polynomials exactly as the kernels  $L_n(x, y)$  from Lemma 2.5. Hence,  $\sum_{j=0}^\ell \Psi_j * \overline{\Phi_j} * f \rightarrow f$  in  $L^p(w)$  ( $1 \leq p \leq \infty$ ). Then (3.16) in  $L^p(w)$  follows as above. The unconditional convergence in  $L^p(w)$ ,  $1 < p < \infty$ , follows by Proposition 4.3 and Theorem 4.5 below.  $\square$

**Remark 3.2.** *It is easy to see that there exists a function  $\hat{a} \geq 0$  satisfying (3.1) – (3.2) such that  $\hat{a}^2(t) + \hat{a}^2(2t) = 1$ ,  $t \in [1/2, 1]$ . Suppose that in the above construction  $\hat{b} = \hat{a}$  and  $\hat{a} \geq 0$ . Then  $\Phi_j = \Psi_j$  and  $\varphi_\xi = \psi_\xi$ . Now (3.16) becomes  $f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi$ . It is easily seen that  $\{\psi_\xi : \xi \in \mathcal{X}\}$  is a tight frame for  $L^2(w)$  (see [12]).*

#### 4. FIRST SCALE OF WEIGHTED TRIEBEL-LIZORKIN SPACES ON $[-1, 1]$

In analogy to the classical case on  $\mathbb{R}^d$  we will define our first scale of weighted Triebel-Lizorkin spaces by means of the Littlewood-Paley expressions employing the kernels  $\Phi_j$ , defined by

$$(4.1) \quad \Phi_0(x, y) := P_0(x)P_0(y) \quad \text{and} \quad \Phi_j(x, y) := \sum_{\nu=0}^{\infty} \hat{a}\left(\frac{\nu}{2^{j-1}}\right) P_\nu(x)P_\nu(y), \quad j \geq 1,$$

where  $\hat{a}$  satisfies the conditions

$$(4.2) \quad \hat{a} \in C^\infty[0, \infty), \quad \text{supp } \hat{a} \subset [1/2, 2],$$

$$(4.3) \quad |\hat{a}(t)| > c > 0, \quad \text{if } t \in [3/5, 5/3].$$

**Definition 4.1.** *Let  $s \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . Then the weighted Triebel-Lizorkin space  $F_p^{s,q} := F_p^{s,q}(w)$  is defined as the set of all  $f \in \mathcal{D}'$  such that*

$$(4.4) \quad \|f\|_{F_p^{s,q}} := \left\| \left( \sum_{j=0}^{\infty} (2^{sj} |\Phi_j * f(\cdot)|)^q \right)^{1/q} \right\|_p < \infty$$

with the usual modification when  $q = \infty$ .

Observe that the above definition is independent of the choice of  $\hat{a}$  as long as it satisfies (4.2)-(4.3) (see Theorem 4.5 below).

**Proposition 4.2.** *For every  $s \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ ,  $F_p^{sq}$  is a quasi-Banach space which is continuously embedded in  $\mathcal{D}'$ .*

**Proof.** We will only prove the continuous embedding of  $F_p^{sq}$  in  $\mathcal{D}'$ . Then the completeness follows by a standard argument (see e.g. [18], p. 49).

Suppose the kernels  $\Phi_j$  are as in the definition of  $F_p^{sq}$  with  $\hat{a}$  satisfying (4.2)-(4.3) which are the same as (3.1)-(3.2). Then as was already mentioned, there is a function  $\hat{b}$  satisfying (3.1)-(3.2) such that (3.3) holds as well. Let  $\Psi_j$  be defined by (3.6). Then by Proposition 3.1  $f \in F_p^{sq}$  has the representation:  $f = \sum_{j=0}^{\infty} \Psi_j * \Phi_j * f$  in  $\mathcal{D}'$ . Hence for  $\phi \in \mathcal{D}$  we have  $\langle f, \phi \rangle = \sum_{j=0}^{\infty} \langle \Psi_j * \bar{\Phi}_j * f, \phi \rangle$ . Using (3.17)-(3.18) it follows that

$$\langle \Psi_j * \bar{\Phi}_j * f, \phi \rangle = \sum_{\nu=0}^{2^j} \overline{\hat{a}\left(\frac{\nu}{2^{j-1}}\right)} \hat{b}\left(\frac{\nu}{2^{j-1}}\right) a_\nu(f) a_\nu(\phi)$$

and

$$\begin{aligned} |\langle \Psi_j * \bar{\Phi}_j * f, \phi \rangle| &\leq c 2^{j/2} \|\Phi_j * f\|_2 \max_{0 \leq \nu \leq 2^j} |a_\nu(\phi)| \\ (4.5) \qquad \qquad \qquad &\leq c 2^{j(2/p+1/2)} \|\Phi_j * f\|_p \max_{0 \leq \nu \leq 2^j} |a_\nu(\phi)| \\ &\leq c 2^{-j} \|f\|_{F_p^{sq}} \mathcal{N}_k(\phi), \end{aligned}$$

where  $k \geq 2/p + 3/2 - s$ ,  $\mathcal{N}_k(\cdot)$  is from (2.28), and we used inequality (2.10). Consequently,  $|\langle f, \phi \rangle| \leq c \|f\|_{F_p^{sq}} \mathcal{N}_k(\phi)$ , which is the desired embedding.  $\square$

It is natural to define the weighted potential space (generalized weighted Sobolev space)  $H_s^p := H_s^p(w)$ ,  $s > 0$ ,  $1 \leq p \leq \infty$ , on  $[-1, 1]$  as the set of all  $f \in \mathcal{D}'$  such that

$$(4.6) \qquad \|f\|_{H_s^p} := \left\| \sum_{n=0}^{\infty} (n+1)^s a_n(f) P_n(\cdot) \right\|_p < \infty,$$

where  $a_n(f) := \langle f, P_n \rangle$  as in Lemma 2.9.

In the next statement we identify certain weighted Triebel-Lizorkin spaces as weighted potential spaces or  $L^p(w)$ .

**Proposition 4.3.** *We have*

$$(4.7) \qquad F_p^{s2} \sim H_s^p, \quad s > 0, \quad 1 < p < \infty,$$

and

$$(4.8) \qquad F_p^{02} \sim H_0^p \sim L^p(w), \quad 1 < p < \infty,$$

with equivalent norms.

One proves this proposition in a standard way using e.g. the multipliers from [3]. The proof can be carried out exactly as in the case of spherical harmonic expansions, given in [10, Proposition 4.3], and will be omitted.

Associated with  $F_p^{sq}$  is the sequence space  $f_p^{sq}$  defined as follows.

**Definition 4.4.** *Let  $s \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . Then  $f_p^{sq}$  is defined as the space of all complex-valued sequences  $h := \{h_\xi\}_{\xi \in \mathcal{X}}$  such that*

$$(4.9) \qquad \|h\|_{f_p^{sq}} := \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} \sum_{\xi \in \mathcal{X}_j} (|h_\xi| \tilde{\mathbb{1}}_{I_\xi}(\cdot))^q \right)^{1/q} \right\|_p < \infty$$

with the usual modification for  $q = \infty$ . Recall that  $\tilde{\mathbb{1}}_{I_\xi} := \mu(I_\xi)^{-1/2} \mathbb{1}_{I_\xi}$ .

We now introduce the ‘‘analysis’’ and ‘‘synthesis’’ operators

$$(4.10) \quad S_\varphi : f \rightarrow \{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}} \quad \text{and} \quad T_\psi : \{h_\xi\}_{\xi \in \mathcal{X}} \rightarrow \sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi.$$

Here is our main result concerning the weighted  $F$ -spaces.

**Theorem 4.5.** *Let  $s \in \mathbb{R}$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . Then the operators  $S_\varphi : F_p^{sq} \rightarrow f_p^{sq}$  and  $T_\psi : f_p^{sq} \rightarrow F_p^{sq}$  are bounded and  $T_\psi \circ S_\varphi = \text{Id}$  on  $F_p^{sq}$ . Consequently, for  $f \in \mathcal{D}'$  we have that  $f \in F_p^{sq}$  if and only if  $\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}} \in f_p^{sq}$ . Furthermore,*

$$(4.11) \quad \|f\|_{F_p^{sq}} \sim \|\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}}\|_{f_p^{sq}} \sim \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} \sum_{\xi \in \mathcal{X}_j} |\langle f, \varphi_\xi \rangle \psi_\xi(\cdot)|^q \right)^{1/q} \right\|_p.$$

In addition, the definition of  $F_p^{sq}$  is independent of the selection of  $\hat{a}$  satisfying (4.2)–(4.3).

For the proof of this theorem we will need several lemmas whose proofs are given in the appendix.

**Lemma 4.6.** *If  $\xi \in \mathcal{X}_j$ ,  $j \geq 0$ , and  $0 < t < \infty$ , then*

$$(4.12) \quad |\varphi_\xi(x)|, |\psi_\xi(x)| \leq c(\mathcal{M}_t \tilde{\mathbb{1}}_{I_\xi})(x), \quad x \in [-1, 1], \quad \text{and}$$

$$(4.13) \quad \tilde{\mathbb{1}}_{I_\xi}(x) \leq c(\mathcal{M}_t \varphi_\xi)(x), c(\mathcal{M}_t \psi_\xi)(x), \quad x \in [-1, 1].$$

**Lemma 4.7.** *For any  $\sigma > 0$  there exists a constant  $c_\sigma > 0$  such that*

$$(4.14) \quad |\Phi_j * \psi_\xi(x)| \leq c_\sigma \frac{2^{j/2}}{\sqrt{\mathcal{W}(2^j; x)}(1 + 2^j d(\xi, x))^\sigma}, \quad \xi \in \mathcal{X}_\nu, \quad j-1 \leq \nu \leq j+1,$$

and  $\Phi_j * \psi_\xi(x) = 0$  for  $\xi \in \mathcal{X}_\nu$ ,  $\nu \geq j+2$  or  $\nu \leq j-2$ . Here  $\mathcal{X}_\nu := \emptyset$  if  $\nu < 0$ .

**Definition 4.8.** *For a collection of complex numbers  $\{h_\xi\}_{\xi \in \mathcal{X}_j}$  we let*

$$(4.15) \quad h_\xi^* := \sum_{\eta \in \mathcal{X}_j} \frac{|h_\eta|}{(1 + 2^j d(\eta, \xi))^\sigma}.$$

Here  $\sigma > 1$  is sufficiently large and will be selected later on.

**Lemma 4.9.** *Suppose that  $P \in \Pi_{2j}$ ,  $j \geq 0$ , and let  $a_\xi := \max_{x \in I_\xi} |P(x)|$ . There exists  $r \geq 1$ , depending only on  $\sigma$ ,  $\alpha$ , and  $\beta$ , such that if*

$$b_\xi := \max\{\min_{x \in I_\eta} |P(x)| : \eta \in \mathcal{X}_{j+r}, I_\xi \cap I_\eta \neq \emptyset\},$$

then

$$(4.16) \quad a_\xi^* \sim b_\xi^*$$

with constants of equivalence independent of  $P$ ,  $j$  and  $\xi$ .

**Lemma 4.10.** *Assume  $t > 0$  and let  $\{b_\xi\}_{\xi \in \mathcal{X}_j}$  ( $j \geq 0$ ) be a collection of complex numbers. Suppose that  $\sigma > (4 \max\{\alpha, \beta\} + 3)/t + 1$  in the definition (4.15) of  $b_\xi^*$ . Then*

$$b_\xi^* \mathbb{1}_{I_\xi}(x) \leq c \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_j} |b_\eta| \mathbb{1}_{I_\eta} \right)(x), \quad x \in I_\xi, \quad \xi \in \mathcal{X}_j.$$

**Proof of Theorem 4.5.** Suppose  $\alpha \geq \beta$ . Fix  $0 < t < \min\{p, q\}$  and let  $\sigma > (4\alpha + 3)/t + 1$ . We first note that the right-hand side equivalence in (4.11) follows immediately from Lemma 4.6 and the maximal inequality (2.24).

Assume that  $\{\Phi_j\}$  are from the definition of weighted Triebel-Lizorkin spaces, i.e.  $\Phi_j$  are defined by (4.1), where  $\hat{a}$  satisfies (4.2)-(4.3), the same as (3.1)-(3.2). As already mentioned, there exists a function  $\hat{b}$  satisfying (3.1)-(3.2) such that (3.3) holds. Let  $\Psi_j$  be defined by (3.6) using this  $\hat{b}$ . Also, let  $\{\varphi_\xi\}_{\xi \in \mathcal{X}}$  and  $\{\psi_\xi\}_{\xi \in \mathcal{X}}$  be the associated needlet systems defined as in (3.7).

Further, let  $\{\tilde{\Phi}_j\}$  be a second sequence of kernels like the kernels  $\{\Phi_j\}$  from above but defined by a different function  $\hat{a}$ . Also, we assume that a sequence of companion kernels  $\{\tilde{\Psi}_j\}$  is constructed as above and let  $\{\tilde{\varphi}_\xi\}$ ,  $\{\tilde{\psi}_\xi\}$  be the associated needlet systems, defined as in (3.5)-(3.7). So, we have two totally different systems of kernels and associated needlet systems.

We first establish the boundedness of  $T_{\tilde{\psi}} : f_p^{sq} \rightarrow F_p^{sq}$ , defined similarly as in (4.10), where the space  $F_p^{sq}$  is defined by  $\{\Phi_j\}$ . Let  $h := \{h_\xi\}_{\xi \in \mathcal{X}}$  be a finitely supported sequence and  $f := \sum_\xi h_\xi \psi_\xi$ . Using (4.14) we have for  $x \in [-1, 1]$ ,

$$\begin{aligned} |\Phi_j * f(x)| &= \left| \sum_{\xi \in \mathcal{X}} h_\xi \Phi_j * \tilde{\psi}_\xi(x) \right| \leq \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} |h_\xi| |\Phi_j * \tilde{\psi}_\xi(x)| \\ &\leq c2^{j/2} \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} \frac{|h_\xi|}{\sqrt{\mathcal{W}(2^\nu; x)}(1 + 2^\nu d(\xi, x))^\sigma}. \end{aligned}$$

Fix  $\eta \in \mathcal{X}_j$  and denote  $\mathcal{Y}_\eta := \{\xi \in \mathcal{X}_{j-1} \cup \mathcal{X}_j \cup \mathcal{X}_{j+1} : I_\xi \cap I_\eta \neq \emptyset\}$  ( $\mathcal{X}_{-1} := \emptyset$ ). Notice that  $\#\mathcal{Y}_\eta \leq \text{constant}$  and  $d(x, \xi) \leq c2^{-j}$  if  $x \in I_\xi$  and  $\xi \in \mathcal{Y}_\eta$ . Hence, we have for  $x \in I_\eta$

$$\begin{aligned} |\Phi_j * f(x)| &\leq c2^{j/2} \sum_{j-1 \leq \nu \leq j+1} \sum_{\omega \in \mathcal{Y}_\eta \cap \mathcal{X}_\nu} \sum_{\xi \in \mathcal{X}_\nu} \frac{|h_\xi| \mathbb{1}_\omega(x)}{\sqrt{\mathcal{W}(2^\nu; \omega)}(1 + 2^\nu d(\xi, \omega))^\sigma} \\ &\leq c2^{j/2} \sum_{\omega \in \mathcal{Y}_\eta} \frac{h_\omega^* \mathbb{1}_\omega(x)}{\sqrt{\mathcal{W}(2^j; \omega)}} \leq c \sum_{\omega \in \mathcal{Y}_\eta} h_\omega^* \tilde{\mathbb{1}}_\omega(x), \end{aligned}$$

where we also used (2.19). We now insert this in (4.4) and use Lemma 4.10 and the maximal inequality (2.24) to obtain

$$\begin{aligned} \|f\|_{F_p^{sq}} &\leq c \left\| \left( \sum_{j=0}^{\infty} \left[ 2^{sj} \sum_{\eta \in \mathcal{X}_j} \sum_{\omega \in \mathcal{Y}_\eta} h_\omega^* \tilde{\mathbb{1}}_{I_\omega}(\cdot) \right]^q \right)^{1/q} \right\|_p \\ (4.17) \quad &\leq c \left\| \left( \sum_{j=0}^{\infty} \left[ 2^{sj} \sum_{\xi \in \mathcal{X}_j} h_\xi^* \tilde{\mathbb{1}}_{I_\xi}(\cdot) \right]^q \right)^{1/q} \right\|_p \\ &\leq c \left\| \left( \sum_{j=0}^{\infty} \left[ \mathcal{M}_t \left( \sum_{\xi \in \mathcal{X}_j} 2^{sj} |h_\xi| \tilde{\mathbb{1}}_{I_\xi}(\cdot) \right)^q \right]^{1/q} \right) \right\|_p \\ &\leq c \| \{h_\xi\} \|_{f_p^{sq}}. \end{aligned}$$

For the second estimate above it was important that  $\#\mathcal{Y}_\eta \leq c$ . This establishes the desired result for finitely supported sequences. Using the continuous embedding of  $F_p^{sq}$  in  $\mathcal{D}'$  (Lemma 4.2) and the density of finitely supported sequences in  $f_p^{sq}$  it follows from (4.17) that for every  $h \in f_p^{sq}$ ,  $T_{\tilde{\psi}} h := \sum_{\xi \in \mathcal{X}} h_\xi \tilde{\psi}_\xi$  is a well defined

distribution in  $\mathcal{D}'$ . Thus a standard density argument shows that  $T_{\tilde{\psi}} : f_p^{sq} \rightarrow F_p^{sq}$  is bounded.

We next prove the boundedness of the operator  $S_\varphi : F_p^{sq} \rightarrow f_p^{sq}$ , where we assume this time that  $F_p^{sq}$  is defined in terms of  $\{\bar{\Phi}_j\}$ . Let  $f \in F_p^{sq}$ . Then  $\bar{\Phi}_j * f \in \Pi_{2^j}$ . For  $\xi \in \mathcal{X}_j$ , we set

$$a_\xi := \max_{x \in I_\xi} |\bar{\Phi}_j * f(x)|, \quad b_\xi := \max_{x \in I_\eta} \{\min_{x \in I_\eta} |\bar{\Phi}_j * f(x)| : \eta \in \mathcal{X}_{j+r}, I_\xi \cap I_\eta \neq \emptyset\}.$$

Assuming that  $r$  above is the constant from Lemma 4.9, it follows by the same lemma that  $a_\xi^* \sim b_\xi^*$ . Therefore,

$$|\langle f, \varphi_\xi \rangle| = c_\xi^{1/2} |\bar{\Phi}_j * f(\xi)| \leq c\mu(I_\xi)^{1/2} a_\xi \leq c\mu(I_\xi)^{1/2} a_\xi^* \leq c\mu(I_\xi)^{1/2} b_\xi^*.$$

From this, taking into account that  $\tilde{\mathbb{1}}_{I_\xi} := \mu(I_\xi)^{-1/2} \mathbb{1}_{I_\xi}$ , we obtain

$$\begin{aligned} \|\langle f, \varphi_\xi \rangle\|_{f_p^{sq}} &:= \left\| \left( \sum_{j \geq 0} 2^{jsq} \sum_{\xi \in \mathcal{X}_j} [|\langle f, \varphi_\xi \rangle| \tilde{\mathbb{1}}_{I_\xi}(\cdot)]^q \right)^{1/q} \right\|_p \\ &\leq c \left\| \left( \sum_{j \geq 0} 2^{jsq} \sum_{\xi \in \mathcal{X}_j} [b_\xi^* \mathbb{1}_{I_\xi}(\cdot)]^q \right)^{1/q} \right\|_p \\ (4.18) \quad &\leq c \left\| \left( \sum_{j \geq 0} 2^{jsq} \left[ \mathcal{M}_t \left( \sum_{\xi \in \mathcal{X}_j} b_\xi \mathbb{1}_{I_\xi} \right) (\cdot) \right]^q \right)^{1/q} \right\|_p \\ &\leq c \left\| \left( \sum_{j \geq 0} 2^{jsq} \sum_{\xi \in \mathcal{X}_j} b_\xi^q \mathbb{1}_{I_\xi}(\cdot) \right)^{1/q} \right\|_p, \end{aligned}$$

where for the second inequality above we used Lemma 4.10 and for the third the maximal inequality (2.24).

Let  $m_\eta := \min_{x \in I_\eta} |\bar{\Phi}_j * f(x)|$  for  $\xi \in \mathcal{X}_{j+r}$  and denote, for  $\xi \in \mathcal{X}_j$ ,

$$\mathcal{X}_{j+r}(\xi) := \{w \in \mathcal{X}_{j+r} : I_w \cap I_\xi \neq \emptyset\}.$$

Evidently,  $\#\mathcal{X}_{j+r}(\xi) \leq \tilde{c}$ ,  $\tilde{c} = \tilde{c}(r)$ . Hence,  $d(w, \eta) \leq c(r)2^{-j-r}$  for  $w, \eta \in \mathcal{X}_{j+r}(\xi)$  and therefore

$$m_w \leq c \frac{m_w}{1 + 2^{j+r} d(w, \eta)} \leq cm_\eta^*.$$

Consequently, for any  $\xi \in \mathcal{X}_j$  and  $\eta \in \mathcal{X}_{j+r}(\xi)$ , we have  $b_\xi = \max_{w \in \mathcal{X}_{j+r}(\xi)} m_w \leq cm_\eta^*$  and hence

$$b_\xi \mathbb{1}_{I_\xi} \leq c \sum_{\eta \in \mathcal{X}_{j+r}(\xi)} m_\eta^* \mathbb{1}_{I_\eta}.$$



Using this estimate in (4.18) we get

$$\begin{aligned}
\|\langle f, \varphi_\xi \rangle\|_{f_p^{sq}} &\leq c \left\| \left( \sum_{j \geq 0} 2^{jsq} \left( \sum_{\eta \in \mathcal{X}_{j+r}} m_\eta^* \mathbb{1}_{I_\eta}(\cdot) \right)^q \right)^{1/q} \right\|_p \\
&\leq c \left\| \left( \sum_{j \geq 0} 2^{jsq} \left[ \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_{j+r}} m_\eta \mathbb{1}_{I_\eta}(\cdot) \right) \right]^q \right)^{1/q} \right\|_p \\
&\leq c \left\| \left( \sum_{j \geq 0} \left( 2^{js} \sum_{\xi \in \mathcal{X}_j} m_\xi \mathbb{1}_{I_\xi}(\cdot) \right)^q \right)^{1/q} \right\|_p \\
&\leq c \left\| \left( \sum_{j \geq 0} (2^{js} |\tilde{\Phi}_j * f|)^q \right)^{1/q} \right\|_p.
\end{aligned}$$

Thus the boundedness of  $S_\varphi : F_p^{sq} \rightarrow f_p^{sq}$  is established.

The identity  $T_\psi \circ S_\varphi = Id$  follows by Proposition 3.1.

It remains to show that  $F_p^{sq}$  is independent of the particular selection of  $\hat{a}$  in the definition of  $\{\Phi_j\}$ . Denote for an instant by  $\|f\|_{F_p^{sq}(\Phi)}$  the F-norm defined by  $\{\Phi_j\}$ . Then by the above proof it follows that

$$\|f\|_{F_p^{sq}(\Phi)} \leq c \|\langle f, \tilde{\varphi}_\xi \rangle\|_{f_p^{sq}} \quad \text{and} \quad \|\langle f, \varphi_\xi \rangle\|_{f_p^{sq}} \leq c \|f\|_{F_p^{sq}(\bar{\Phi})}$$

and hence

$$\|f\|_{F_p^{sq}(\Phi)} \leq c \|\langle f, \tilde{\varphi}_\xi \rangle\|_{f_p^{sq}} \leq c \|f\|_{F_p^{sq}(\bar{\Phi})}.$$

Now the desired independence follows by reversing the roles of  $\{\Phi_j\}, \{\tilde{\Phi}_j\}$ , and their complex conjugates.  $\square$

## 5. SECOND SCALE OF WEIGHTED TRIEBEL-LIZORKIN SPACES ON $[-1, 1]$

We introduce our second scale of Triebel-Lizorkin spaces by utilizing again the kernels  $\Phi_j$  defined by (4.1) with  $\hat{a}$  satisfying (4.2)-(4.3) (compare with §4).

**Definition 5.1.** *Let  $s \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . Then the weighted Triebel-Lizorkin space  $\mathcal{F}_p^{sq} := \mathcal{F}_p^{sq}(w)$  is defined as the set of all  $f \in \mathcal{D}'$  such that*

$$(5.1) \quad \|f\|_{\mathcal{F}_p^{sq}} := \left\| \left( \sum_{j=0}^{\infty} \left[ 2^{sj} \mathcal{W}(2^j; \cdot)^{-s} |\Phi_j * f(\cdot)| \right]^q \right)^{1/q} \right\|_p < \infty$$

with the usual modification when  $q = \infty$ .

Observe that the above definition is independent of the choice of  $\hat{a}$  as long as it satisfies (4.2)-(4.3) (see Theorem 5.3 below). Following in the footsteps of the development from §4, it is easy to show that  $\mathcal{F}_p^{sq}$  is a complete quasi-Banach space, which is embedded continuously in  $\mathcal{D}'$ . For the latter one proceeds as in the proof of Proposition 4.2, where in (4.5) one, in addition, uses the obvious estimate  $\|g\|_2 \leq cn^\gamma \|\mathcal{W}(n; \cdot)^s g(\cdot)\|_2$ , where  $\gamma := (2 \min\{\alpha, \beta\} + 1)s_+$ , which is immediate from  $c_1 n^{-2 \min\{\alpha, \beta\} - 1} \leq \mathcal{W}(n; x) \leq c_2$ ,  $x \in [-1, 1]$ . We skip the details.

The sequence space  $\mathbf{f}_p^{sq}$  associated with  $\mathcal{F}_p^{sq}$  is now defined as follows.

**Definition 5.2.** *Let  $s \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . Then  $\mathbf{f}_p^{sq}$  is defined as the space of all complex-valued sequences  $h := \{h_\xi\}_{\xi \in \mathcal{X}}$  such that*

$$(5.2) \quad \|h\|_{\mathbf{f}_p^{sq}} := \left\| \left( \sum_{\xi \in \mathcal{X}} \left[ \mu(I_\xi)^{-s} |h_\xi| \tilde{\mathbb{1}}_{I_\xi}(\cdot) \right]^q \right)^{1/q} \right\|_p < \infty$$

with the usual modification when  $q = \infty$ .

To characterize the Triebel-Lizorkin spaces  $\mathcal{F}_p^{sq}$  we use again the operators  $S_\varphi$  and  $T_\psi$  defined in (4.10) and the sequence spaces  $\mathbf{f}_p^{sq}$ .

**Theorem 5.3.** *Let  $s \in \mathbb{R}$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . Then the operators  $S_\varphi : \mathcal{F}_p^{sq} \rightarrow \mathbf{f}_p^{sq}$  and  $T_\psi : \mathbf{f}_p^{sq} \rightarrow \mathcal{F}_p^{sq}$  are bounded and  $T_\psi \circ S_\varphi = Id$  on  $\mathcal{F}_p^{sq}$ . Consequently, for  $f \in \mathcal{D}'$  we have that  $f \in \mathcal{F}_p^{sq}$  if and only if  $\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}} \in \mathbf{f}_p^{sq}$ . Furthermore,*

$$(5.3) \quad \|f\|_{\mathcal{F}_p^{sq}} \sim \|\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}}\|_{\mathbf{f}_p^{sq}} \sim \left\| \left( \sum_{\xi \in \mathcal{X}} \left[ \mu(I_\xi)^{-s} |\langle f, \varphi_\xi \rangle \psi_\xi(\cdot)| \right]^q \right)^{1/q} \right\|_p.$$

In addition, the definition of  $\mathcal{F}_p^{sq}$  is independent of the selection of  $\hat{a}$  satisfying (4.2)–(4.3).

The proof of this theorem is similar to the proof of Theorem 4.5. The only new ingredient is the following lemma.

**Lemma 5.4.** *Let  $t > 0$  and  $s \in \mathbb{R}$ . Suppose  $\{b_\xi\}_{\xi \in \mathcal{X}_j}$  ( $j \geq 0$ ) is a collection of complex numbers and let  $\sigma > (4 \max\{\alpha, \beta\} + 3)(1/t + |s|) + 1$  in the definition (4.15) of  $b_\xi^*$ . Then*

$$(5.4) \quad \mu(I_\xi)^{-s} b_\xi^* \mathbb{1}_{I_\xi}(x) \leq c \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_j} \mu(I_\eta)^{-s} |b_\eta| \mathbb{1}_{I_\eta} \right)(x), \quad x \in I_\xi, \quad \xi \in \mathcal{X}_j,$$

**Proof.** For  $\xi \in \mathcal{X}_j$ ,  $\mu(I_\xi) \sim 2^{-j} \mathcal{W}(2^j; \xi)$  and hence, using (2.22),

$$\mu(I_\xi)^{-s} b_\xi^* \leq c \sum_{\eta \in \mathcal{X}_j} \frac{2^{js} \mathcal{W}(2^j; \xi)^{-s} |b_\eta|}{(1 + 2^j d(\xi, \eta))^\sigma} \leq c \sum_{\eta \in \mathcal{X}_j} \frac{2^{js} \mathcal{W}(2^j; \eta)^{-s} |b_\eta|}{(1 + 2^j d(\xi, \eta))^{\sigma_1}} \leq c \left( \mu(I_\eta)^{-s} |b_\eta| \right)^*,$$

where  $\sigma_1 := \sigma - (2 \max\{\alpha, \beta\} + 1)|s| > (4 \max\{\alpha, \beta\} + 3)/t + 1$ . Now (5.4) follows by Lemma 4.10.  $\square$

Now the proof of Theorem 5.3 can be carried out as the proof of Theorem 4.5, using Lemma 5.4 in place of Lemma 4.10 and selecting  $\sigma$  in the definitions of  $h_\xi^*$  and  $a_\xi^*$ ,  $b_\xi^*$  sufficiently large. We skip the further details.

In a sense the spaces  $\mathcal{F}_p^{sq}$  are more natural than the spaces  $F_p^{sq}$  from §4 since they scale (are embedded) “correctly” with respect to the smoothness index  $s$ .

**Proposition 5.5.** *Let  $0 < p < p_1 < \infty$ ,  $0 < q, q_1 \leq \infty$ , and  $0 < s_1 < s < \infty$ . Then we have the continuous embedding*

$$(5.5) \quad \mathcal{F}_p^{sq} \subset \mathcal{F}_{p_1}^{s_1 q_1} \quad \text{if} \quad s - 1/p = s_1 - 1/p_1.$$

The proof of this embedding result can be carried out similarly as in the classical case on  $\mathbb{R}^n$  using inequality (2.11) and Theorem 5.3 (see e.g. [18], page 129). It will be omitted.

## 6. FIRST SCALE OF WEIGHTED BESOV SPACES ON $[-1, 1]$

To introduce the first scale of weighted Besov spaces we use the kernels  $\Phi_j$  defined in (4.1) with  $\hat{a}$  satisfying (4.2)–(4.3) (see [15, 18]).

**Definition 6.1.** Let  $s \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . Then the weighted Besov space  $B_p^{sq} := B_p^{sq}(w)$  is defined as the set of all  $f \in \mathcal{D}'$  such that

$$\|f\|_{B_p^{sq}} := \left( \sum_{j=0}^{\infty} \left( 2^{sj} \|\Phi_j * f\|_p \right)^q \right)^{1/q} < \infty,$$

where the  $\ell_q$ -norm is replaced by the sup-norm if  $q = \infty$ .

Note that as in the case of weighted Triebel-Lizorkin spaces the above definition is independent of the choice of  $\hat{a}$  satisfying (4.2)-(4.3) (see Theorem 6.5). Also, the Besov space  $B_p^{sq}(w)$  is a quasi-Banach space which is continuously embedded in  $\mathcal{D}'$ .

Our next goal is to link the weighted Besov spaces with best polynomial approximation in  $L^p(w)$ . Recall that  $E_n(f)_p$  denotes the best approximation of  $f \in L^p(w)$  from  $\Pi_n$  (see (2.7)).

**Proposition 6.2.** Let  $s > 0$ ,  $1 \leq p \leq \infty$ , and  $0 < q \leq \infty$ . Then  $f \in B_p^{sq}$  if and only if

$$(6.1) \quad \|f\|_{B_p^{sq}}^A := \|f\|_p + \left( \sum_{j=0}^{\infty} (2^{sj} E_{2^j}(f)_p)^q \right)^{1/q} < \infty.$$

Moreover,

$$(6.2) \quad \|f\|_{B_p^{sq}}^A \sim \|f\|_{B_p^{sq}}.$$

**Proof.** Let  $f \in B_p^{sq}$ . It is easy and standard to show that under the assumptions on  $s$ ,  $p$ , and  $q$  the space  $B_p^{sq}$  is continuously imbedded in  $L^p(w)$ , i.e.  $f$  can be identified as a function in  $L^p(w)$  and  $\|f\|_p \leq c\|f\|_{B_p^{sq}}$ .

It is easy to construct (see e.g. [6]) a function  $\hat{a} \geq 0$  satisfying (4.2)-(4.3) such that  $\hat{a}(t) + \hat{a}(2t) = 1$  for  $t \in [1/2, 1]$  and hence

$$(6.3) \quad \sum_{\nu=0}^{\infty} \hat{a}(2^{-\nu}t) = 1, \quad t \in [1, \infty).$$

Assume that  $\{\Phi_j\}$  are defined by (4.1) with such a function  $\hat{a}$ . As in Proposition 3.1, it is easy to see that  $f = \sum_{j=0}^{\infty} \Phi_j * f$  in  $L^p(w)$ . Hence, since  $\Phi_j * f \in \Pi_{2^j}$ ,

$$(6.4) \quad E_{2^\ell}(f)_p \leq \sum_{j=\ell+1}^{\infty} \|\Phi_j * f\|_p, \quad \ell \geq 0.$$

Now, a standard argument using (6.4) shows that  $\|f\|_{B_p^{sq}}^A \leq c\|f\|_{B_p^{sq}}$ .

To prove the estimate in the other direction, we note that  $\Phi_j * f = \Phi_j * (f - Q)$  for  $Q \in \Pi_{2^{j-2}}$  ( $j \geq 2$ ). Hence, as in Lemma 2.5,  $\|\Phi_j * f\|_p \leq c\|f - Q\|_p$ . Therefore,

$$\|\Phi_j * f\|_p \leq cE_{2^{j-2}}(f)_p, \quad j \geq 2, \quad \text{and} \quad \|\Phi_j * f\|_p \leq c\|f\|_p,$$

which imply  $\|f\|_{B_p^{sq}} \leq c\|f\|_{B_p^{sq}}^A$ .

Above we used that the definition of  $B_p^{sq}$  is independent of the selection of  $\hat{a}$ , satisfying (4.2)-(4.3).  $\square$

**Remark 6.3.** Proposition 6.2 shows that when  $s > 0$  and  $1 \leq p \leq \infty$  the weighted Besov spaces  $B_p^{sq}$  can be identified as approximation spaces induced by best polynomial approximation in  $L^p(w)$ . Also, it is worth mentioning that  $E_n(f)_p$  can

be characterized via the weighted moduli of smoothness of Ditzian-Totik [4]. Consequently, the weighted moduli of smoothness can be used for characterization of weighted Besov spaces as well.

It is natural to associate with the weighted Besov space  $B_p^{sq}$  the sequence space  $b_p^{sq}$  defined as follows.

**Definition 6.4.** Let  $s \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . Then  $b_p^{sq} := b_p^{sq}(w)$  is defined to be the space of all complex-valued sequences  $h := \{h_\xi\}_{\xi \in \mathcal{X}}$  such that

$$\|h\|_{b_p^{sq}} := \left( \sum_{j=0}^{\infty} 2^{jsq} \left[ \sum_{\xi \in \mathcal{X}_j} \left( \mu(I_\xi)^{1/p-1/2} |h_\xi| \right)^{p \cdot q/p} \right]^{1/q} \right)^{1/q} < \infty$$

with the usual modification for  $p = \infty$  or  $q = \infty$ .

Our main result in this section is the following characterization of weighted Besov spaces, which employs the operators  $S_\varphi$  and  $T_\psi$  defined in (4.10).

**Theorem 6.5.** Let  $s \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . The operators  $S_\varphi : B_p^{sq} \rightarrow b_p^{sq}$  and  $T_\psi : b_p^{sq} \rightarrow B_p^{sq}$  are bounded and  $T_\psi \circ S_\varphi = Id$  on  $B_p^{sq}$ . Consequently, for  $f \in \mathcal{D}'$  we have that  $f \in B_p^{sq}$  if and only if  $\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}} \in b_p^{sq}$ . Moreover,

$$(6.5) \quad \|f\|_{B_p^{sq}} \sim \|\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}}\|_{b_p^{sq}} \sim \left( \sum_{j=0}^{\infty} 2^{sjq} \left[ \sum_{\xi \in \mathcal{X}_j} \|\langle f, \varphi_\xi \rangle \psi_\xi\|_p^p \right]^{q/p} \right)^{1/q}.$$

In addition, the definition of  $B_p^{sq}$  is independent of the selection of  $\widehat{a}$  satisfying (4.2)–(4.3).

To prove this theorem we will need the following lemma whose proof is presented in the appendix.

**Lemma 6.6.** For every  $P \in \Pi_{2^j}$ ,  $j \geq 0$ , and  $0 < p \leq \infty$

$$(6.6) \quad \left( \sum_{\xi \in \mathcal{X}_j} \max_{x \in I_\xi} |P(x)|^p \mu(I_\xi) \right)^{1/p} \leq c \|P\|_p.$$

**Proof of Theorem 6.5.** Note first that the right-hand side of (6.5) follows immediately from (3.12).

As in the proof of Theorem 4.5, assume that the kernels  $\Phi_j$  are defined by (4.1), where  $\widehat{a}$  satisfies (4.2)–(4.3). Let  $\widehat{b}$  be such that (3.1)–(3.3) hold and let  $\Psi_j$  be defined by (3.6) using this  $\widehat{b}$ . Also, let  $\{\varphi_\xi\}_{\xi \in \mathcal{X}}$  and  $\{\psi_\xi\}_{\xi \in \mathcal{X}}$  be the associated needlet systems defined as in (3.7). Further, assume that  $\{\widetilde{\Phi}_j\}$ ,  $\{\widetilde{\Psi}_j\}$ ,  $\{\widetilde{\varphi}_\xi\}$ ,  $\{\widetilde{\psi}_\xi\}$  is a second set of kernels and needlets.

We first prove the boundedness of the operator  $T_{\widetilde{\psi}} : b_p^{sq} \rightarrow B_p^{sq}$ , where  $B_p^{sq}$  is defined via  $\{\widetilde{\Phi}_j\}$ . Let  $0 < t < \min\{p, 1\}$  and  $\sigma \geq (2\alpha + 2)/t + \alpha + 1/2$ . Assume that  $h = \{h_\xi\}$  is a finitely supported sequence and set  $f := \sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi$ . Employing

Lemmata 2.7, 4.7, and (2.22) we get

$$\begin{aligned}
|\Phi_j * f(x)| &\leq \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} |h_\xi| |\Phi_\mu * \tilde{\psi}_\xi(x)| \\
&\leq c \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} |h_\xi| \frac{2^{j/2}}{\sqrt{\mathcal{W}(2^j; x)(1 + 2^j d(\xi, x))^\sigma}} \\
&\leq c \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} |h_\xi| \frac{2^{j/2}}{\sqrt{\mathcal{W}(2^j; \xi)(1 + 2^j d(\xi, x))^{\sigma - \alpha - 1/2}}} \\
&\leq c \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} |h_\xi| \mu(I_\xi)^{-1/2} \mathcal{M}_t(\mathbb{1}_{I_\xi})(x),
\end{aligned}$$

where we also used that  $\sigma \geq (2\alpha + 2)/t + \alpha + 1/2$ . Using the maximal inequality (2.24) it follows that

$$\begin{aligned}
\|\Phi_j * f\|_p^p &\leq \left\| \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} |h_\xi| \mu(I_\xi)^{-1/2} \mathcal{M}_t(\mathbb{1}_{I_\xi})(\cdot) \right\|_p^p \\
&\leq c \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} |h_\xi|^p \mu(I_\xi)^{-p/2} \int_{-1}^1 \mathbb{1}_{I_\xi}(x) w(x) dx \\
&\leq c \sum_{j-1 \leq \nu \leq j+1} \sum_{\xi \in \mathcal{X}_\nu} |h_\xi|^p \mu(I_\xi)^{1-p/2}.
\end{aligned}$$

Multiplying by  $2^{js}$  and summing over  $j \geq 0$  we get  $\|f\|_{B_p^{s,q}} \leq c \|\{h_\xi\}\|_{b_p^{s,q}}$ . To extend the result to an arbitrary sequence  $h = \{h_\xi\} \in b_p^{s,q}$  one proceeds similarly as in the Triebel-Lizorkin case by using the embedding of  $B_p^{s,q}$  in  $\mathcal{D}'$  and the density of finitely supported sequences in  $b_p^{s,q}$ .

We next prove the boundedness of the operator  $S_\varphi : B_p^{s,q} \rightarrow b_p^{s,q}$ , where we assume that  $B_p^{s,q}$  is defined in terms of  $\{\overline{\Phi}_j\}$ . Note first that

$$|\langle f, \varphi_\xi \rangle| \sim \mu(I_\xi)^{1/2} |\overline{\Phi}_j * f(\xi)|, \quad \xi \in \mathcal{X}_j.$$

Since  $\overline{\Phi}_j * f \in \Pi_{2^j}$ , then using Lemma 6.6

$$\sum_{\xi \in \mathcal{X}_j} \mu(I_\xi)^{1-p/2} |\langle f, \varphi_\xi \rangle|^p \leq c \sum_{\xi \in \mathcal{X}_j} \mu(I_\xi) \sup_{x \in I_\xi} |\overline{\Phi}_j * f(x)|^p \leq c \|\overline{\Phi}_j * f\|_p^p,$$

which yields  $\|\{f, \varphi\}\|_{b_p^{s,q}} \leq c \|f\|_{B_p^{s,q}}$ .

The identity  $T_\psi \circ S_\varphi = Id$  follows by Proposition 3.1.

The independence of  $B_p^{s,q}$  of the particular selection of  $\widehat{a}$  in the definition of  $\{\overline{\Phi}_j\}$  follows from above exactly as in the Triebel-Lizorkin case (see the proof of Theorem 4.5).  $\square$

## 7. SECOND SCALE OF WEIGHTED BESOV SPACES ON $[-1, 1]$

We introduce a second scale of weighted Besov spaces by using again as in §6 the kernels  $\Phi_j$ , defined by (4.1) with  $\widehat{a}$  satisfying (4.2)-(4.3).

**Definition 7.1.** *Let  $s \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . Then the weighted Besov space  $\mathcal{B}_p^{s,q} := \mathcal{B}_p^{s,q}(w)$  is defined as the set of all  $f \in \mathcal{D}'$  such that*

$$\|f\|_{\mathcal{B}_p^{s,q}} := \left( \sum_{j=0}^{\infty} \left[ 2^{sj} \|\mathcal{W}(2^j; \cdot)^{-s} \Phi_j * f(\cdot)\|_p \right]^q \right)^{1/q} < \infty,$$

where the  $\ell_q$ -norm is replaced by the sup-norm if  $q = \infty$ .

As for the other weighted Besov and Triebel-Lizorkin spaces considered here the above definition is independent of the choice of  $\hat{a}$  satisfying (4.2)-(4.3). Also, the Besov space  $\mathcal{B}_p^{sq}(w)$  is a quasi-Banach space which is continuously embedded in  $\mathcal{D}'$ .

The main advantages of the spaces  $\mathcal{B}_p^{sq}$  over  $B_p^{sq}$  are that, first, they scale (are embedded) “correctly” with respect to the smoothness index  $s$ , and secondly, the right smoothness spaces in nonlinear  $n$ -term weighted approximation from needles are defined in terms of spaces  $\mathcal{B}_p^{sq}$  (see §8 below).

**Proposition 7.2.** *Let  $0 < p \leq p_1 < \infty$ ,  $0 < q \leq q_1 \leq \infty$ , and  $0 < s_1 \leq s < \infty$ . Then we have the continuous embedding*

$$(7.1) \quad \mathcal{B}_p^{sq} \subset \mathcal{B}_{p_1}^{s_1 q_1} \quad \text{if} \quad s - 1/p = s_1 - 1/p_1.$$

This embedding result follows readily by applying inequality (2.11).

We now define the companion to  $\mathcal{B}_p^{sq}(w)$  sequence space  $\mathbf{b}_p^{sq}(w)$ .

**Definition 7.3.** *Let  $s \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . Then  $\mathbf{b}_p^{sq} := \mathbf{b}_p^{sq}(w)$  is defined to be the space of all complex-valued sequences  $h := \{h_\xi\}_{\xi \in \mathcal{X}}$  such that*

$$\|h\|_{\mathbf{b}_p^{sq}} := \left( \sum_{j=0}^{\infty} \left[ \sum_{\xi \in \mathcal{X}_j} \left( \mu(I_\xi)^{-s+1/p-1/2} |h_\xi| \right)^p \right]^{q/p} \right)^{1/q} < \infty$$

with the usual modification for  $p = \infty$  or  $q = \infty$ .

For the characterization of weighted Besov spaces  $\mathcal{B}_p^{sq}$ , we again employ the operators  $S_\varphi$  and  $T_\psi$  defined in (4.10).

**Theorem 7.4.** *Let  $s \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . The operators  $S_\varphi : \mathcal{B}_p^{sq} \rightarrow \mathbf{b}_p^{sq}$  and  $T_\psi : \mathbf{b}_p^{sq} \rightarrow \mathcal{B}_p^{sq}$  are bounded and  $T_\psi \circ S_\varphi = \text{Id}$  on  $\mathcal{B}_p^{sq}$ . Consequently, for  $f \in \mathcal{D}'$  we have that  $f \in \mathcal{B}_p^{sq}$  if and only if  $\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}} \in \mathbf{b}_p^{sq}$ . Moreover,*

$$(7.2) \quad \|f\|_{\mathcal{B}_p^{sq}} \sim \|\{\langle f, \varphi_\xi \rangle\}\|_{\mathbf{b}_p^{sq}} \sim \left( \sum_{j=0}^{\infty} \left[ \sum_{\xi \in \mathcal{X}_j} \mu(I_\xi)^{-sp} \|\langle f, \varphi_\xi \rangle \psi_\xi\|_p^p \right]^{q/p} \right)^{1/q}.$$

In addition, the definition of  $\mathcal{B}_p^{sq}$  is independent of the selection of  $\hat{a}$  satisfying (4.2)-(4.3).

The following additional lemma is needed for the proof of Theorem 7.4.

**Lemma 7.5.** *For every  $P \in \Pi_{2^j}$ ,  $j \geq 0$ , and  $0 < p \leq \infty$*

$$(7.3) \quad \left( \sum_{\xi \in \mathcal{X}_j} \mathcal{W}(2^j; \xi)^{-sp} \sup_{x \in I_\xi} |P(x)|^p \mu(I_\xi) \right)^{1/p} \leq c \|\mathcal{W}(2^j; \cdot) P(\cdot)\|_p.$$

The proof of this lemma is similar to the proof of Lemma 6.6, where one uses Lemma 5.4 in place of Lemma 4.10. We skip it.

For the proof of Theorem 7.4, one proceeds as in the proof of Theorem 6.5, using Lemma 7.5 instead of Lemma 6.6. The proof will be omitted.

## 8. APPLICATION OF WEIGHTED BESOV SPACES TO NONLINEAR APPROXIMATION

We consider here nonlinear  $n$ -term approximation for a needlet system  $\{\psi_\eta\}_{\eta \in \mathcal{X}}$  with  $\varphi_\eta = \psi_\eta$ , defined as in (3.5)-(3.8) with  $\widehat{b} = \widehat{a}$ ,  $\widehat{a} \geq 0$ . Then  $\widehat{a}$  satisfies

$$\widehat{a}^2(t) + \widehat{a}^2(2t) = 1, \quad t \in [1/2, 1].$$

Hence  $\{\psi_\eta\}$  are real-valued.

Denote by  $\Sigma_n$  the nonlinear set consisting of all functions  $g$  of the form

$$g = \sum_{\xi \in \Lambda} a_\xi \psi_\xi,$$

where  $\Lambda \subset \mathcal{X}$ ,  $\#\Lambda \leq n$ , and  $\Lambda$  is allowed to vary with  $g$ . Let  $\sigma_n(f)_p$  denote the error of best  $L^p(w)$ -approximation to  $f \in L^p(w)$  from  $\Sigma_n$ :

$$\sigma_n(f)_p := \inf_{g \in \Sigma_n} \|f - g\|_p.$$

The approximation will take place in  $L^p(w)$ ,  $0 < p < \infty$ . Assume in the following that  $0 < p < \infty$ ,  $s > 0$ , and  $1/\tau := s + 1/p$ . Denote briefly  $\mathcal{B}_\tau^s := \mathcal{B}_\tau^{s\tau}$ .

By Theorem 7.4 and (3.12) it follows that

$$(8.1) \quad \|f\|_{\mathcal{B}_\tau^s} \approx \left( \sum_{\xi \in \mathcal{X}} |\langle f, \psi_\xi \rangle \psi_\xi|_p^\tau \right)^{1/\tau}.$$

The embedding of  $\mathcal{B}_\tau^s$  into  $L^p(w)$  plays an important role here.

**Proposition 8.1.** *If  $f \in \mathcal{B}_\tau^s$ , then  $f$  can be identified as a function  $f \in L^p(w)$  and*

$$(8.2) \quad \|f\|_p \leq \left\| \sum_{\xi \in \mathcal{X}} |\langle f, \psi_\xi \rangle \psi_\xi(\cdot)| \right\|_p \leq c \|f\|_{\mathcal{B}_\tau^s}.$$

We now state our main result in this section.

**Theorem 8.2. [Jackson estimate]** *If  $f \in \mathcal{B}_\tau^s$ , then*

$$(8.3) \quad \sigma_n(f)_p \leq cn^{-s} \|f\|_{\mathcal{B}_\tau^s},$$

where  $c$  depends only on  $s$ ,  $p$ , and the parameters of the needlet system.

The proofs of this theorem and Proposition 8.1 can be carried out exactly as the proofs of the respective Jackson estimate and embedding result in [10] and will be omitted.

It is an open problem to prove the companion to (8.3) Bernstein estimate:

$$(8.4) \quad \|g\|_{\mathcal{B}_\tau^s} \leq cn^s \|g\|_p \quad \text{for } g \in \Sigma_n, \quad 1 < p < \infty.$$

This would enable one to characterize the rates (approximation spaces) of nonlinear  $n$ -term approximation in  $L^p(w)$  ( $1 < p < \infty$ ) from needlet systems.

## 9. APPENDIX

**Proof of Proposition 2.2.** We need the following integral representation of  $L_n(x, y)$  from [12] (see (2.15)):

$$(9.1) \quad L_n(x, y) = c_{\alpha, \beta} \int_0^\pi \int_0^1 L_n^{\alpha, \beta}(t(x, y, r, \psi)) dm_{\alpha, \beta}(r, \psi),$$

where  $L_n^{\alpha, \beta}(t)$  is defined by (2.1),

$$t(x, y, r, \psi) := \frac{1}{2}(1+x)(1+y) + \frac{1}{2}(1-x)(1-y)r^2 + r\sqrt{1-x^2}\sqrt{1-y^2}\cos\psi - 1,$$

the integral is against

$$dm_{\alpha,\beta}(r, \psi) := (1 - r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin \psi)^{2\beta} dr d\psi,$$

and the constant  $c_{\alpha,\beta}$  is determined from

$$c_{\alpha,\beta} \int_0^\pi \int_0^1 1 dm_{\alpha,\beta}(r, \psi) = 1.$$

For any  $u \in [-1, 1]$  we will denote by  $\theta_u$  the only angle in  $[0, \pi]$  such that  $u = \cos \theta_u$ .

We will need the following lemma contained in the proof of Theorem 2.4 in [12].

**Lemma 9.1.** *Let  $\alpha, \beta > -1/2$  and  $k \geq 2\alpha + 2\beta + 3$ . Then there is a constant  $c_k > 0$  depending only on  $k, \alpha,$  and  $\beta$  such that for  $x, y \in [-1, 1]$*

$$\int_0^\pi \int_0^1 \frac{n^{2\alpha+1} dm_{\alpha,\beta}(r, \psi)}{\left(1 + n\sqrt{1 - t(x, y, r, \psi)}\right)^k} \leq c_k \frac{1}{\sqrt{\mathcal{W}_{\alpha,\beta}(n; x)} \sqrt{\mathcal{W}_{\alpha,\beta}(n; y)} (1 + n|\theta_x - \theta_y|)^\sigma},$$

where  $\sigma = k - 2\alpha - 2\beta - 3$ .

Identity (9.1) yields

$$\begin{aligned} & |L_n(x, y) - L_n(\xi, y)| \\ (9.2) \quad & \leq c \int_0^\pi \int_0^1 |L_n^{\alpha,\beta}(t(x, y, r, \psi)) - L_n^{\alpha,\beta}(t(\xi, y, r, \psi))| dm_{\alpha,\beta}(r, \psi) \\ & \leq c \int_0^\pi \int_0^1 \|\partial L_n^{\alpha,\beta}(\cdot)\|_{L^\infty(I_{r,\psi})} |t(x, y, r, \psi) - t(\xi, y, r, \psi)| dm_{\alpha,\beta}(r, \psi), \end{aligned}$$

where  $\partial f = f'$  and  $I_{r,\psi}$  is the interval with end points  $t(x, y, r, \psi)$  and  $t(\xi, y, r, \psi)$ .

From estimate (2.16) in [12] and Markov's inequality, for any  $k$  there exists a constant  $c_k > 0$  such that

$$\begin{aligned} & \|\partial L_n^{\alpha,\beta}(\cdot)\|_{L^\infty(I_{r,\psi})} \leq c_k \max_{u \in I_{r,\psi}} \frac{n^{2\alpha+4}}{\left(1 + n\sqrt{1 - u}\right)^k} \\ (9.3) \quad & \leq c_k n^{2\alpha+4} \left[ \left(1 + n\sqrt{1 - t(x, y, r, \psi)}\right)^{-k} + \left(1 + n\sqrt{1 - t(\xi, y, r, \psi)}\right)^{-k} \right]. \end{aligned}$$

For the rest of the proof we assume that  $k > 0$  is sufficiently large.

From the definition of  $t(x, y, r, \psi)$  one easily obtains

$$1 - t(x, y, r, \psi) = 2 \sin^2 \frac{\theta_x - \theta_y}{2} + 2 \sin^2 \frac{\theta_x}{2} \sin^2 \frac{\theta_y}{2} (1 - r^2) + \sin \theta_x \sin \theta_y (1 - r \cos \psi),$$

which implies

$$\begin{aligned} & t(x, y, r, \psi) - t(\xi, y, r, \psi) = \cos(\theta_\xi - \theta_y) - \cos(\theta_x - \theta_y) \\ & + (\cos \theta_\xi - \cos \theta_x) \sin^2 \frac{\theta_y}{2} (1 - r^2) + (\sin \theta_\xi - \sin \theta_x) \sin \theta_y (1 - r \cos \psi). \end{aligned}$$

It is readily seen that

$$\begin{aligned} & |\cos(\theta_\xi - \theta_y) - \cos(\theta_x - \theta_y)| = 2 \left| \sin \frac{\theta_x + \theta_\xi - 2\theta_y}{2} \sin \frac{\theta_\xi - \theta_x}{2} \right| \\ & \leq |\theta_\xi - \theta_x| (|\theta_z - \theta_y| + cn^{-1}), \end{aligned}$$



where we used that  $|\theta_z - \theta_\xi| \leq cn^{-1}$  and  $|\theta_x - \theta_\xi| \leq cn^{-1}$ . Therefore,

$$(9.4) \quad \begin{aligned} & |t(x, y, r, \psi) - t(\xi, y, r, \psi)| \\ & \leq |\theta_\xi - \theta_x| [ (|\theta_z - \theta_y| + cn^{-1}) + \sin^2 \frac{\theta_y}{2} (1 - r^2) + \sin \theta_y (1 - r \cos \psi) ]. \end{aligned}$$

We use this and (9.3) in (9.2) to obtain

$$|L_n(x, y) - L_n(\xi, y)| \leq c|\theta_\xi - \theta_x|(A_1 + B_1 + A_2 + B_2 + A_3 + B_3),$$

where  $A_j$  and  $B_j$  are integrals of the same type with  $A_j$  involving  $t(x, y, r, \psi)$  and  $B_j$  involving  $t(\xi, y, r, \psi)$ ; the indices  $j = 1, 2, 3$  correspond to the three terms in the right-hand side of (9.4). We will estimate them separately.

**Case 1.** We first estimate the integral

$$A_1 := n^{2\alpha+4} \int_0^\pi \int_0^1 \frac{|\theta_z - \theta_y| + cn^{-1}}{(1 + n\sqrt{1 - t(x, y, r, \psi)})^k} dm_{\alpha, \beta}(r, \psi)$$

and the integral  $B_1$ , the same as  $A_1$  but involving  $t(\xi, y, r, \psi)$  in place of  $t(x, y, r, \psi)$ .

Using the estimate in Lemma 9.1 and the fact that  $|\theta_z - \theta_y| \sim |\theta_x - \theta_y| + cn^{-1}$ , we have

$$\begin{aligned} A_1 & \leq c \frac{n^3(|\theta_z - \theta_y| + cn^{-1})}{\sqrt{\mathcal{W}(n; x)}\sqrt{\mathcal{W}(n; y)}(1 + n|\theta_x - \theta_y|)^\sigma} \\ & \leq c \frac{n^2}{\sqrt{\mathcal{W}(n; x)}\sqrt{\mathcal{W}(n; y)}(1 + n|\theta_z - \theta_y|)^{\sigma-1}}. \end{aligned}$$

On account of (2.22) this gives the desired estimate.

The integral  $B_1$  is estimated similarly with the same bound.

**Case 2.** We now estimate the integrals

$$A_2 := n^{2\alpha+4} \int_0^\pi \int_0^1 \frac{\sin^2 \frac{\theta_y}{2} (1 - r^2)}{(1 + n\sqrt{1 - t(x, y, r, \psi)})^k} dm_{\alpha, \beta}(r, \psi)$$

and  $B_2$  which is the same but involves  $t(\xi, y, r, \psi)$  in place of  $t(x, y, r, \psi)$ .

By the definition of  $dm_{\alpha, \beta}(r, \psi)$  we have  $(1 - r^2)dm_{\alpha, \beta}(r, \psi) = dm_{\alpha+1, \beta}(r, \psi)$ . Then using the estimate from Lemma 9.1 with  $\alpha$  replaced by  $\alpha + 1$ , we get

$$\begin{aligned} A_2 & \leq c \frac{n \sin^2 \frac{y}{2}}{\sqrt{\mathcal{W}_{\alpha+1, \beta}(n; x)}\sqrt{\mathcal{W}_{\alpha+1, \beta}(n; y)}(1 + n|\theta_x - \theta_y|)^\sigma} \\ & \leq c \frac{n^2}{\sqrt{\mathcal{W}_{\alpha, \beta}(n; x)}\sqrt{\mathcal{W}_{\alpha, \beta}(n; y)}(1 + n|\theta_x - \theta_y|)^\sigma}, \end{aligned}$$

where we used the fact that  $\mathcal{W}_{\alpha+1, \beta}(n; y) = \mathcal{W}_{\alpha, \beta}(n; y)(\sin^2 \frac{\theta_y}{2} + n^{-2})$  and hence  $\mathcal{W}_{\alpha+1, \beta}(n; x) \geq \mathcal{W}_{\alpha, \beta}(n; x)n^{-2}$ . The equivalence  $|\theta_z - \theta_y| \sim |\theta_x - \theta_y| + cn^{-1}$  and (2.22) then give the desired estimate. The integral  $B_2$  is estimated similarly.

**Case 3.** We finally estimate the integrals

$$A_3 := n^{2\alpha+4} \int_0^\pi \int_0^1 \frac{\sin \theta_y (1 - r \cos \psi)}{(1 + n\sqrt{1 - t(x, y, r, \psi)})^k} dm_{\alpha, \beta}(r, \psi)$$

and  $B_3$  which involves  $t(\xi, y, r, \psi)$  in place of  $t(x, y, r, \psi)$ .

Assume first that  $|\sin \theta_x| \geq n^{-1}$ . Using the fact that

$$1 - t(x, y, r, \psi) \geq \sin \theta_x \sin \theta_y (1 - r \cos \psi),$$

we conclude that

$$\begin{aligned} A_3 &\leq \frac{n^{2\alpha+2}}{\sin \theta_x} \int_0^\pi \int_0^1 \frac{1}{(1+n\sqrt{1-t(x,y,r,\psi)})^{k-2}} dm_{\alpha,\beta}(r,\psi) \\ &\leq cn^{2\alpha+3} \int_0^\pi \int_0^1 \frac{1}{(1+n\sqrt{1-t(x,y,r,\psi)})^{k-2}} dm_{\alpha,\beta}(r,\psi). \end{aligned}$$

Now the estimate from Lemma 9.1 can be applied to get the desired estimate.

Let  $|\sin \theta_x| \leq n^{-1}$ . We have

$$|\sin \theta_y| \leq |\sin \theta_y - \sin \theta_x| + |\sin \theta_x| \leq |\theta_y - \theta_x| + n^{-1}$$

and use the fact that

$$1 - t(x, y, r, \psi) \geq 2 \sin^2 \frac{\theta_x - \theta_y}{2} \geq c(\theta_x - \theta_y)^2$$

to conclude that

$$A_3 \leq cn^{2\alpha+3} \int_0^\pi \int_0^1 \frac{1}{(1+n\sqrt{1-t(x,y,r,\psi)})^{k-2}} dm_{\alpha,\beta}(r,\psi).$$

Applying the estimated from Lemma 9.1 we obtain the desired result.  $B_3$  is estimated in the same way.

Putting the above estimates together completes the proof of Theorem 2.2.  $\square$

**Proof of Proposition 2.4.** Note first that it suffices to prove (2.4) only for  $n \geq n_0$ , where  $n_0$  is sufficiently large. This follows from the fact that  $P_n^{(\alpha,\beta)}$  and  $P_{n+1}^{(\alpha,\beta)}$  do not have common zeros and  $\mathcal{W}(n;x) \sim 1$  if  $n \leq \text{constant}$ . Furthermore, since  $P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\beta,\alpha)}(x)$ , it is sufficient to consider only the case  $x \in [0, 1]$ .

Note that the Jacobi polynomials are normalized by  $P_k^{(\alpha,\beta)}(1) = \binom{k+\alpha}{k} \sim k^\alpha$  and using Markov's inequality it follows that  $P_k^{(\alpha,\beta)}(x) \geq ck^\alpha$  for  $1 - \delta k^{-2} \leq x \leq 1$ , where  $\delta > 0$  is a sufficiently small constant. From this one readily infers that (2.6) holds for  $1 - \delta_1 n^{-2} \leq x \leq 1$ ,  $\delta_1 > 0$ . Define  $\theta \in [0, \pi]$  from  $x = \cos \theta$ . Then the latter condition on  $x$  is apparently equivalent to  $0 \leq \theta \leq \delta_2 n^{-1}$  with  $\delta_2$  being a positive constant.

To estimate  $\Lambda_n(\cos \theta)$  for  $c^* n^{-1} \leq \theta \leq \pi/2$  with  $c^* > 0$  sufficiently large, we need the following asymptotic formula of the Jacobi polynomials: For  $\alpha, \beta > -1$ ,

$$\begin{aligned} \left(\sin \frac{\theta}{2}\right)^\alpha \left(\cos \frac{\theta}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos \theta) &= N^{-\alpha} \frac{\Gamma(n+\alpha+1)}{n!} \left(\frac{\theta}{\sin \theta}\right)^{1/2} J_\alpha(N\theta) \\ &\quad + \theta^{1/2} \mathcal{O}(n^{-3/2}) \end{aligned}$$

if  $c_0 n^{-1} \leq \theta \leq \pi/2$ , where  $N = n + \eta$  with  $\eta := (\alpha + \beta + 1)/2$ ,  $J_\alpha$  is the Bessel function, and  $c_0 > 0$  is an arbitrary but fixed constant (see [17, Theorem 8.21.12, p. 195]).

Using that  $2/\pi \leq \sin \theta/\theta \leq 1$  and  $\cos \theta/2 \sim 1$  on  $[0, \pi/2]$ , and also  $\Gamma(n+\alpha+1)/n! \sim n^\alpha$ , we infer from above

$$\left(\sin \frac{\theta}{2}\right)^{2\alpha} \left[P_k^{(\alpha,\beta)}(\cos \theta)\right]^2 \geq c_1 [J_\alpha((k+\eta)\theta)]^2 - c_2 k^{-3/2} \theta^{1/2} |J_\alpha((k+\eta)\theta)|.$$

Recall the well-known asymptotic formula

$$J_\alpha(z) = \left(\frac{2}{\pi z}\right)^{1/2} [\cos(z+\gamma) + \mathcal{O}(z^{-1})], \quad z \rightarrow \infty,$$

where  $\gamma = -\alpha\pi/2 - \pi/4$ .

All of the above leads to

$$(9.5) \quad \begin{aligned} \left(\sin \frac{\theta}{2}\right)^{2\alpha} \Lambda_n(\cos \theta) &\geq \sum_{k=n}^{n+[\varepsilon n]} \left( c_1 [J_\alpha((k+\eta)\theta)]^2 - c_2 k^{-3/2} \theta^{1/2} |J_\alpha((k+\eta)\theta)| \right) \\ &\geq \frac{c}{n\theta} \sum_{k=n}^{n+[\varepsilon n]} [\cos^2(k\theta + b(\theta)) - c'(n\theta)^{-1}] - c''\varepsilon n^{-1}, \end{aligned}$$

for  $c_0 n^{-1} \leq \theta \leq \pi/2$ , where  $b(\theta) = ((\alpha + \beta + 1)/2)\theta + \gamma$ . We now use the well known identities for the Dirichlet kernel and its conjugate to obtain, for  $m > n$ ,

$$\sum_{k=n}^m \cos^2(k\theta + b) = \frac{1}{2}(m - n + 1) + (\cos 2b + \sin 2b) \frac{\sin(m - n + 1)\theta \cos(n + m)\theta}{2 \sin \theta}.$$

Therefore,

$$\sum_{k=n}^{n+[\varepsilon n]} \cos^2(k\theta + b(\theta)) \geq \frac{1}{2}([\varepsilon n] + 1) \left( 1 - \frac{2}{([\varepsilon n] + 1) \sin \theta} \right) \geq \frac{1}{2} \varepsilon n \left( 1 - \frac{\pi}{\varepsilon n \theta} \right) \geq \frac{\varepsilon n}{4},$$

whenever  $(2\pi/\varepsilon)n^{-1} \leq \theta \leq \pi/2$ . Substituting this in (9.5) we obtain

$$(9.6) \quad \begin{aligned} \left(\sin \frac{\theta}{2}\right)^{2\alpha} \Lambda_n(\cos \theta) &\geq \frac{c}{n\theta} \left( \frac{\varepsilon n}{4} - \frac{c'\varepsilon n}{n\theta} \right) - \frac{c''\varepsilon}{n} \\ &\geq \frac{c\varepsilon}{\theta} \left( \frac{1}{4} - \frac{c'}{c^*} \right) - \frac{c''\varepsilon}{n} \geq \frac{c^\diamond}{\theta}, \quad c^\diamond > 0, \end{aligned}$$

if  $c^* n^{-1} \leq \theta \leq \pi/2$  with  $c^* := \max\{c_0, 8c', 2\pi/\varepsilon\}$  and  $n$  is sufficiently large. Hence,

$$(9.7) \quad \Lambda_n(\cos \theta) \geq \frac{c^\diamond}{\theta} \left( \sin \frac{\theta}{2} \right)^{-2\alpha} \geq c\theta^{-(2\alpha+1)}, \quad c^* n^{-1} \leq \theta \leq \pi/2,$$

for sufficiently large  $n$ , which yields (2.6) in this case.

For the remaining case  $\delta_2 n^{-1} \leq \theta \leq c^* n^{-1}$ , we need further properties of Jacobi polynomials. Let  $x_{\nu,n} = \cos \theta_{\nu,n}$  denote the zeros of Jacobi polynomial  $P_n^{(\alpha,\beta)}$ , where

$$0 < \theta_{1,n} < \theta_{2,n} < \dots < \theta_{n,n} < \pi.$$

It is well known that  $\theta_{\nu,n} \sim \nu/n$ , but we will need much more precise asymptotic representation for  $\theta_{\nu,n}$ , see below. The Jacobi polynomials satisfy the following relation (see e.g. [11, Theorem 3.3, p. 171]),

$$(9.8) \quad P_n^{(\alpha,\beta)}(\cos \theta) \sim n^{1/2} |\theta - \theta_{\nu_\theta,n}| \frac{n^{\alpha+1/2}}{\nu_\theta^{\alpha+1/2}}, \quad \theta \in [0, \pi],$$

where  $\nu_\theta$  denotes the index of the zero  $x_{\nu,n}$ ,  $1 \leq \nu \leq n$ , which is (one of) the closest to  $x$  ( $x = \cos \theta$ ).

We will need the asymptotics of the zeros of the Jacobi polynomials from [5]:

$$(9.9) \quad \theta_{\nu,n} = \frac{j_{\alpha,\nu}}{N} + \frac{1}{4N^2} \left[ \left( \alpha^2 - \frac{1}{4} \right) \frac{1 - t \cot t}{2t} - \frac{\alpha^2 - \beta^2}{4} \tan \frac{t}{2} \right] + t^2 \mathcal{O}(n^{-3}),$$

where  $N = n + \eta$  as before,  $j_{\alpha,\nu}$  is the  $\nu$ th positive zero of the Bessel function  $J_\alpha(x)$  and  $t = j_{\alpha,\nu}/N$ . Here the  $\mathcal{O}$ -term is uniformly bounded for  $\nu = 1, 2, \dots, [\gamma n]$ , where

$\gamma \in (0, 1)$ . It is easy to verify that  $(1 - t \cot t)/t = \mathcal{O}(t)$  as  $t \rightarrow 0$  and obviously  $1/(n + \eta) - 1/n = \mathcal{O}(n^{-2})$ . Hence

$$(9.10) \quad \theta_{\nu, n} = \frac{j_{\alpha, \nu}}{n} + \mathcal{O}(n^{-2}), \quad \nu = 1, 2, \dots, [\gamma n].$$

We will also use that

$$0 < j_{\alpha, 1} < j_{\alpha, 2} < \dots \quad \text{and} \quad j_{\alpha, \nu} \rightarrow \infty.$$

Let  $j_{\alpha, \nu_{\max}} := \max\{j_{\alpha, \nu} : j_{\alpha, \nu} \leq (1 + \varepsilon)c^*\}$  and denote  $\mathcal{J} := \{j_{\alpha, 1}, j_{\alpha, 2}, \dots, j_{\alpha, \nu_{\max}}\}$ . Notice that  $\nu_{\max}$  is a constant independent of  $n$ . Suppose that  $\mathcal{J} \neq \emptyset$  (the case  $\mathcal{J} = \emptyset$  is easier).

Fix  $\delta_2 n^{-1} \leq \theta \leq c^* n^{-1}$ . Then by (9.8) it follows that

$$P_k^{(\alpha, \beta)}(\cos \theta) \sim k^{\alpha+1} |\theta - \theta_{\nu_\theta, k}|,$$

where the  $\nu_\theta$ 's involved are bounded by a constant independent of  $n$ . Hence, (9.10) can be used to represent  $\theta_{\nu_\theta, k}$  for  $n \leq k \leq n + [\varepsilon n]$  if  $n$  is sufficiently large. Using the above we get

$$\begin{aligned} \Lambda_n(\cos \theta) &\geq cn^{2\alpha+2} \sum_{k=n}^{n+[\varepsilon n]} |\theta - \theta_{\nu_\theta, k}|^2 \geq cn^{2\alpha} \sum_{k=n}^{n+[\varepsilon n]} |k\theta - k\theta_{\nu_\theta, k}|^2 \\ &\geq cn^{2\alpha} \sum_{k=n}^{n+[\varepsilon n]} \left( |k\theta - j_{\alpha, \nu_\theta}|^2 - c' k^{-1} |k\theta - j_{\alpha, \nu_\theta}| \right) \\ &\geq cn^{2\alpha} \left( \sum_{k=n}^{n+[\varepsilon n]} |k\theta - j_{\alpha, \nu_\theta}|^2 - c' c^* \varepsilon \right), \end{aligned}$$

where we used (9.10). Therefore,

$$(9.11) \quad \Lambda_n(\cos \theta) \geq cn^{2\alpha} \left( \sum_{k=n}^{n+[\varepsilon n]} \text{dist}(k\theta, \mathcal{J})^2 - c^\diamond \right), \quad c, c^\diamond > 0,$$

where  $\text{dist}(k\theta, \mathcal{J})$  denotes the distance of  $k\theta$  from the set  $\mathcal{J}$ , that is the distance of  $k\theta$  from the nearest zero of the Bessel function  $J_\alpha(x)$ .

It remains to estimate the sum in (9.11). Denote  $\mathcal{K} := \{n, n+1, \dots, n+[\varepsilon n]\}$  and let  $\mathcal{K}_0$  be the set of all indices  $k \in \mathcal{K}$  such that  $\text{dist}(k\theta, \mathcal{J}) < m\theta$ , where  $m := [\varepsilon n / (6\nu_{\max})]$ . Evidently

$$\#\mathcal{K}_0 \leq (2m+1)\nu_{\max} \leq (2[\varepsilon n / (6\nu_{\max})] + 1)\nu_{\max} \leq \varepsilon n / 2 \quad \text{if } n \geq 6\nu_{\max} \varepsilon^{-1}.$$

Then  $\#\mathcal{K} \setminus \mathcal{K}_0 \geq [\varepsilon n] + 1 - \varepsilon n / 2 \geq \varepsilon n / 2$  and hence

$$\sum_{k=n}^{n+[\varepsilon n]} \text{dist}(k\theta, \mathcal{J})^2 \geq \sum_{k \in \mathcal{K} \setminus \mathcal{K}_0} (m\theta)^2 \geq c \sum_{k \in \mathcal{K} \setminus \mathcal{K}_0} (n\theta)^2 \geq c\delta_2^2 \varepsilon n \geq c_* n, \quad c_* > 0.$$

Inserting this in (9.11) we obtain

$$\Lambda_n(\cos \theta) \geq cn^{2\alpha} (c_* n - c^\diamond) \geq \tilde{c} n^{2\alpha+1}$$

for sufficiently large  $n$ . This implies the stated inequality (2.6) with  $x = \cos \theta$  in the case  $\delta_2 n^{-1} \leq \theta \leq c^* n^{-1}$ . The proof of Proposition 2.4 is complete.  $\square$

**Proof of Proposition 2.6.** Suppose  $\alpha \geq \beta$  and let  $1 < q < \infty$ . By Lemma 2.5, (i) we have  $g = L_n * g$  and using Hölder's inequality, (2.3), and that  $\mathcal{W}(n; x) \geq cn^{-2\alpha-1}$  we obtain

$$|g(x)| \leq \|g\|_q \left( \frac{n}{\mathcal{W}(n; x)} \right)^{1/q} \leq cn^{(2\alpha+2)/q} \|g\|_q, \quad x \in [-1, 1],$$

which leads to

$$(9.12) \quad \|g\|_\infty \leq cn^{(2\alpha+2)/q} \|g\|_q, \quad 1 < q \leq \infty.$$

If  $0 < q \leq 1$ , then the above inequality with  $q = 2$  gives

$$\|g\|_\infty^2 \leq cn^{2\alpha+2} \int_{-1}^1 |g(y)|^{2-q} |g(y)|^q w(y) dy \leq cn^{2\alpha+2} \|g\|_\infty^{2-q} \|g\|_q^q$$

which shows that (9.12) holds for  $0 < q \leq 1$  as well.

Let  $0 < q < p < \infty$  (the case  $p = \infty$  is contained in (9.12)). Then using (9.12) we obtain

$$\begin{aligned} \|g\|_p &= \left( \int_{-1}^1 |g(x)|^{p-q} |g(x)|^q w(x) dx \right)^{1/p} \\ &\leq cn^{(2\alpha+2)(\frac{1}{q}-\frac{1}{p})} \|g\|_q^{\frac{p-q}{p}} \|g\|_q^{\frac{q}{p}} = cn^{(2\alpha+2)(\frac{1}{q}-\frac{1}{p})} \|g\|_q. \end{aligned}$$

On the other hand, by [11, p. 114]

$$\|g\|_p \leq cn^{2(\frac{1}{q}-\frac{1}{p})} \|g\|_q.$$

Putting the above two estimates together gives (2.10).

To prove (2.11) we will need the following inequality

$$(9.13) \quad \int_{-1}^1 \frac{w(y)}{\mathcal{W}(n; y)^{p/2+\gamma} (1+nd(x, y))^\sigma} dy \leq c \frac{1}{n\mathcal{W}(n; x)^{p/2+\gamma-1}}, \quad x \in [-1, 1],$$

where  $\gamma \in \mathbb{R}$  and  $\sigma$  is sufficiently large. The proof of (9.13) is contained in the proof of Proposition 1 in [12]. Assume  $1 < q < \infty$ . Then using Lemma 2.5, (i), Hölder's inequality ( $1/q + 1/q' = 1$ ), and (1.10) we have, for  $x \in [-1, 1]$ ,

$$\begin{aligned} |g(x)| &\leq \|\mathcal{W}(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q \left( \int_{-1}^1 \left| L_n(x, y) \mathcal{W}(n; y)^{-s-\frac{1}{p}+\frac{1}{q}} \right|^{q'} w(y) dy \right)^{1/q'} \\ &\leq c \frac{n}{\mathcal{W}(n; x)^{1/2}} \left( \int_{-1}^1 \frac{w(y) dy}{\mathcal{W}(n; y)^{\frac{q'}{2}+\gamma} (1+nd(x, y))^\sigma} \right)^{1/q'} \|\mathcal{W}(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q \end{aligned}$$

with  $\gamma = q'(s + \frac{1}{p} - \frac{1}{q})$ . Now, applying (9.13) we infer

$$|g(x)| \leq c \frac{n^{1/q}}{\mathcal{W}(n; x)^{s+1/p}} \|\mathcal{W}(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q,$$

which implies

$$(9.14) \quad \|\mathcal{W}(n; \cdot)^{s+1/p} g(\cdot)\|_\infty \leq cn^{1/q} \|\mathcal{W}(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q, \quad 1 < q \leq \infty.$$

If  $0 < q \leq 1$ , then by (9.14) with  $q = 2$

$$\begin{aligned} \|\mathcal{W}(n; \cdot)^{s+1/p} g(\cdot)\|_\infty &\leq cn^{1/2} \|\mathcal{W}(n; \cdot)^{s+\frac{1}{p}-\frac{1}{2}} g(\cdot)\|_2 \\ &\leq cn^{1/2} \|\mathcal{W}(n; \cdot)^{s+1/p} g(\cdot)\|_\infty^{1-q/2} \|\mathcal{W}(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q^{q/2}, \end{aligned}$$

which shows that (9.14) holds for  $0 < q \leq 1$  as well. Let  $p < \infty$ . Using (9.14), we get

$$\begin{aligned} \|\mathcal{W}(n; \cdot)^s g(\cdot)\|_p &= \left( \int_{-1}^1 [\mathcal{W}(n; x)^s g(x)]^{p-q} [\mathcal{W}(n; x)^s g(x)]^q w(x) dx \right)^{1/p} \\ &\leq cn^{\frac{1}{q} - \frac{1}{p}} \|\mathcal{W}(n; \cdot)^{s + \frac{1}{p} - \frac{1}{q}} g(\cdot)\|_q^{p-q} \left( \int_{-1}^1 \frac{[\mathcal{W}(n; x)^s g(x)]^q}{\mathcal{W}(n; x)^{\frac{p-q}{p}}} w(x) dx \right)^{1/p} \\ &= cn^{\frac{1}{q} - \frac{1}{p}} \|\mathcal{W}(n; \cdot)^{s + \frac{1}{p} - \frac{1}{q}} g(\cdot)\|_q. \end{aligned}$$

Thus (2.11) is established if  $p < \infty$ . When  $p = \infty$  (2.11) follows from (9.14).  $\square$

**Proof of estimate (2.22).** We only consider the case when  $-1/2 \leq x, y \leq 1$  since the other cases are similar or simpler. Choose  $\theta, \phi \in [0, 2\pi/3]$  so that  $x = \cos \theta$ ,  $y = \cos \phi$ . Then  $d(x, y) = |\theta - \phi|$ . We have

$$\begin{aligned} \sin \theta + n^{-1} &\leq |\sin \theta - \sin \phi| + \sin \phi + n^{-1} \leq |\theta - \phi| + \sin \phi + n^{-1} \\ &\leq (1 + n|\theta - \phi|)(\sin \phi + n^{-1}). \end{aligned}$$

Then (2.22) follows using (2.20).  $\square$

**Proof of Lemma 2.7.** We first show that for  $y \in [0, 1]$  and  $0 < r \leq \pi$

$$(9.15) \quad \mu(B_y(r)) := \int_{B_y(r)} w(x) dx \sim r(d(y, 1) + r)^{2\alpha+1}.$$

Indeed, choose  $0 \leq \zeta \leq \pi/2$  so that  $y = \cos \zeta$  and consider the case when  $\zeta + r \leq 2\pi/3$  (the case  $\zeta + r > 2\pi/3$  is trivial; then  $\mu(B_y(r)) \sim 1$ ). We have

$$\begin{aligned} \mu(B_y(r)) &\sim \int_{\max\{\zeta-r, 0\}}^{\zeta+r} (1 - \cos u)^\alpha \sin u du \sim \int_{\max\{\zeta-r, 0\}}^{\zeta+r} u^{2\alpha+1} du \\ &\sim (\zeta + r - \max\{\zeta - r, 0\})(\zeta + r)^{2\alpha+1} \sim r(\zeta + r)^{2\alpha+1}, \end{aligned}$$

which yields (9.15).

We now proceed with the proof of (2.25). Denote briefly  $J_\eta := B_\eta(\varepsilon)$  which is an interval. Equivalence (2.25) is trivial when  $x \in J_\eta$ .

Assume  $x \in [-1, 1] \setminus J_\eta$ . Denote by  $I_{x,\eta}$  the interval with end points  $x$  and  $\eta$ . By the definition of the maximal operator in (2.23) it readily follows that

$$\left( \frac{\mu(J_\eta)}{\mu(J_\eta \cup I_{x,\eta})} \right)^{1/t} \leq (\mathcal{M}_t \mathbb{1}_{J_\eta})(x) \leq \left( \frac{\mu(J_\eta)}{\mu(I_{x,\eta})} \right)^{1/t}$$

and since  $\mu(I_{x,\eta}) \leq \mu(J_\eta \cup I_{x,\eta}) \leq \mu(J_\eta) + \mu(I_{x,\eta}) \leq c\mu(I_{x,\eta})$ , we have

$$(9.16) \quad (\mathcal{M}_t \mathbb{1}_{J_\eta})(x) \sim \left( \frac{\mu(J_\eta)}{\mu(I_{x,\eta})} \right)^{1/t}.$$

We will only consider the case when  $x \in [-1/2, 1]$ , since the case  $x \in [-1, -1/2]$  is simpler. Choose  $\gamma \in [0, \pi/2]$  and  $\phi \in [0, 2\pi/3]$  so that  $\eta = \cos \gamma$  and  $x = \cos \phi$ . Then  $d(\eta, x) = |\gamma - \phi|$ . By (9.15) and (9.16) it follows that

$$[(\mathcal{M}_t \mathbb{1}_{J_\eta})(x)]^t \sim \frac{\varepsilon(d(\eta, 1) + \varepsilon)^{2\alpha+1}}{\frac{|\gamma-\phi|}{2} \left( \frac{\gamma+\phi}{2} + \frac{|\gamma-\phi|}{2} \right)^{2\alpha+1}} \sim \frac{\varepsilon}{d(\eta, x)} \left( \frac{d(\eta, 1) + \varepsilon}{d(\eta, x) + d(\eta, 1)} \right)^{2\alpha+1},$$

which implies (2.25). Estimates (2.26) are immediate from (2.25).  $\square$

**Proof of (3.12)-(3.13).** The equivalence  $\|\tilde{\mathbb{1}}_{I_\xi}\|_p \sim (2^{-j}\mathcal{W}(2^j; \xi))^{1/p-1/2}$  follows by (2.19).

From (2.3) and (2.15) it follows that, for  $\xi \in \mathcal{X}_j$  and  $0 < p < \infty$ ,

$$(9.17) \quad \|\varphi_\xi\|_p, \|\psi_\xi\|_p \leq cc_\xi^{1/2} \left( \frac{2^j}{\mathcal{W}(2^j; \xi)} \right)^{1-1/p} \leq c \left( \frac{2^j}{\mathcal{W}(2^j; \xi)} \right)^{1/2-1/p}.$$

When  $p = \infty$ , similar estimates follow by (3.11).

To estimate  $\|\varphi_\xi\|_p, \|\psi_\xi\|_p$  from below, we first note that by (2.5) and (9.17) it follows that  $\|\varphi_\xi\|_2 \sim \|\psi_\xi\|_2 \sim 1$ . Let  $2 < p < \infty$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ . Using Hölder's inequality and (9.17) we obtain

$$0 < c \leq \|\varphi_\xi\|_2^2 \leq \|\varphi_\xi\|_p \|\varphi_\xi\|_{p'} \leq \|\varphi_\xi\|_p \left( \frac{2^j}{\mathcal{W}(2^j; \xi)} \right)^{1/2-1/p'}$$

and similarly for  $\psi$ . Hence

$$(9.18) \quad \|\varphi_\xi\|_p, \|\psi_\xi\|_p \geq c \left( \frac{2^j}{\mathcal{W}(2^j; \xi)} \right)^{1/2-1/p}.$$

In the case  $p = \infty$ , we proceed similarly as above and obtain the same estimate.

If  $0 < p < 2$ , then

$$0 < c \leq \|\varphi_\xi\|_2^2 \leq \|\varphi_\xi\|_p^p \|\varphi_\xi\|_\infty^{2-p} \leq c \|\varphi_\xi\|_p^p \left( \frac{2^j}{\mathcal{W}(2^j; \xi)} \right)^{1-p/2},$$

which implies (9.18) and similarly for  $\|\psi_\xi\|_p$ .

Finally, (3.13) follows by the lower bound from (9.18) with  $p = \infty$  and (3.10).

□

**Proof of Lemma 4.6.** Estimate (4.12) is immediate from (3.11) and Lemma 2.7 (see (2.26) and (2.19)).

For the proof of (4.13) we first observe that by (3.13) there exists a point  $\zeta \in B_\xi(c^*2^{-j})$  such that  $|\varphi_\xi(\zeta)| \geq c^\diamond (2^j/\mathcal{W}(2^j; \xi))^{1/2}$ . By (2.4) it follows that

$$|\varphi_\xi(x) - \varphi_\xi(\zeta)| \leq c \frac{2^{3j/2}d(x, \zeta)}{\sqrt{\mathcal{W}(2^j; \xi)}}, \quad x \in B_\zeta(c2^{-j}),$$

and hence for sufficiently small constant  $c^\flat > 0$  we have

$$|\varphi_\xi(x)| \geq \frac{c^\diamond}{2} (2^j/\mathcal{W}(2^j; \xi))^{1/2} \quad \text{for } x \in B_\zeta(c^\flat 2^{-j}) \cap [-1, 1].$$

Therefore, there exists an interval  $J_\eta := B_\eta(c2^{-j}) \subset [-1, 1]$  such that

$$\|\varphi_\xi\|_{L^\infty(J_\eta)} \geq c(2^j/\mathcal{W}(2^j; \xi))^{1/2} \quad \text{and} \quad d(\eta, \xi) \leq c2^{-j}.$$

Hence,

$$(\mathcal{M}_t \varphi_\xi)(x) \geq c(\mathcal{M}_t \tilde{\mathbb{1}}_{J_\eta})(x) \geq c \tilde{\mathbb{1}}_{I_\xi}(x),$$

where for the last estimate we used (2.25). Thus (4.13) is established for  $\varphi_\xi$ . The proof for  $\psi_\xi$  is the same. □

**Proof of Lemma 4.7.** From the orthogonality of Jacobi polynomials it follows that  $\Phi_j * \psi_\xi(x) = 0$  if  $\xi \in \mathcal{X}_\nu$ , where  $\nu \geq j+2$  or  $\nu \leq j-2$ .

Assume that  $\xi \in \mathcal{X}_\nu$ ,  $j-1 \leq \nu \leq j+1$ . From the localization of the kernels  $\Phi_j, \Psi_\nu$  (see (3.9)) and the definition of  $c_\xi$  (see (2.14)) we get that for any  $\sigma > 0$  there is a constant  $c_\sigma > 0$  such that

$$\begin{aligned} |\Phi_j * \psi_\xi(x)| &= \sqrt{c_\xi} \left| \int_{-1}^1 \Phi_j(x, y) \Psi_\nu(y, \xi) w(y) dy \right| \\ &\leq c 2^{3j/2} \mathcal{W}(2^j; x)^{-1/2} \int_{-1}^1 \frac{w(y)}{\mathcal{W}(2^j; y) (1 + 2^j d(x, y))^\sigma (1 + 2^j d(y, \xi))^\sigma} dy. \end{aligned}$$

Setting  $\xi = \cos \theta$ ,  $x = \cos \eta$  for some  $0 \leq \theta, \eta \leq \pi$  and applying the substitution  $y = \cos \phi$ , we obtain

$$\begin{aligned} |\Phi_j * \psi_\xi(x)| &\leq c 2^{3j/2} \mathcal{W}(2^j; x)^{-1/2} \int_0^\pi \frac{w(\cos \phi) \sin \phi}{\mathcal{W}(2^j; \cos \phi) (1 + 2^j |\eta - \phi|)^\sigma (1 + 2^j |\theta - \phi|)^\sigma} d\phi \\ &\leq c 2^{3j/2} \mathcal{W}(2^j; x)^{-1/2} \int_0^\pi \frac{1}{(1 + 2^j |\eta - \phi|)^\sigma (1 + 2^j |\theta - \phi|)^\sigma} d\phi \\ &\leq c 2^{j/2} \mathcal{W}(2^j; x)^{-1/2} (1 + 2^j |\eta - \theta|)^{-\sigma}, \end{aligned}$$

where we used the inequality

$$2^j \int_{\mathbb{R}} \frac{1}{(1 + 2^j |\eta - \phi|)^\sigma (1 + 2^j |\theta - \phi|)^\sigma} d\phi \leq c (1 + 2^j |\eta - \theta|)^{-\sigma}. \quad \square$$

For the proof of Lemma 4.9 we will need this lemma.

**Lemma 9.2.** *Let  $P \in \Pi_{2^j}$ ,  $j \geq 0$  and  $\xi \in \mathcal{X}_j$ . Suppose  $x_1, x_2 \in [-1, 1]$  and  $d(x_\nu, \xi) \leq c_* 2^{-j}$ ,  $\nu = 1, 2$ . Then for any  $\sigma > 0$*

$$|P(x_1) - P(x_2)| \leq c_\sigma 2^j d(x_1, x_2) \sum_{\eta \in \mathcal{X}_j} \frac{|P(\eta)|}{(1 + 2^j d(\xi, \eta))^\sigma},$$

where  $c_\sigma > 0$  depends only on  $\sigma$ ,  $\alpha$ ,  $\beta$ , and  $c_*$ .

**Proof.** Let  $P \in \Pi_{2^j}$ . Suppose  $L_{2^j}(x, y)$  is the reproducing kernel from Lemma 2.5 with  $n = 2^j$ . Then  $L_{2^j} * P = P$ . Since  $L_{2^j}(x, \cdot)P(\cdot) \in \Pi_{2^{j+2}}$ , using that the quadrature (2.14) is exact for all polynomials from  $\Pi_{2^{j+2}}$  we get

$$P(x) = \int_{-1}^1 L_{2^j}(x, y) P(y) w(y) dy = \sum_{\eta \in \mathcal{X}_j} c_\eta L_{2^j}(x, \eta) P(\eta), \quad x \in [-1, 1].$$

Recall that  $c_\eta \sim 2^{-j} \mathcal{W}(2^j; \eta)$ . Now, using Theorem 2.2 we obtain for  $x_1, x_2 \in [-1, 1]$  with  $d(x_\nu, \xi) \leq c_* 2^{-j}$ ,  $\nu = 1, 2$ ,

$$\begin{aligned} |P(x_1) - P(x_2)| &= \left| \int_{-1}^1 [L_{2^j}(x_1, y) - L_{2^j}(x_2, y)] P(y) w(y) dy \right| \\ &\leq \sum_{\eta \in \mathcal{X}_j} |c_\eta| |L_{2^j}(x_1, \eta) - L_{2^j}(x_2, \eta)| |P(\eta)| \\ &\leq c 2^j d(x_1, x_2) \sum_{\eta \in \mathcal{X}_j} \left( \frac{\mathcal{W}(2^j; \eta)}{\mathcal{W}(2^j; \xi)} \right)^{1/2} \frac{|P(\eta)|}{(1 + 2^j d(\xi, \eta))^\sigma} \\ &\leq c 2^j d(x_1, x_2) \sum_{\eta \in \mathcal{X}_j} \frac{|P(\eta)|}{(1 + 2^j d(\xi, \eta))^{\sigma - \max\{\alpha, \beta\} - 1/2}}, \end{aligned}$$



where for the last inequality we used (2.22). Since  $\sigma > 0$  can be arbitrarily large the result follows.  $\square$

**Proof of Lemma 4.9.** Clearly  $a_\xi \leq b_\xi + d_\xi$ , where

$$d_\xi := \max\{|P(x_1) - P(x_2)| : x_1 \in I_\xi, d(x_1, x_2) \leq c_2 2^{-(j+r)}\},$$

and  $c_2$  is the constant appearing in (2.18). By Lemma 9.2 it follows that

$$d_\xi \leq c_2^{-r} \sum_{\eta \in \mathcal{X}_j} \frac{|P(\eta)|}{(1 + 2^j d(\xi, \eta))^\sigma}, \quad \xi \in \mathcal{X}_j.$$

Then recalling the definition of  $d_\xi^*$  in (4.15) we infer

$$\begin{aligned} d_\xi^* &\leq c_2^{-r} \sum_{w \in \mathcal{X}_j} \sum_{\eta \in \mathcal{X}_j} \frac{|P(\eta)|}{(1 + 2^j d(w, \eta))^\sigma (1 + 2^j d(\xi, w))^\sigma} \\ &\leq c_2^{-r} \sum_{\eta \in \mathcal{X}_j} \frac{|P(\eta)|}{(1 + 2^j d(\eta, \xi))^\sigma} \leq c_2^{-r} a_\xi^*, \end{aligned}$$

where for the second inequality we switched the order of summation and used the simple fact that for  $\sigma > 1$

$$\sum_{w \in \mathcal{X}_j} \frac{1}{(1 + 2^j d(w, \eta))^\sigma (1 + 2^j d(\xi, w))^\sigma} \leq c \frac{1}{(1 + 2^j d(\eta, \xi))^\sigma}.$$

Consequently,  $a_\xi^* \leq b_\xi^* + d_\xi^* \leq b_\xi^* + c_2^{-r} a_\xi^*$  with  $c > 0$  independent of  $r$ . Choosing  $r$  sufficiently large we obtain  $a_\xi^* \leq c b_\xi^*$ . The estimate in the other direction is trivial.  $\square$

**Proof of Lemma 4.10.** We may assume that  $\alpha \geq \beta$ . Fix  $\xi \in \mathcal{X}_j$  and define  $Y_0 := \{\eta \in \mathcal{X}_j : d(\eta, \xi) \leq c_2 2^{-j}\}$  and

$$Y_m := \{\eta \in \mathcal{X}_j : c_2 2^{-j+m-1} < d(\eta, \xi) \leq c_2 2^{-j+m}\}, \quad m \geq 1.$$

where  $c_2 > 0$  is from (2.18). Using (2.18) we have  $\#Y_m \leq c_2^m$ . Also, let

$$J_m := B_\xi(c_2(2^m + 1)2^{-j}) = \{x \in [-1, 1] : d(\eta, \xi) \leq c_2(2^m + 1)2^{-j}\}, \quad m \geq 0.$$

Evidently,  $J_m$  is an interval and  $I_\eta \subset J_m$  if  $\eta \in Y_\nu$ ,  $0 \leq \nu \leq m$ .

We next show that

$$(9.19) \quad \mu(J_m) \leq c_2^{m(4\alpha+3)} \mu(I_\eta) \quad \text{for all } \eta \in Y_m.$$

Suppose  $\xi \in [0, 1]$ ; the case  $\xi \in [-1, 0]$  is the same. Let  $J_m =: [y_1, y_2]$  and chose  $\phi_1, \phi_2 \in [0, \pi]$  so that  $y_1 =: \cos \phi_1$  and  $y_2 =: \cos \phi_2$  ( $\phi_1 > \phi_2$ ). Exactly as in the proof of Lemma 2.7

$$\mu(J_m) = \int_{J_m} w(y) dy \leq c(\phi_1 - \phi_2) \phi_1^{2\alpha+1} \leq c_2^{-j+m} \mathcal{W}(2^j, y_1)$$

and using (2.22)

$$(9.20) \quad \mu(J_m) \leq c_2^{-j+m} \mathcal{W}(2^j, \xi) (1 + 2^j d(y_1, \xi))^{2\alpha+1} \leq c_2^{-j+(2\alpha+2)m} \mathcal{W}(2^j, \xi).$$

On the other hand, using again (2.22)

$$\mu(I_\eta) \geq c_2^{-j} \mathcal{W}(2^j, \eta) \geq c_2^{-j} \mathcal{W}(2^j, \xi) (1 + 2^j d(\eta, \xi))^{-2\alpha-1} \geq c_2^{-j-(2\alpha+1)m} \mathcal{W}(2^j, \xi).$$

Combining this with (9.20) gives (9.19).

Let  $\rho := \max\{0, 1 - \frac{1}{t}\} < 1$ . Using Hölder's inequality if  $t > 1$  and the  $t$ -triangle inequality if  $0 < t \leq 1$ , we have

$$b_\xi^* = \sum_{\eta \in \mathcal{X}_j} \frac{|b_\eta|}{(1 + 2^j d(\eta, \xi))^\sigma} \leq c \sum_{m \geq 0} 2^{-m\sigma} \sum_{\eta \in Y_m} |b_\eta| \leq c \sum_{m \geq 0} 2^{-m(\sigma-\rho)} \left( \sum_{\eta \in Y_m} |b_\eta|^t \right)^{1/t}.$$

We next use (9.19) to obtain, for  $x \in I_\xi$ ,

$$\begin{aligned} b_\xi^* &= c \sum_{m \geq 0} 2^{-m(\sigma-1)} \left( \int_{-1}^1 \left[ \sum_{\eta \in Y_m} |b_\eta| \mu(I_\eta)^{-1/t} \mathbb{1}_{I_\eta}(x) \right]^t w(x) dx \right)^{1/t} \\ &\leq c \sum_{m \geq 0} 2^{-m(\sigma-1)} \left( \frac{1}{\mu(J_m)} \int_{J_m} \left[ \sum_{\eta \in Y_m} \left( \frac{\mu(J_m)}{\mu(I_\eta)} \right)^{1/t} |b_\eta| \mathbb{1}_{I_\eta}(x) \right]^t w(x) dx \right)^{1/t} \\ &\leq c \sum_{m \geq 0} 2^{-m(\sigma-1-(4\alpha+3)/t)} \left( \frac{1}{\mu(J_m)} \int_{J_m} \left[ \sum_{\eta \in Y_m} |b_\eta| \mathbb{1}_{I_\eta}(x) \right]^t w(x) dx \right)^{1/t} \\ &\leq c \mathcal{M}_t \left( \sum_{w \in \mathcal{X}_j} |b_w| \mathbb{1}_{I_w} \right)(x) \sum_{m \geq 0} 2^{-m(\sigma-1-(4\alpha+3)/t)} \leq c \mathcal{M}_t \left( \sum_{w \in \mathcal{X}_j} |b_w| \mathbb{1}_{I_w} \right)(x), \end{aligned}$$

where for the last inequality we used that  $\sigma > (4\alpha + 3)/t + 1$ .  $\square$

**Proof of Lemma 6.6.** For  $\xi \in \mathcal{X}_j$ , we set  $a_\xi := \max_{x \in I_\xi} |P(x)|$ ,  $m_\xi := \min_{x \in I_\xi} |P(x)|$  and

$$b_\xi := \max_{x \in I_w} \{ \min |P(x)| : w \in \mathcal{X}_{j+r}, I_w \cap I_\xi \neq \emptyset \},$$

where  $r \geq 1$  (sufficient large) is the constant from Lemma 4.9. If  $0 < t < p$  then

$$\begin{aligned} \left( \sum_{\xi \in \mathcal{X}_j} a_\xi^p \mu(I_\xi) \right)^{1/p} &= \left\| \sum_{\xi \in \mathcal{X}_j} a_\xi \mathbb{1}_{I_\xi}(\cdot) \right\|_p \leq c \left\| \sum_{\xi \in \mathcal{X}_j} b_\xi^* \mathbb{1}_{I_\xi}(\cdot) \right\|_p \\ (9.21) \quad &\leq c \left\| \mathcal{M}_t \left( \sum_{\xi \in \mathcal{X}_j} b_\xi \mathbb{1}_{I_\xi}(\cdot) \right) \right\|_p \leq c \left\| \sum_{\xi \in \mathcal{X}_j} b_\xi \mathbb{1}_{I_\xi}(\cdot) \right\|_p, \end{aligned}$$

where for the first inequality we used Lemma 4.9 and for the second Lemma 4.10. Also, for  $\xi \in \mathcal{X}_j$  let  $\mathcal{X}_{j+r}(\xi) := \{w \in \mathcal{X}_{j+r} : I_w \cap I_\xi \neq \emptyset\}$ . Evidently,  $\#\mathcal{X}_{j+r}(\xi) \leq c$ . Then, for  $w, \eta \in \mathcal{X}_{j+r}(\xi)$  we have  $d(w, \eta) \leq c(r)2^{-j-r}$  and hence

$$m_w \leq c \frac{m_w}{1 + 2^{j+r} d(w, \eta)} \leq c m_\eta^*.$$

Therefore, for any  $\xi \in \mathcal{X}_j$  and  $\eta \in \mathcal{X}_{j+r}(\xi)$  we have  $b_\xi = \max_{w \in \mathcal{X}_{j+r}(\xi)} m_w \leq c m_\eta^*$  and hence

$$b_\xi \mathbb{1}_{I_\xi} \leq \sum_{\eta \in \mathcal{X}_{j+r}(\xi)} m_\eta^* \mathbb{1}_{I_\eta}.$$

Using this in (9.21) we get

$$\begin{aligned} \left( \sum_{\xi \in \mathcal{X}_j} a_\xi^p \mu(I_\xi) \right)^{1/p} &\leq c \left\| \sum_{\eta \in \mathcal{X}_{j+r}} m_\eta^* \mathbb{1}_{I_\eta}(\cdot) \right\|_p \leq c \left\| \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_{j+r}} m_\eta \mathbb{1}_{I_\eta}(\cdot) \right) \right\|_p \\ &\leq c \left\| \sum_{\eta \in \mathcal{X}_{j+r}} m_\eta \mathbb{1}_{I_\eta}(\cdot) \right\|_p \leq c \|P\|_p, \end{aligned}$$

which completes the proof.  $\square$

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