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Localized Tight Frames on Spheres *†

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Abstract

In this paper we wish to present a new class of tight frames on the sphere. These frames have excellent pointwise localization and approximation properties. These properties are based on pointwise localization of kernels arising in the spectral calculus for certain pseudo-differential operators, and on a positive-weight quadrature formula for the sphere that the authors have recently developed. Improved bounds on the weights in this formula are another by-product of our analysis.

1 Introduction

Frames were introduced in the 1950s by Duffin and Schaeffer [2] to represent functions via over-complete sets. Let us review the basic facts and terminology for frames when the target functions belong to a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. In that case, a set $\{\psi_j\}_{j\in\mathcal{J}}$ is a *frame* if there are constants $c, C > 0$ such that for all $f \in \mathcal{H}$

$$c\|f\|^2 \leq \sum_{j\in\mathcal{J}} |\langle f, \psi_j\rangle|^2 \leq C\|f\|^2.$$

The smallest C and largest c are called upper and lower *frame bounds.* If $C = c$, we say the frame is *tight.* If $C = c = 1$, then the frame is *normalized*, and if in addition $\|\psi_j\| = 1$ for all j, then the frame is an orthonormal basis.

**Key words:* Tight frames, n-sphere, localization, pseudo-differential operator.

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Frames, including tight ones, arise naturally in wavelet analysis on $\mathbb{R}^n$ when continuous wavelet transforms are discretized. They provide a redundancy that helps reduce the effect of noise in data, and they have been constructed, studied, and employed extensively in both theoretical and applied problems [1, 5, 6, 9, 11]. Tight frames, which are very close to orthonormal bases, are especially useful in signal and image processing, for they simplify the task of synthesizing a function from processed data [1].

On $\mathbb{S}^n$, the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, both wavelets and frames have been developed (see [12, 15] for references and more discussion) and applied to geoscience problems [7, 18].

Tight, well-localized frames are another matter. Although there have been theoretical proofs of the existence of (normalized) tight frames on $\mathbb{S}^n$ (see Dykema et al. [3]), there is no computationally implementable method of finding them.

The purpose of this paper is to construct and study a class of computationally implementable, well-localized, tight frames on $\mathbb{S}^n$. Central to this construction is a class of extremely well-localized frame functions, which arise in connection with kernels for certain pseudo-differential operators on $\mathbb{S}^n$. This construction also has an interesting connection to wavelet masks. The frame functions are of interest in their own right. In particular, they can be used in the construction and characterization of many of the classical Banach spaces including $L^p(\mathbb{S}^n)$, Besov and Triebel-Lizorkin. We will discuss this in a subsequent paper that is now in preparation. Finally, the localized frame functions give rise to improved quadrature formulas on the sphere, and they allow us to obtain explicit upper and lower bound estimates on the quadrature coefficients. We mention also that our the pseudo-differential operators that we use provide a foundation of extending our construction to other Riemannian manifolds.

This paper is organized as follows. In Section 2, an infinite series of Hilbert space operators that forms a resolution of the identity is defined in terms of a class of wavelet-like masks. Such a construction leads to simple decomposition and reconstruction formulas within a Hilbert space setting. In section 3, attention is focused on $L^2(\mathbb{S}^n)$ where the partial sums of the resolution of the identity are seen to be certain pseudo-differential operators when the projections involved are interpreted as those associated with spherical harmonics. These ΨDO operators are well-localized, provide good approximation orders. We discuss discretizing them, via quadrature, in section 3.2; these discretized kernels lead to the construction of tight frames on $\mathbb{S}^n$. The heart of the paper is in section 4, where a detailed analysis of the (pointwise) localization of the kernels associated with the ΨDO operators is given. Finally in section 5, much improved upper and lower bounds for quadrature weights are obtained using the kernel approach of this paper. A key to this is a new and extended Marcinkiewicz-Zygmund formula for classes of functions which include polynomials.

2 Operator Frames

Let $\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2 \ldots$ be a set of orthogonal projections on a Hilbert space $\mathcal{H}$ for which $\mathsf{P}_j \mathsf{P}_k = \delta_{j,k} \mathsf{P}_k$ and $\sum_{\nu=0}^{\infty} \mathsf{P}_\nu = I$. In other words, the projections form a resolution of the identity in $\mathcal{H}$. Take $a(t)$ to be in $C(\mathbb{R})$, with support in $[\frac{1}{2}, 2]$ and let $\{\mu_\nu\}_{\nu=0}^{\infty}$ be an unbounded, monotonically increasing sequence, with $\mu_0 \geq 0$; set $j_\nu := \lfloor \log_2(\mu_\nu) \rfloor$. For $j = 0, 1 \ldots$, define the operators:

$$\mathsf{A}_j := \begin{cases} \delta_{j,j_1} \mathsf{P}_0 + \displaystyle\sum_{\nu=0}^{\infty} a\left(\frac{\mu_\nu}{2^{j+j_1}}\right) \mathsf{P}_\nu\,, & \mu_0 = 0, \\ \displaystyle\sum_{\nu=0}^{\infty} a\left(\frac{\mu_\nu}{2^{j+j_0}}\right) \mathsf{P}_\nu\,, & \mu_0 > 0. \end{cases} \tag{1}$$

The split between $\mu_0 = 0$ and $\mu_0 > 0$ occurs later in connection with a split between $\mathbb{S}^1$ and $\mathbb{S}^n$, $n \geq 2$.

For fixed j, the sum is finite; when $\mu_0 = 0$, it will be over those ν for which j_ν satisfies $j - 1 \leq j_\nu - j_1 \leq j + 1$, and when $\mu_0 > 0$, $j - 1 \leq j_\nu - j_0 \leq j + 1$. We have the following result.

Proposition 2.1 *If $a(t), \tilde{a}(t) \in C(\mathbb{R})$, both with support in $[\frac{1}{2}, 2]$, and satisfy $a(t)\overline{\tilde{a}(t)} + a(2t)\overline{\tilde{a}(2t)} \equiv 1$ on $[\frac{1}{2}, 1]$, then $\sum_{j=0}^{\infty} \mathsf{A}_j \widetilde{\mathsf{A}}_j^* = I$, where convergence is in the strong operator topology.*

Proof: We will do the case when $\mu_0 > 0$. The other case is nearly identical to it. Let $J > 0$ be an integer. Multiply the terms in $\sum_{j=0}^{J} \mathsf{A}_j \widetilde{\mathsf{A}}_j^*$. Note that the coefficient of P_ν in this sum is $\sum_{j=0}^{J} a\left(\frac{\mu_\nu}{2^{j+j_0}}\right) \overline{\tilde{a}\left(\frac{\mu_\nu}{2^{j+j_0}}\right)}$. Because the support requirements on a and $\tilde{a}$ imply that the only nonzero contributions to this coefficient are from $j = j_\nu - j_0$ and $j = j_\nu - j_0 + 1$, we have that

$$\text{Coefficient of } \mathsf{P}_\nu = \begin{cases} 0 & j_\nu > J + j_0 \\ a\left(\frac{\mu_\nu}{2^{j_\nu}}\right) \overline{\tilde{a}\left(\frac{\mu_\nu}{2^{j_\nu}}\right)} & j_\nu = J + j_0 \\ a\left(\frac{\mu_\nu}{2^{j_\nu}}\right) \overline{\tilde{a}\left(\frac{\mu_\nu}{2^{j_\nu}}\right)} + a\left(\frac{\mu_\nu}{2^{j_\nu+1}}\right) \overline{\tilde{a}\left(\frac{\mu_\nu}{2^{j_\nu+1}}\right)}, & j_\nu < J + j_0 \end{cases}$$

Applying the identity, with $t = \frac{\mu_\nu}{2^{j_\nu+1}}$, to the expression on the right for $j_\nu < J + j_0$ shows that it is 1. We then have

$$\begin{aligned}\sum_{j=0}^{J} \mathsf{A}_j \widetilde{\mathsf{A}}_j^* &= \sum_{j_\nu<J+j_0} \mathsf{P}_\nu + \sum_{j_\nu=J+j_0} a\left(\frac{\mu_\nu}{2^{j_\nu}}\right) \overline{\tilde{a}\left(\frac{\mu_\nu}{2^{j_\nu}}\right)} \mathsf{P}_\nu \\ &= \sum_{j_\nu<J+j_0} \mathsf{P}_\nu + \sum_{j_\nu=J+j_0} a\left(\frac{\mu_\nu}{2^{J+j_0}}\right) \overline{\tilde{a}\left(\frac{\mu_\nu}{2^{J+j_0}}\right)} \mathsf{P}_\nu \end{aligned} \tag{2}$$

from which it easily follows that

$$\|f - \sum_{j=0}^{J} \mathsf{A}_j \widetilde{\mathsf{A}}_j^* f\|^2 \leq (1 + \|a\|_\infty \|\tilde{a}\|_\infty)^2 \sum_{j_\nu \geq J+j_0} \|\mathsf{P}_\nu f\|^2.$$

Since the projections form a resolution of the identity, the right side will vanish as $J \to \infty$, and so $\sum_{j=0}^{J} \mathsf{A}_j \widetilde{\mathsf{A}}_j^*$ converges strongly to I. □

There is an interesting way to view the partial sums from the proof. Define the function

$$b(t) := \begin{cases} 1 & t \leq 1 \\ a(t)\overline{\tilde{a}(t)} & t > 1, \end{cases} \tag{3}$$

where a and $\tilde{a}$ satisfy the conditions in the proposition. Then from (2), we see that

$$\mathsf{B}_J := \sum_{j=0}^{J} \mathsf{A}_j \widetilde{\mathsf{A}}_j^* = \sum_{\nu=0}^{\infty} b\left(\frac{\mu_\nu}{2^{J+j_0}}\right) \mathsf{P}_\nu \,. \tag{4}$$

The point is that if we define the unbounded self-adjoint operator $\mathsf{M} = \sum_{\nu=0}^{\infty} \mu_\nu \mathsf{P}_\nu$, then both A_j and B_J can be defined via spectral theory as $\mathsf{A}_j = a(2^{-j-j_0}\,\mathsf{M})$ and $\mathsf{B}_J = b(2^{-J-j_0}\,\mathsf{M})$. In section 3.1, the operator M will be an order 1 pseudodifferential operator, essentially the square root of the Laplace-Beltrami operator on $\mathbb{S}^n$, and we will be working with ΨDO's on $\mathbb{S}^n$.

We can construct functions a and $\tilde{a}$ that satisfy the conditions of the proposition using *biorthogonal wavelet masks* [1, §8.3], $m_0(\xi)$ and $\widetilde{m}_0(\xi)$, which are continuous, 2π-periodic functions that satisfy $m_0(\pi) = \widetilde{m}_0(\pi) = 0$, $m_0(0) = \widetilde{m}_0(0) = 1$, along with $m_0(\xi)\overline{\widetilde{m}_0(\xi)} + m_0(\xi+\pi)\overline{\widetilde{m}_0(\xi+\pi)} \equiv 1$. Given m_0, we define

$$a(t) := \begin{cases} m_0(\pi \log_2(t)), & \frac{1}{2} \leq t \leq 2, \\ 0 & t < \frac{1}{2} \text{ or } t > 2, \end{cases} \tag{5}$$

and similarly for $\tilde{a}$ and $\widetilde{m}_0$. Both a and $\tilde{a}$ are obviously continuous, have support in $[\frac{1}{2}, 2]$, and satisfy the identity $a(t)\overline{\tilde{a}(t)} + a(2t)\overline{\tilde{a}(2t)} \equiv 1$ on $[\frac{1}{2}, 1]$.

Indeed, for any integer $k \geq 0$ it is possible to construct masks and corresponding dual masks that are at least in C^{k+1} and generate wavelets and dual wavelets with $k+1$ or more vanishing moments [1, §8.3], although the number of vanishing moments may not be the same for m_0 and $\widetilde{m}_0$. Since such $m_0(\xi)$ and $\widetilde{m}_0$ have zeros at least of order $k+1$ at $\xi = \pm\pi$, the functions $a(t)$ and $\tilde{a}(t)$ will have derivatives to order k that will join smoothly to 0 at $t = 1/2$ and $t = 2$. Both functions will be in $C^k(\mathbb{R})$. Of course, we can also use one of the Daubechies masks that generate orthogonal wavelets. For these $m_0 = \widetilde{m}_0$, in which case $a = \tilde{a}$.

There is another way to generate a function for which $a = \tilde{a}$. Let $g \in C^\infty(\mathbb{R})$ have support in $[\frac{1}{2}, 2]$ and be such that $|g(t)| > 0$ when $\frac{1}{2} < t < 2$. It is easy to check that the function $a(t)$ defined by

$$a(t) := \begin{cases} \dfrac{g(t)}{\sqrt{|g(t/2)|^2 + |g(t)|^2 + |g(2t)|^2}}, & \frac{1}{2} < t < 2, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

is C^∞ and satisfies $|a(t)|^2 + |a(2t)|^2 \equiv 1$ when $\frac{1}{2} \leq t \leq 1$. We summarize these observations below.

Proposition 2.2 *Let $k \geq 0$ be an integer. There exist functions $a, \tilde{a} \in C^k(\mathbb{R})$ that are supported on $[\frac{1}{2}, 2]$ and satisfy $a(t)\overline{\tilde{a}(t)} + a(2t)\overline{\tilde{a}(2t)} \equiv 1$ on $[\frac{1}{2}, 1]$. In addition, there exist $a \in C^k(\mathbb{R})$, including $k = \infty$, such that $\tilde{a} = a$, in which case the identity above becomes $|a(t)|^2 + |a(2t)|^2 \equiv 1$ for $t \in [\frac{1}{2}, 1]$.*

We now can define decomposition and reconstruction operators for $f \in \mathcal{H}$. We define the *decomposition operator* to be

$$f \to w_j = \widetilde{\mathsf{A}}_j^* f.$$

From the proposition above, we have that $f = If = \sum_{j=0}^\infty \mathsf{A}_j \widetilde{\mathsf{A}}_j^* f$. Consequently, we also see that f can be written as the sum

$$f = \sum_{j=0}^\infty \mathsf{A}_j w_j,$$

which then gives us our *reconstruction operator*.

Proposition 2.3 *If $\tilde{a} = a$, then $\widetilde{\mathsf{A}}_j^* f = \mathsf{A}_j^* f$ and the operator frame that we have constructed is tight in the sense that*

$$\|f\|^2 = \sum_{j=0}^{\infty} \|\mathsf{A}_j^* f\|^2.$$

Proof: This follows immediately from the decomposition and reconstruction formulas above. $\square$

3 Frames on $\mathbb{S}^n$

3.1 ΨDO Frames on $\mathbb{S}^n$

We turn to the situation in which the underlying Hilbert space is $\mathcal{H} = L^2(\mathbb{S}^n)$, with $d\mu$ being the usual measure on the n-sphere. Throughout the paper, we will let $\lambda_n := \frac{n-1}{2}$ and $\{Y_{\ell,m} : \ell = 0, 1, \ldots, m = 1 \ldots d_\ell^n\}$ be the usual orthonormal set of spherical harmonics [16, 21] associated with $\mathbb{S}^n$, where for $n \geq 2$,

$$d_\ell^n = \frac{\ell + \lambda_n}{\lambda_n} \binom{\ell + n - 2}{\ell}. \tag{7}$$

Denote by $\mathbf{H}_\ell$ the span of the spherical harmonics with fixed order ℓ, and let $\Pi_L = \bigoplus_{\ell=0}^{L} \mathbf{H}_\ell$ be the span of all spherical harmonics of order at most L. The orthogonal projection P_ℓ onto $\mathbf{H}_\ell$ is given by

$$\mathsf{P}_\ell f = \sum_{m=1}^{d_\ell^n} \langle f, Y_{\ell,m} \rangle Y_{\ell,m} \,.$$

Using the addition formula for spherical harmonics, one can write the kernel for this projection as

$$P_\ell(\xi, \eta) = \sum_{m=1}^{d_\ell^n} Y_{\ell,m}(\xi) \overline{Y_{\ell,m}(\eta)} = \frac{\ell + \lambda_n}{\lambda_n \omega_n} P_\ell^{(\lambda_n)}(\xi \cdot \eta),$$

where $\lambda_n = \frac{n-1}{2}$ and $P_\ell^{(\lambda_n)}$ is the ultraspherical polynomial of order λ_n and degree ℓ. We regard $\mathbb{S}^n$ as being the unit sphere in $\mathbb{R}^{n+1}$, and we let the quantity $\xi \cdot \eta$ denote the usual "dot" product for $\mathbb{R}^{n+1}$.

On the sphere, an operator K with a kernel of the form $K(\xi \cdot \eta)$ can be written as a convolution on $\mathbb{S}^n$; that is, $\mathsf{K}f = K * f$, where

$$K * f(\xi) = \int_{\mathbb{S}^n} K(\xi \cdot \eta) f(\eta) d\mu(\eta)\,.$$

Because of the form of the convolution, these operators commute with rotations. Depending on the properties of the kernel, one may (and will!) apply these operators to spaces other than $L^2(\mathbb{S}^n)$.

The spherical harmonic $Y_{\ell,m}$ is an eigenfunction corresponding to the eigenvalue $-\ell(\ell+n-1) = \lambda_n^2 - (\ell+\lambda_n)^2$ for Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ on $\mathbb{S}^n$. It follows that $\ell + \lambda_n$ is an eigenvalue corresponding to the eigenfunctions $Y_{\ell,m}\,, m = 1 \ldots d_\ell^n$, of the pseudo-differential operator

$$\mathsf{L}_n := \sqrt{\lambda_n^2 - \Delta_{\mathbb{S}^n}}$$

The operators A_j introduced in section 2 can be written in terms of L_n if the projections are interpreted as those associated with the spherical harmonics. Choosing the projections this way, setting $\mu_\ell = \ell + \lambda_n$, and using the spectral theorem for L_n, we see that the A_j's become

$$\mathsf{A}_j = \begin{cases} \delta_{j,0}\mathsf{P}_0 + a(2^{-j}\mathsf{L}_1) & n = 1 \\ \quad a(2^{-j-j_0}\mathsf{L}_n) & n \geq 2,\ j_0 = \lfloor \log_2(\lambda_n) \rfloor, \end{cases}$$

where $a \in C^k(\mathbb{R})$ is defined by (5). These are pseudo-differential operators with kernels

$$A_j(\xi\cdot\eta) = \begin{cases} \frac{1}{2\pi}\delta_{j,0} + \frac{1}{\pi}\sum_{\ell=1}^{\infty} a(2^{-j}\ell)\cos(\ell\theta), & n = 1,\ \xi\cdot\eta = \cos\theta \\ \sum_{\ell=0}^{\infty} a\left(\frac{\ell+\lambda_n}{2^{j+j_0}}\right) \frac{\ell+\lambda_n}{\lambda_n \omega_n} P_\ell^{(\lambda_n)}(\xi\cdot\eta), & n \geq 2,\ j_0 = \lfloor \log_2(\lambda_n) \rfloor. \end{cases} \quad (8)$$

We note that the assumption on the support of a implies that the orders of the spherical harmonics in the kernels satisfy $2^{j+j_0-1} < \ell + \lambda_n < 2^{j+j_0+1}$.

Virtually identical formulas and comments apply to the operators $\widetilde{\mathsf{A}}_j$ and their corresponding kernels $\widetilde{A}_j(\xi\cdot\eta)$. (Merely putting $\tilde{}$ over appropriate a's, A_j's, and A_j's will suffice!)

The partial sum $\mathsf{B}_J = \sum_{j=0}^{J} \mathsf{A}_j \widetilde{\mathsf{A}}_j^*$, which is given in terms of b defined in (3), has the spectral form

$$\mathsf{B}_J = b(2^{-J-j_0}\mathsf{L}_n)\,.$$

The kernel for B_J is thus

$$B_J(\xi\cdot\eta) = \begin{cases} \frac{1}{2\pi}b(0) + \frac{1}{\pi}\sum_{\ell=1}^{\infty} b(2^{-J}\ell)\cos(\ell\theta), & n=1,\ \xi\cdot\eta=\cos\theta \\ \sum_{\ell=0}^{\infty} b\left(\frac{\ell+\lambda_n}{2^{J+j_0}}\right)\frac{\ell+\lambda_n}{\lambda_n\omega_n}P_\ell^{(\lambda_n)}(\xi\cdot\eta), & n\ge 2,\ j_0=\lfloor\log_2(\lambda_n)\rfloor. \end{cases} \tag{9}$$

We will study and establish various properties of such ΨDO kernels in section 4. Here we will use those results to discuss the approximation properties of these operator frames.

Theorem 3.1 *Let a, $\tilde{a}$ be as in Proposition 2.2, with $k > \max\{n,2\}$, and let b be defined by (3). If $f\in L^p(\mathbb{S}^n)$, $1\le p\le\infty$, and if $L>0$ is an integer such that $2^{-J-j_0}\le(L+\lambda_n)^{-1}$, then*

$$\|f-\mathsf{B}_Jf\|_p \le C_{b,k,n}E_L(f)_p\,,\ E_L(f)_p := \mathrm{dist}_{L^p}(f,\Pi_L). \tag{10}$$

In addition, for $1\le p<\infty$ or, if $p=\infty$, for $f\in C(\mathbb{S}^n)$, we have

$$\lim_{J\to\infty}\mathsf{B}_Jf = f\,. \tag{11}$$

Proof: Apply Corollary 4.10 with $\kappa=b$, k as above, and $\varepsilon=2^{-J-j_0}$. □

The theorem implies that B_Jf approximates f to within an error comparable to $E_L(f)_p$, which is that for the best approximation to f from Π_L in L^p. Much work [10, 17, 19, 20, 24] has been done on estimating this error for various smoothness classes and spaces. This work allows us to obtain rates of approximation when f has additional smoothness requirements. A typical result [10] is this: If $f\in L^p(\mathbb{S}^n)$, with $\|f\|_p=1$, belongs to a smoothness class $W_p^\alpha(\mathbb{S}^n)$, which is analogous to a Sobolev space, then $E_L(f)_p\sim L^{-\alpha}$. Choosing f similarly and taking $L\sim 2^J$, we get a corresponding result for our case: $\|f-\mathsf{B}_Jf\|_p\sim 2^{-\alpha J}$.

3.2 Discretization: Tight Frames on $\mathbb{S}^n$

The frames themselves will be obtained by discretizing these formulas. Let X be a finite set of distinct points in $\mathbb{S}^n$; we will call these the *centers*. There are several important quantities associated with this set: the *mesh norm*, $h_X=\sup_{y\in\mathbb{S}^n}\inf_{\xi\in X}d(\xi,y)$, where $d(\cdot,\cdot)$ is the geodesic distance between points on the sphere; the *separation radius*, $q_X=\frac{1}{2}\min_{\xi\neq\xi'}\,d(\xi,\xi')$; and the *mesh ratio*, $\rho_X := h_X/q_X \geq 1$. The set of centers X is called ρ-uniform if $\rho_X \leq \rho$. For $\rho \geq n+1$, there exists a ρ–uniform X with h_X arbitrarily small [13, Proposition 3.2]. Let $\mathcal{X}$ be the Voronoi partition of $\mathbb{S}^n$ for X. (Other partitions will do as well.) The region containing ξ will be called R_ξ.

The following quadrature formula is essential to our construction.

Theorem 3.2 ([13, 14]) *There exists a constant $c^\diamond > 0$ (depending only on n) such that for any $L \geq 1$ and a ρ-uniform set X in $\mathbb{S}^n$ with $h_X \leq c^\diamond/L$, there exist positive coefficients $\{c_\xi\}_{\xi\in X}$ such that the quadrature formula*

$$\int_{\mathbb{S}^n} f(\eta)\, d\mu(\eta) \doteq \sum_{\xi\in X} c_\xi f(\xi)$$

is exact for all spherical polynomials of degree $\leq L$. In addition, $c_\xi \approx L^{-n}$ with constants of equivalence depending only on n.

We remark that the in the papers [13, 14] only upper bounds on the weights were established. Lower bounds, and a more precise version of the theorem can be found in Theorem 5.4 and the remarks following it.

Fix $\rho \geq n+1$. Pick a sequence of ρ-uniform sets X_j so that $h_{X_j} \leq c^\diamond 2^{-j-j_0-2}$. Then the quadrature formula above is exact for all spherical harmonics of degree $\ell \leq 2^{j+j_0+2}$. Also, $c_\xi \approx 2^{-(j+j_0)n}$ and $\#X \approx 2^{(j+j_0)n}$.

The frame transform has the form $w_j(\eta) = \mathsf{A}_j^* f(\eta) = \langle f(\zeta), A_j(\zeta\cdot\eta)\rangle$. The point is that $w_j(\eta)$ is a spherical polynomial of degree less than 2^{j+j_0+1}, because $A_j(\zeta\cdot\eta)$ is a spherical polynomial with degree less than 2^{j+j_0+1}. In the reconstruction formula this then contributes the term

$$\mathsf{A}_j w_j(\omega) = \int_{\mathbb{S}^n} A_j(\omega\cdot\eta) w_j(\eta) d\mu(\eta).$$

The product $A_j(\omega\cdot\eta)w_j(\eta)$ is a spherical polynomial of degree less than $2^{j+j_0+1}+2^{j+j_0+1} = 2^{j+j_0+2}$. It can thus be integrated *exactly* with the quadrature formula, so that

$$\begin{aligned} \mathsf{A}_j w_j(\omega) &= \sum_{\xi\in X_j} c_\xi A_j(\xi\cdot\omega) w_j(\omega) = \sum_{\xi\in X_j} \langle f, \psi_{j,\xi}\rangle \psi_{j,\xi}, && (12)\\ \text{where}\quad \psi_{j,\xi}(\eta) &:= \sqrt{c_\xi} A_j(\eta\cdot\xi),\ \xi\in X_j, && (13)\end{aligned}$$

is the analysis frame function at level j. We can now prove this result.

Theorem 3.3 *Let $\tilde{a} = a$ be as in Proposition 2.2, with $k > \max\{n, 2\}$, and let A_j be the kernel in (8). If $f \in C(\mathbb{S}^n)$ or, for $1 \le p < \infty$, if $f \in L^p(\mathbb{S}^n)$, then $f = \sum_{j=0}^{\infty} \sum_{\xi \in X_j} \langle f, \psi_{j,\xi} \rangle \psi_{j,\xi}$, with convergence being in the appropriate space. In addition, if $f \in L^2(\mathbb{S}^n)$, the frame $\{\psi_{j,\xi}\}_{j \in \mathbb{Z}_+, \xi \in X_j}$ is tight,*

$$\|f\|^2 = \sum_{j=0}^{\infty} \sum_{\xi \in X_j} |\langle f, \psi_{j,\xi} \rangle|^2.$$

Finally, the frame functions have vanishing moments that increase with j.

Proof: From (12), we have that $\mathsf{B}_J f = \sum_{j=0}^{J} \sum_{\xi \in X_j} \langle f, \psi_{j,\xi} \rangle \psi_{j,\xi}$. By Theorem 3.1 this converges to f in all of the spaces mentioned. To prove that the frame is tight, just observe that for $f \in L^2(\mathbb{S}^n)$, we have $\langle \mathsf{B}_J f, f \rangle = \sum_{j=0}^{J} \sum_{\xi \in X_j} \left| \langle f, \psi_{j,\xi} \rangle \right|^2$. Taking the limit as $J \to \infty$ then yields the equation for $\|f\|^2$. The statement concerning vanishing moments follows from the structure of the A_j's. □

The next section is devoted to proving localization properties concerning families of ΨDO's depending on a small parameter. Applying the results from there yields excellent localization properties for the level j frame function defined in (13).

Proposition 3.4 *Let $\tilde{a} = a$ be as in Proposition 2.2, with $k > \max\{n, 2\}$, let A_j and B_J be the kernels in (8) and (9), respectively, and let $\psi_{j,\xi}$ be given by (13). If $\theta := \cos^{-1}(\eta \cdot \xi)$, then for all $\theta \in [0, \pi]$ there are constants C and C', which depend on k, n, a, and X, such that these hold:*

$$|\psi_{j,\xi}(\eta)| \le \frac{2^{n(j+j_0)/2}\, C}{1 + (2^{j+j_0}\, \theta)^k} \text{ and } |B_J(\eta \cdot \xi)| \le \frac{2^{n(J+j_0)}\, C'}{1 + (2^{J+j_0}\, \theta)^k}.$$

Proof: For the bound on $\psi_{j,\xi}$, apply Theorem 4.5 to $A_j(\eta \cdot \xi)$, with $\kappa = a$ and $\varepsilon = 2^{-j-j_0}$, then use the resulting bound, that $c_\xi \approx 2^{-(j+j_0)n}$, and (13) to obtain the estimate. To bound $B_J(\xi \cdot \eta)$, apply Theorem 4.5 with $\kappa = b$ and $\varepsilon = 2^{-J-j_0}$. □

4 Localization of ΨDO Kernels on $\mathbb{S}^n$

We want to study the localization properties of ΨDO kernels related to the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ on the sphere. As we did earlier, let $\mathsf{L}_n := \sqrt{\lambda_n^2 - \Delta_{\mathbb{S}^n}}$ and let $\kappa(t) \in C^k(\mathbb{R})$, with $k \ge \max\{2, n-1\}$, be even and satisfy

$$|\kappa^{(r)}(t)| \le C_\kappa (1+|t|)^{r-\alpha} \text{ for all } t \in \mathbb{R},\ r = 0, \dots, k, \tag{14}$$

where $\alpha > n+k$ and $C_\kappa > 0$ are fixed constants. We remark that all compactly supported, even C^k functions satisfy (14), as do even functions in the Schwartz class, $\mathcal{S}$. Even functions in $\mathcal{S}$ satisfy (14) for arbitrarily large k and α. Define the family of ΨDOs

$$\mathsf{K}_{\varepsilon,n} := \kappa(\varepsilon \mathsf{L}_n) = \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))\mathsf{P}_\ell,\ 0 < \varepsilon \le 1,$$

along with the associated family of kernels

$$K_{\varepsilon,n}(\underbrace{\xi \cdot \eta}_{\cos\theta}) := \begin{cases} \frac{1}{2\pi}\kappa(0) + \frac{1}{\pi}\sum_{\ell=1}^{\infty} \kappa(\varepsilon\ell)\cos \ell\theta, & n = 1, \\ \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))\frac{\ell+\lambda_n}{\lambda_n\omega_n} P_\ell^{(\lambda_n)}(\cos\theta), & n \ge 2, \end{cases} \tag{15}$$

where $\cos\theta = \xi \cdot \eta$ and $0 < \varepsilon \le 1$.

Our aim in this section is to obtain uniform bounds on the kernel $K_\varepsilon(\xi\cdot\eta)$ for small ε, with the bounds being explicitly dependent on ε.

The simple estimates given below in section 4.1 on the terms in the series used to define the kernels $K_{\varepsilon,n}$ confirm that, under mild conditions, these series are uniformly convergent. Let $n \ge 2$. Consider the ultraspherical identity [22, (4.7.14)] with $\lambda = \lambda_n$,

$$\frac{d}{dx} P_\ell^{(\lambda_n)}(x) = 2\lambda_n P_{\ell-1}^{(\lambda_n+1)}(x).$$

Since $\lambda_n + 1 = \lambda_{n+2}$ and $\omega_n = \lambda_{n+2}\omega_{n+2}/\pi$, we have, for $\ell \ge 1$,

$$\begin{aligned} \frac{d}{dx}\left\{ \left(\frac{\ell+\lambda_n}{\lambda_n\omega_n} \right) P_\ell^{(\lambda_n)}(x) \right\} &= 2\left(\frac{\ell-1+\lambda_{n+2}}{\omega_n} \right) P_{\ell-1}^{(\lambda_{n+2})}(x) \\ &= 2\pi \left(\frac{\ell-1+\lambda_{n+2}}{\lambda_{n+2}\omega_{n+2}} \right) P_{\ell-1}^{(\lambda_{n+2})}(x). \end{aligned}$$

Multiply both sides by $\kappa(\varepsilon(\ell+\lambda_n))$ and sum on ℓ from 1 to ∞. Adjust the summation index on the right side and on the left use $\frac{d}{dx}P_0^{(\lambda_n)}(x) = 0$ to arrive at the identity

$$\frac{d}{dx}K_{\varepsilon,n}(x) = 2\pi K_{\varepsilon,n+2}(x). \tag{16}$$

As can be directly verified, this holds even when $n = 1$.

4.1 Convergence issues and an L^∞ estimate on $K_{\varepsilon,n}$

The series defining the kernels are uniformly and absolutely convergent, by the M-test. This is easy to see for $n = 1$. For $n \geq 2$, start with the bound [22, Eqns. (4.7.3) & (7.33.1)]

$$|P_\ell^{(\lambda_n)}(\cos\theta)| \leq \binom{\ell+n-2}{\ell} = P_\ell^{(\lambda_n)}(1), \tag{17}$$

and note that

$$\frac{\ell+\lambda_n}{\lambda_n}\binom{\ell+n-2}{\ell} \leq 2\binom{\ell+n-1}{\ell} \leq 2(1+\ell)^{n-1}.$$

From this and the assumptions on $\kappa(t)$, the terms in the series satisfy the bound,

$$|\kappa(\varepsilon(\ell+\lambda_n))|\frac{\ell+\lambda_n}{\lambda_n\omega_n}\left|P_\ell^{(\lambda_n)}(\cos\theta)\right| \leq \frac{2\,C_\kappa(1+\ell)^{n-1}}{\omega_n(1+\varepsilon(\ell+\lambda_n))^\alpha} \leq \frac{2\,C_\kappa\varepsilon^{-(n-1)}}{\omega_n(1+\varepsilon\ell)^{\alpha-n+1}},$$

which suffices for the M-test, since $\alpha > n+k \geq n+2$ implies the series on the right above is convergent. Note that the estimate holds even when $n = 1$, provided the terms on the right are properly adjusted.

It is easy to take this a step further and obtain an estimate on $\|K_{\varepsilon,n}\|_\infty$, which we will need later on anyway.

Proposition 4.1 *If κ satisfies (14), then*

$$\|K_{\varepsilon,n}\|_\infty \leq \frac{3\,C_\kappa}{\omega_n}\varepsilon^{-n}. \tag{18}$$

Proof: From the series definition of the kernel and the estimate on each term, we get this chain of inequalities:

$$\begin{aligned}\|K_{\varepsilon,n}\|_\infty &\le \sum_{\ell=0}^{\infty}\frac{2\,C_\kappa\varepsilon^{-(n-1)}}{\omega_n(1+\varepsilon\ell)^{\alpha-n+1}}\\ &\le \frac{2\,C_\kappa\varepsilon^{-(n-1)}}{\omega_n}+\int_0^\infty\frac{2\,C_\kappa\varepsilon^{-(n-1)}du}{\omega_n(1+\varepsilon u)^{\alpha-n+1}}\\ &\le \frac{2\,C_\kappa\varepsilon^{-n}}{\omega_n}\left(\varepsilon+\frac{1}{\alpha-n}\right)\end{aligned}$$

Using $\varepsilon \le 1$ and $\alpha - n > k \ge 2$ in the previous inequality and simplifying, we obtain (18). □

4.2 Integral representations

We now wish to obtain integral representations for the kernels $K_\varepsilon(\cos\theta)$. We begin with the Dirichlet-Mehler integral representation for the Gegenbauer polynomials [4, p. 177],

$$P_\ell^{(\lambda)}(\cos\theta)=\frac{2^\lambda\Gamma(\lambda+\frac{1}{2})\Gamma(\ell+2\lambda)}{\sqrt{\pi}\ell!\Gamma(\lambda)\Gamma(2\lambda)(\sin\theta)^{2\lambda-1}}\int_\theta^\pi\frac{\cos\big((\ell+\lambda)\varphi-\lambda\pi\big)}{(\cos\theta-\cos\varphi)^{1-\lambda}}d\varphi,$$

which holds for any real $\lambda > 0$. We will take $\lambda = \lambda_n = \frac{n-1}{2}$, with $n \ge 2$ throughout this section. Multiply both sides of the previous equation by $\frac{\ell+\lambda_n}{\lambda_n\omega_n}$, and then simplify to get this:

$$\frac{\ell+\lambda_n}{\lambda_n\omega_n}P_\ell^{(\lambda_n)}(\cos\theta)=\frac{\gamma_n(\ell+\lambda_n)(\ell+n-2)!}{\ell!(\sin\theta)^{n-2}}\int_\theta^\pi\frac{\cos\big((\ell+\lambda_n)\varphi-\lambda_n\pi\big)}{(\cos\theta-\cos\varphi)^{1-\lambda_n}}d\varphi, \tag{19}$$

where

$$\gamma_n:=\frac{2^{\lambda_n}\Gamma(\lambda_n+\frac{1}{2})}{\sqrt{\pi}\lambda_n\omega_n\Gamma(\lambda_n)\Gamma(2\lambda_n)}. \tag{20}$$

Using the expression on the right in equation (19) in the series definition of $K_{\varepsilon,n}$, we get this representation:

$$K_{\varepsilon,n}(\cos\theta)=\frac{\gamma_n}{(\sin\theta)^{n-2}}\int_\theta^\pi\frac{C_{\varepsilon,n}(\varphi)}{(\cos\theta-\cos\varphi)^{1-\lambda_n}}d\varphi, \tag{21}$$

where $C_{\varepsilon,n}$ is given by the series

$$C_{\varepsilon,n}(\varphi) := \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))\frac{(\ell+\lambda_n)(\ell+n-2)!}{\ell!}\begin{cases} \sin(\lambda_n\pi)\sin(\ell+\lambda_n)\varphi & n \text{ even} \\ \cos(\lambda_n\pi)\cos(\ell+\lambda_n)\varphi & n \text{ odd} \end{cases} \tag{22}$$

We want to put this series in a more convenient form. To begin, the factor $\frac{(\ell+\lambda_n)(\ell+n-2)!}{\ell!}$ is the product $(\ell+\lambda_n)(\ell+n-2)(\ell+n-3)\cdots(\ell+1)$, which can be rewritten as

$$\frac{(\ell+\lambda_n)(\ell+n-2)!}{\ell!} = \prod_{r=1}^{\lfloor\frac{n-1}{2}\rfloor}\left((\ell+\lambda_n)^2-(\lambda_n-r)^2\right)\times\begin{cases} \ell+\lambda_n & \text{even,} \\ 1 & \text{odd.} \end{cases}$$

From this, we see that if we define the degree $n-1$ polynomial

$$Q_{n-1}(z) := \prod_{r=1}^{\lfloor\frac{n-1}{2}\rfloor}\left(z^2-(\lambda_n-r)^2\right)\times\begin{cases} z\sin(\lambda_n\pi) & n \text{ even,} \\ \cos(\lambda_n\pi) & n \text{ odd,} \end{cases} \tag{23}$$

then we have that

$$C_{\varepsilon,n}(\varphi) := \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))Q_{n-1}(\ell+\lambda_n)\begin{cases} \sin(\ell+\lambda_n)\varphi & n \text{ even} \\ \cos(\ell+\lambda_n)\varphi & n \text{ odd} \end{cases} \tag{24}$$

We want to make a few observations about the polynomial Q_{n-1}. First, by direct calculation we have that $Q_{n-1}(-z) = (-1)^{n-1}Q_{n-1}(z)$, so that Q_{n-1} is an even function for odd n and an odd function for even n. Second, the zeros of Q_{n-1} are located at $\pm(\lambda_n - r)$, for $r = 1, \dots, \lfloor\frac{n-1}{2}\rfloor$. This means that the function

$$g(t) := \kappa(\varepsilon t)Q_{n-1}(t)\begin{cases} \sin(t\varphi) & n \text{ even} \\ \cos(t\varphi) & n \text{ odd} \end{cases},$$

is even in t and has its zeros at $t = \pm(\lambda_n - r)$ for $r = 1\dots,\lfloor\lambda_n\rfloor$. In addition, we have defined g so that

$$C_{\varepsilon,n}(\varphi) = \sum_{\ell=0}^{\infty} g(\ell+\lambda_n)$$

We want to apply the Poisson summation formula (PSF),

$$\sum_{\mu\in\mathbb{Z}} f(\mu) = \sum_{\nu\in\mathbb{Z}} \hat{f}(2\pi\nu), \quad \hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-i\omega t}dt,$$

which holds for "nice" f, to $f(t) = g(t+\lambda_n)$. Using the evenness of g and what we said about its zeros, we see that the left side of the PSF becomes

$$\sum_{\mu\in\mathbb{Z}} g(\mu+\lambda_n) = 2\sum_{\ell=0}^{\infty} g(\ell+\lambda_n) = 2C_{\varepsilon,n}(\varphi)$$

Employing elementary properties of the Fourier transform, we can show that

$$\hat{f}(\omega) = e^{i\lambda_n\omega}\hat{g}(\omega) = \varepsilon^{-1}e^{i\lambda_n\omega}Q_{n-1}(i\tfrac{d}{d\omega})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon})$$

and so the right side of the PSF is

$$\begin{aligned}\sum_{\nu\in\mathbb{Z}} \hat{f}(2\pi\nu) &= \varepsilon^{-1}\sum_{\nu\in\mathbb{Z}} e^{2\pi\nu i\lambda_n}Q_{n-1}(i\tfrac{d}{d\omega})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon})|_{\omega=2\pi\nu} \\ &= \varepsilon^{-1}\sum_{\nu\in\mathbb{Z}}(-1)^{(n-1)\nu}Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+2\pi\nu}{\varepsilon})\end{aligned}$$

Equating the two sides of the PSF and dividing by 2, we arrive at the following result.

Proposition 4.2 *If κ satisfies (14), then for $n \geq 2$ equation (21) holds with $C_{\varepsilon,n}$ given by*

$$C_{\varepsilon,n}(\varphi) = (2\varepsilon)^{-1}\sum_{\nu\in\mathbb{Z}}(-1)^{(n-1)\nu}Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+2\pi\nu}{\varepsilon}) \tag{25}$$

In addition, for the $n = 1$ case we have

$$K_{\varepsilon,1}(\cos\theta) = (2\pi\varepsilon)^{-1}\sum_{\nu\in\mathbb{Z}}\hat{\kappa}(\tfrac{\theta+2\pi\nu}{\varepsilon}) \tag{26}$$

4.3 Estimates on $C_{\varepsilon,n}$

We need to obtain bounds on the kernels $C_{\varepsilon,n}$ from the previous section. The key to obtaining these bounds is this result.

Lemma 4.3 *Let κ satisfy (14). If $0 \le j \le n-1$ and $0 \le r \le k$ are integers, then, $\frac{d^r}{dt^r}\{t^j\kappa\} \in L^1$ and*

$$|\omega|^r|\hat{\kappa}^{(j)}(\omega)| \le \|\tfrac{d^r}{dt^r}\left\{t^j\kappa\right\}\|_{L^1}.$$

Proof: Since $\kappa \in C^k$, the derivative $\frac{d^r}{dt^r}\left\{t^j\kappa\right\}$ is a linear combination of terms of the form $t^p\kappa^{(q)}$, each of which is bounded by a multiple of $(1+|t|)^{p+q-\alpha}$. This is in L^1 because $\alpha - p - q > \alpha - (n-1) - k > 1$. This allows us to apply standard properties of the Fourier transform to obtain the formula $(-i)^{r+j}\omega^r\hat{\kappa}^{(j)}(\omega) = \widehat{\frac{d^r}{dt^r}\{t^j\kappa\}}$, which immediately implies the inequality. $\square$

Consider the function below,

$$\left(\tfrac{\varphi+\omega}{\varepsilon}\right)^r Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon}) = \sum_{j=0}^{n-1}\varepsilon^{-j}q_{j,n}\left(\tfrac{\varphi+\omega}{\varepsilon}\right)^r\hat{\kappa}^{(j)}(\tfrac{\varphi+\omega}{\varepsilon}),$$

where $Q_{n-1}(z) = \sum_{j=0}^{n-1} q_{j,n}z^j$ is defined in (23). From Lemma 4.3, we have that

$$\begin{aligned}\left|\left(\tfrac{\varphi+\omega}{\varepsilon}\right)^r Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon})\right| &\le \textstyle\sum_{j=0}^{n-1}\varepsilon^{-j}|q_{j,n}|\,\|\tfrac{d^r}{dt^r}\left\{t^j\kappa\right\}\|_{L^1}\\ &\le B_{n,k,\kappa}\varepsilon^{-(n-1)},\end{aligned}$$

where

$$B_{n,k,\kappa} := \left(\sum_{j=0}^{n-1}|q_{n,j}|\right)\max_{j<n,\,r\le k}\|\tfrac{d^r}{dt^r}\left\{t^j\kappa\right\}\|_{L^1} \tag{27}$$

Adding the inequalities we get for $r = 0$ and $r = k$ and manipulating the result, we get that

$$\left|Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon})\right| \le \frac{2B_{n,k,\kappa}\varepsilon^{-(n-1)}}{1+\left|\frac{\varphi+\omega}{\varepsilon}\right|^k}.$$

We can use this inequality in conjunction with the series for $C_{\varepsilon,n}$ in (25) to arrive at the bound,

$$|C_{\varepsilon,n}(\varphi)| \le (2\varepsilon)^{-1}\sum_{\nu\in\mathbb{Z}}\frac{2B_{n,k,\kappa}\varepsilon^{-(n-1)}}{1+\left|\frac{\varphi+2\pi\nu}{\varepsilon}\right|^k} = \sum_{\nu\in\mathbb{Z}}\frac{B_{n,k,\kappa}\varepsilon^{-n}}{1+\left|\frac{\varphi+2\pi\nu}{\varepsilon}\right|^k}, \tag{28}$$

which holds for all $\varphi \in \mathbb{R}$ and $0 < \varepsilon \le 1$. If we restrict φ to be in the interval $[0,\pi]$, then the dominant term in the series on the right comes from

$\nu = 0$. The other terms are each bounded above by $B_{n,k,\kappa}\varepsilon^{k-n}((2|\nu|-1)\pi)^{-k}$. Summing them and then estimating the resulting series by an integral gives us

$$\sum_{\nu\in\mathbb{Z},\nu\neq 0} \frac{B_{n,k,\kappa}\varepsilon^{-n}}{1+\left|\frac{\varphi+2\pi\nu}{\varepsilon}\right|^k} \leq B_{n,k,\kappa}\varepsilon^{k-n}\pi^{-k}\frac{2k-1}{k-1}$$

Multiply top and bottom on the left above by $1+\left(\frac{\varphi}{\varepsilon}\right)^k$ and use $0 \leq \varphi \leq \pi$ and $k \geq 2$ to get

$$\sum_{\nu\in\mathbb{Z},\nu\neq 0} \frac{B_{n,k,\kappa}\varepsilon^{-n}}{1+\left|\frac{\varphi+2\pi\nu}{\varepsilon}\right|^k} \leq \frac{6B_{n,k,\kappa}\varepsilon^{-n}}{1+\left(\frac{\varphi}{\varepsilon}\right)^k}.$$

Combining this bound with that from equation (28) yields the result below.

Proposition 4.4 *Let κ satisfy (14), with $k \geq 2$ and $n \geq 2$. If $0 \leq \varphi \leq \pi$, then the kernel $C_{\varepsilon,n}$ defined in (22) satisfies the bound,*

$$|C_{\varepsilon,n}(\varphi)| \leq \frac{7B_{n,k,\kappa}\varepsilon^{-n}}{1+\left(\frac{\varphi}{\varepsilon}\right)^k}. \tag{29}$$

In addition, for the case $n = 1$ we have

$$|K_{\varepsilon,1}(\cos\theta)| \leq \frac{7B_{1,k,\kappa}\varepsilon^{-1}}{1+\left(\frac{\theta}{\varepsilon}\right)^k}. \tag{30}$$

Proof: Only the second inequality requires comment. The proof we gave works for the $n = 1$ case because it has the form given in equation (26), which is essentially the same as that for the $C_{\varepsilon,n}$'s. □

4.4 Estimates on $K_{\varepsilon,n}$

We now turn to obtaining explicit bounds on the ΨDO kernels $K_{\varepsilon,n}$ similar to the bound on $K_{\varepsilon,1}$ in (30). From the integral representation in (21) and the bound on $C_{\varepsilon,n}$, we have that

$$|K_{\varepsilon,n}(\cos\theta)| \leq \frac{7B_{n,k,\kappa}\gamma_n\varepsilon^{-n}}{(\sin\theta)^{n-2}}\int_\theta^\pi \frac{(\cos\theta-\cos\varphi)^{\frac{n-3}{2}}d\varphi}{1+\left(\frac{\varphi}{\varepsilon}\right)^k}. \tag{31}$$

The two values of θ that present difficulties are $\theta = 0$ and $\theta = \pi$. The form of the inequality above is adequate for the $\theta = 0$ case, but needs to be reformulated for the $\theta = \pi$ case. To do that, we begin by denoting the angle supplementary to an angle α by $\tilde{\alpha}$, so throughout this section we will let $\tilde{\theta} = \pi - \theta$ and $\tilde{\varphi} = \pi - \varphi$. Changing variables in the integral on the right above and using $\sin\tilde{\alpha} = \sin\alpha$ and $\cos\tilde{\alpha} = -\cos\alpha$, we have this reformulation of (31):

$$|K_{\varepsilon,n}(\cos\theta)| \leq \frac{7B_{n,k,\kappa}\gamma_n\varepsilon^{-n}}{(\sin\tilde{\theta})^{n-2}} \int_0^{\tilde{\theta}} \frac{(\cos\tilde{\varphi} - \cos\tilde{\theta})^{\frac{n-3}{2}} d\tilde{\varphi}}{1 + \left(\frac{\pi-\tilde{\varphi}}{\varepsilon}\right)^k}. \tag{32}$$

The next step is to bound both of these integrals. Recall the sum-to-product identity, $\cos\alpha - \cos\beta \equiv 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$, which holds for all α and β. Assuming that $\pi \geq \beta > \alpha \geq \pi/2$ and using the fact that $\frac{\sin t}{t}$ is decreasing for $0 \leq t \leq \pi$, we have that

$$6 < 8\frac{\sin(3\pi/4)}{3\pi/4}\frac{\sin(\pi/4)}{\pi/4} \leq \frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2} = 8\frac{\sin\frac{\alpha+\beta}{2}}{\frac{\alpha+\beta}{2}}\frac{\sin\frac{\beta-\alpha}{2}}{\frac{\beta-\alpha}{2}} \leq 8,$$

and so

$$\left(\frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2}\right)^{\frac{n-3}{2}} \leq 2^{\frac{3(n-3)}{2}} \times \begin{cases} \frac{2}{\sqrt{3}} & n = 2 \\ 1 & n \geq 3 \end{cases} \leq 2 \cdot 2^{\frac{3(n-3)}{2}} \tag{33}$$

Assume that $\varepsilon \leq \theta \leq \pi/2$, and apply (33) to (31) to get the chain of inequalities below.

$$\begin{aligned} |K_{\varepsilon,n}(\cos\theta)| &\leq \frac{14 \cdot 2^{\frac{3(n-3)}{2}} B_{n,k,\kappa}\gamma_n\varepsilon^{-n}}{(\sin\theta)^{n-2}} \int_\theta^\pi \frac{(\theta^2 - \varphi^2)^{\frac{n-3}{2}} d\varphi}{1 + \left(\frac{\varphi}{\varepsilon}\right)^k} \\ &\leq 14 \cdot 2^{\frac{3(n-3)}{2}} B_{n,k,\kappa}\gamma_n\varepsilon^{-n} \left(\frac{\theta}{\sin\theta}\right)^{n-2} \int_1^{\pi/\theta} \frac{(t^2-1)^{\frac{n-3}{2}} dt}{1 + (\theta/\varepsilon)^k t^k} \\ &\leq \frac{14 \cdot 2^{\frac{3(n-3)}{2}} B_{n,k,\kappa}\gamma_n\varepsilon^{-n}(\pi/2)^{n-2}}{\left(\frac{\theta}{\varepsilon}\right)^k} \int_1^\infty \frac{(t^2-1)^{\frac{n-3}{2}} dt}{t^k} \end{aligned}$$

Use $2(\theta/\varepsilon)^k \geq 1 + (\theta/\varepsilon)^k$, change variables of integration from $t \to 1/t$, and note that because $k \geq \max\{2, n-1\} \geq n-1$, the resulting integral on the right is bounded above by $\int_0^1 (1-t^2)^{\frac{n-3}{2}} dt = 2^{n-3}\Gamma(\lambda_n)^2/\Gamma(2\lambda_n)$ [23, p. 255]. After simplifying, we arrive at this estimate:

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{14 \cdot 2^{\frac{3(n-3)}{2}} \pi^{n-2} B_{n,k,\kappa} \gamma_n \Gamma(\lambda_n)^2/\Gamma(2\lambda_n)}{1+(\frac{\theta}{\varepsilon})^k} \varepsilon^{-n}.$$

The messy quantity in the numerator can be simplified considerably. This requires employing the definition of γ_n in (20), the formula for ω_n, the familiar properties of the Γ-function, along with the less familiar duplication formula [23, p. 240], $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$, and manipulating the expressions involved. The result is that

$$2^{\frac{3(n-3)}{2}} \pi^{n-2} \gamma_n \Gamma(\lambda_n)^2/\Gamma(2\lambda_n) = \frac{\omega_{n-1}}{4\sqrt{\pi}}\,,\ \omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

Thus we can rewrite the previous inequality, which holds for $\varepsilon \le \theta \le \pi/2$, as

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{7\omega_{n-1} B_{n,k,\kappa}}{2\sqrt{\pi}(1+(\frac{\theta}{\varepsilon})^k)} \varepsilon^{-n}.$$

If we now apply (33) to (32), with $0 \le \tilde{\theta} \le \pi/2$ (or, equivalently, $\pi/2 \le \theta \le \pi$), then

$$\begin{aligned} |K_{\varepsilon,n}(\cos\theta)| &\le \frac{7 B_{n,k,\kappa} \gamma_n \varepsilon^{-n}}{(\sin\tilde{\theta})^{n-2}} \int_0^{\tilde{\theta}} \frac{(\tilde{\theta}^2 - \tilde{\varphi}^2)^{\frac{n-3}{2}} d\tilde{\varphi}}{1+\left(\frac{\pi-\tilde{\varphi}}{\varepsilon}\right)^k} \\ &\le \frac{14 \cdot 2^{\frac{3(n-3)}{2}} B_{n,k,\kappa} \gamma_n \varepsilon^{-n}}{(1+\left(\frac{\theta}{\varepsilon}\right)^k)(\sin\tilde{\theta})^{n-2}} \int_0^{\tilde{\theta}} (\tilde{\theta}^2 - \tilde{\varphi}^2)^{\frac{n-3}{2}} d\tilde{\varphi} \end{aligned}$$

Carrying out manipulations analogous to those for the previous case, we obtain

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{7\omega_{n-1} B_{n,k,\kappa}}{4\sqrt{\pi}(1+(\frac{\theta}{\varepsilon})^k)} \varepsilon^{-n}.$$

The final case concerns $0 \le \theta \le \varepsilon$. For such θ, we have, from the L^∞ bound in (18), that

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{3\,C_\kappa}{\omega_n} \varepsilon^{-n} \le \frac{3\,C_\kappa}{\omega_n} \left(\frac{1+(\frac{\theta}{\varepsilon})^k}{1+(\frac{\theta}{\varepsilon})^k} \right) \varepsilon^{-n} \le \frac{6\,C_\kappa}{\omega_n(1+(\frac{\theta}{\varepsilon})^k)} \varepsilon^{-n},$$

which, when combined with equation (26) for $n = 1$, gives us the main result of this section.

Theorem 4.5 *Let κ satisfy (14), with $k \geq \max\{2, n-1\}$. If $0 \leq \theta \leq \pi$, then the kernel $K_{\varepsilon,n}$ satisfies the bound,*

$$|K_{\varepsilon,n}(\cos\theta)| \leq \frac{\beta_{n,k,\kappa}}{1+(\frac{\theta}{\varepsilon})^k}\varepsilon^{-n}, \tag{34}$$

where

$$\beta_{n,k,\kappa} := \begin{cases} 7B_{1,k,\kappa} & \text{if } n=1, \\ \max\left\{\frac{6\,C_\kappa}{\omega_n}, \frac{7\omega_{n-1}B_{n,k,\kappa}}{2\sqrt{\pi}}\right\} & \text{if } n \geq 2. \end{cases} \tag{35}$$

We conclude this section with an application of this theorem to obtaining a bound on the L^1 norm of $K_{\varepsilon,n}(\xi\cdot\eta)$, with η fixed. By the Funk-Hecke formula [16, Theorem 6], this norm is given by

$$\int_{\mathbb{S}^n} |K_{\varepsilon,n}(\xi\cdot\eta)|d\mu(\xi) = \omega_{n-1}\int_0^\pi |K_{\varepsilon,n}(\cos\theta)|\sin^{n-1}\theta\, d\theta,$$

which is of course independent of η. For that reason we will drop any reference to η and denote the norm by $\|K_{\varepsilon,n}\|_1$. Here is the bound we want.

Corollary 4.6 *Let $n \geq 1$. If κ satisfies (14), with $k > \max\{2, n\}$, then*

$$\|K_{\varepsilon,n}\|_1 \leq 2\omega_{n-1}\beta_{n,k,\kappa}$$

Proof: By Theorem 4.5 and the remarks above, we have

$$\|K_{\varepsilon,n}\|_1 \leq \omega_{n-1}\int_0^\pi |K_{\varepsilon,n}(\cos\theta)|\sin^{n-1}\theta\, d\theta \leq \beta_{n,k,\kappa}\omega_{n-1}\varepsilon^{-n}\int_0^\pi \frac{\sin^{n-1}\theta\, d\theta}{1+(\frac{\theta}{\varepsilon})^k}.$$

The integral on the right above can be estimated this way:

$$\begin{aligned} \int_0^\pi \frac{\sin^{n-1}\theta\, d\theta}{1+(\frac{\theta}{\varepsilon})^k} &< \varepsilon^n \int_0^{\pi/\varepsilon} \frac{t^{n-1}dt}{1+t^k} \\ &< \varepsilon^n \left\{\int_0^1 t^{n-1}dt + \int_1^\infty \frac{dt}{t^{k+1-n}}\right\} \\ &< \frac{k\varepsilon^n}{n(k-n)} \leq 2\varepsilon^n. \end{aligned}$$

The corollary then follows immediately from the estimate. □

4.5 Operator properties of $\mathsf{K}_{\varepsilon,n}$

We now turn to the operator properties of $\mathsf{K}_{\varepsilon,n}$. Our first result is calculating the norm of the map of $\mathsf{K}_{\varepsilon,n} : L^p \to L^q$. After that we will prove a lemma showing that for certain κ the operator $\mathsf{K}_{\varepsilon,n}$ will be a reproducing reproducing kernel on Π_L. We will close the section with a result showing that for such κ and $\varepsilon \leq (L+\lambda_n)^{-1}$ then the norm of $f - \mathsf{K}_{\varepsilon,n} f$ is comparable to the distance from f to Π_L, in appropriate norms.

Theorem 4.7 *If κ satisfies (14), with $k > \max\{2, n\}$, then, for all $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, the operator $\mathsf{K}_{\varepsilon,n}\colon L^p(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$ is bounded and its norm satisfies*

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \leq 2\omega_{n-1}\beta_{n,k,\kappa}(4\omega_{n-1}\varepsilon^n)^{-(\frac{1}{p}-\frac{1}{q})_+}$$

where $\beta_{n,k,\kappa}$ is defined in (35) and $(x)_+ = x$ for $x > 0$ and $(x)_+ = 0$ otherwise.

Proof: The operators are all of the form $K_{\varepsilon,n} * f$ and so, for the (p,q) pairs $(1,1)$, (∞,∞), $(\infty,1)$, all satisfy $\|K_{\varepsilon,n} * f\|_q \leq \|K_{\varepsilon,n}\|_1 \|f\|_p$. By the Riesz-Thorin theorem [25, p. 95] and Corollary 4.6, we then have for $1 \leq q \leq p \leq \infty$

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \leq \|K_{\varepsilon,n}\|_1 \leq 2\omega_{n-1}\beta_{n,k,\kappa}\,.$$

For the pair $(1,\infty)$, we have $\|K_{\varepsilon,n} * f\|_\infty \leq \|K_{\varepsilon,n}\|_\infty \|f\|_1$. By (18) and (35), we have $\|K_{\varepsilon,n}\|_\infty \leq \frac{1}{2}\beta_{n,k,\kappa}\varepsilon^{-n}$, and so $\|K_{\varepsilon,n} * f\|_\infty \leq \frac{1}{2}\beta_{n,k,\kappa}\varepsilon^{-n}\|f\|_1$. Apply Riesz-Thorin to the pairs (p,q), where $\frac{1}{p} = (1-t)\alpha + t$ and $\frac{1}{q} = (1-t)\alpha$, where $0 < t < 1$ and $0 < \alpha < 1$, $(\frac{1}{\alpha}, \frac{1}{\alpha})$ and $(1,\infty)$ to get

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \leq (2\beta_{n,k,\kappa}\omega_{n-1})^{1-t}(\frac{1}{2}\beta_{n,k,\kappa}\varepsilon^{-n})^t = 2\omega_{n-1}\beta_{n,k,\kappa}(4\omega_{n-1}\varepsilon^n)^{-t}$$

Since $\frac{1}{p} = (1-t)\alpha + t = \frac{1}{q} + t$, $t = \frac{1}{p} - \frac{1}{q}$. Thus, for $q > p$, we have

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \leq 2\omega_{n-1}\beta_{n,k,\kappa}(4\omega_{n-1}\varepsilon^n)^{-(\frac{1}{p}-\frac{1}{q})}$$

Putting the last inequality together with the one for $q \leq p$ then yields the result. □

Lemma 4.8 *Let $L > 0$ be an integer and let $0 < \varepsilon \leq (L+\lambda_n)^{-1}$. If κ satisfies (14), with $k \geq \max\{2, n-1\}$, and if $\kappa(t) \equiv 1$ on $[0,1]$, then $K_{\varepsilon,n}(\xi \cdot \eta)$ is a reproducing kernel on Π_L, the space of spherical harmonics having degree at most L.*

Proof: Let $S(\eta) \in \Pi_L$. Then, for $\ell \leq L$, we have that the projections $\mathsf{P}_\ell S(\eta) = \sum_{m=1}^{d_\ell^n} \langle S, Y_{\ell,m}\rangle Y_{\ell,m}(\eta)$ and for $\ell > L$, $\mathsf{P}_\ell S = 0$. From the definition of $\mathsf{K}_{\varepsilon,n}$ we see that

$$\mathsf{K}_{\varepsilon,n} S(\eta) = \sum_{\ell=0}^{L} \kappa(\varepsilon(\ell+\lambda_n)) \sum_{m=1}^{d_\ell^n} \langle S, Y_{\ell,m}\rangle Y_{\ell,m}(\eta).$$

However, $\varepsilon(\ell+\lambda_n) \leq \varepsilon(L+\lambda_n) \leq 1$, so $\kappa(\varepsilon(\ell+\lambda_n)) = 1$ for all such ℓ. It follows that

$$\mathsf{K}_{\varepsilon,n} S(\eta) = \sum_{\ell=0}^{L} \sum_{m=1}^{d_\ell^n} \langle S, Y_{\ell,m}\rangle Y_{\ell,m}(\eta) = S(\eta),$$

and so $K_{\varepsilon,n}$ is a reproducing kernel on the space of such polynomials. □

Remark 4.9 Let $L > 0$ be an integer. If we choose ε so that $L = \lfloor \varepsilon^{-1} - \lambda_n \rfloor$, then by combining the previous theorem and lemma we get a familiar result about harmonic polynomials: *If $S \in \Pi_L$, then $\|S\|_q \leq C_n L^{n(\frac{1}{p}-\frac{1}{q})_+}\|S\|_p$.*

We let $E_L(f)_p$ denote the distance of $f \in L^p(\mathbb{S}^n)$ to Π_L, i.e.,

$$E_L(f)_p := \inf_{S\in\Pi_L} \|f - S\|_p \,. \tag{36}$$

Corollary 4.10 *Let κ satisfy(14), with $k > \max\{2, n\}$, and in addition suppose $\kappa(t) \equiv 1$ on $[0, 1]$. If $f \in L^p(\mathbb{S}^n)$, $1 \leq p \leq \infty$, and $\varepsilon \leq (L + \lambda_n)^{-1}$, then*

$$\|f - K_{\varepsilon,n} * f\|_p \leq (1 + 2\omega_{n-1}\beta_{n,k,\kappa})E_L(f)_p \,. \tag{37}$$

In addition, for $1 \leq p < \infty$ or, if $p = \infty$, for $f \in C(\mathbb{S}^n)$, we have

$$\lim_{\varepsilon\downarrow 0} K_{\varepsilon,n} * f = f \,. \tag{38}$$

Proof: By Lemma 4.8, the kernel $K_{\varepsilon,n}$ reproduces harmonic polynomials in Π_L. Consequently, if $S \in \Pi_L$, then $K_{\varepsilon,n} * S = S$, and

$$f - K_{\varepsilon,n} * f = f - S + K_{\varepsilon,n} * S - K_{\varepsilon,n} * f = (I + \mathsf{K}_\varepsilon)(f - S).$$

By Theorem 4.7 and this equation, we have that

$$\|f - K_{\varepsilon,n} * f\|_p \leq (1 + \|\mathsf{K}_\varepsilon\|_{p,p})\|f - S\|_p \leq (1 + 2\omega_{n-1}\beta_{n,k,\kappa})\|f - S\|_p \,.$$

Taking the infimum over all $S \in \Pi_L$ yields (37). The limit in (38) follows from (37) together with the fact that the spherical harmonics are dense in L^p for $1 \leq p < \infty$ and in $C(\mathbb{S}^n)$ in the usual L^∞ norm [21, § IV.2]. □

The estimate in (37) is useful for obtaining rates of approximation, simply because rates of approximation by spherical harmonics are well known for many classes of functions. See the discussion following Theorem 3.1.

5 Applications to Quadrature on $\mathbb{S}^n$

5.1 Marcinkiewicz-Zygmund Inequalities

In this section we wish to give inequalities resembling the Marcinkiewicz-Zygmund ones for trigonometric polynomials. These inequalities provide equivalences between norms defined through integrals and discrete norms stemming from sampled points and certain weights. Here, instead of polynomials, we will work with functions of the form $K_{\varepsilon,n} * f$ for $f \in L^1(\mathbb{S}^n)$. Out of this will come the inequalities derived in [13], with constants that can be evaluated and via a proof that avoids the theory of doubling weights and delayed means.

The place to start is with a decomposition of the sphere into a finite number of non-overlapping, connected regions R_ξ, each containing an interior point ξ that will serve for function evaluations as well as labeling. For example, in the decomposition described in section 3.2, the centers in X play the role of the special points, and the Voronoi region containing ξ plays the role of R_ξ. Indeed, we will let X be the set of the ξ's used for labels and $\mathcal{X} = \{R_\xi \subset \mathbb{S}^n \,|\, \xi \in X\}$. In addition, let $\|\mathcal{X}\| = \max_{\xi \in X}\{\mathrm{diam}(R_\xi)\}$. The quantity that we wish to estimate first is the magnitude of the difference between the continuous and discrete norms for $g = K_{\varepsilon,n} * f$,

$$E_{\mathcal{X}} := \big|\|g\|_1 - \sum_{\xi \in X} |g(\xi)|\mu(R_\xi)\big|,$$

where we assume that $f \in L^1(\mathbb{S}^n)$. It is straightforward to show that

$$E_X \leq \sum_{\xi \in X} \int_{R_\xi} |g(\eta) - g(\xi)| d\mu(\eta) \leq \sup_{\zeta \in \mathbb{S}^n} F_{\varepsilon,\mathcal{X}}(\zeta) \|f\|_1 \,,$$

where $F_{\varepsilon,\mathcal{X}}(\zeta) := \sum_{\xi\in X}\int_{R_\xi}\left|K_{\varepsilon,n}(\eta\cdot\zeta)-K_{\varepsilon,n}(\xi\cdot\zeta)\right|d\mu(\eta)$, which is the quantity we need to estimate.

Choose ζ to be the north pole of $\mathbb{S}^n$ and let θ be the colatitude in spherical coordinates; set $\theta_\eta = \cos^{-1}(\eta\cdot\zeta)$ and $\theta_\xi = \cos^{-1}(\xi\cdot\zeta)$. Denote by θ_ξ^+ and θ_ξ^-, respectively, the high and low values for θ over R_ξ. Using equation (16) for the derivative of $K_{\varepsilon,n}$, we can write $F_{\varepsilon,\mathcal{X}}(\zeta)$ as

$$\begin{aligned} F_{\varepsilon,\mathcal{X}}(\zeta) &= 2\pi\sum_{\xi\in X}\int_{R_\xi}\left|\int_{\theta_\zeta}^{\theta_\eta} K_{\varepsilon,n+2}(\cos t)\sin t dt\right| d\mu(\eta) \\ &\leq 2\pi\sum_{\xi\in X}\mu(R_\xi)\int_{\theta_\xi^-}^{\theta_\xi^+} |K_{\varepsilon,n+2}(\cos t)|\sin t\, dt. \end{aligned}$$

Divide $\mathbb{S}^n$ into $M = \lfloor \pi/\|\mathcal{X}\|\rfloor$ equal bands in which $(m-1)\pi/M \leq \theta \leq m\pi/M$, $m = 1,\dots,M$. To avoid trivial situations and simplify later inequalities, we will assume that $M \geq 3$. Call these bands $B_1,\dots,B_M$. Each R_ξ can have non-trivial intersection with at most two adjacent bands, because $\operatorname{diam}(R_\xi) \leq \|\mathcal{X}\| \leq \pi/M$. So if $R_\xi \subset B_m\cup B_{m+1}$, then $(m-1)\pi/M \leq \theta_\xi^- \leq \theta_\xi^+ \leq (m+1)\pi/M$. In addition, the sum of the contributions from all $R_\xi \subset B_m \cup B_{m+1}$ is bounded above by the quantity,

$$2\pi\mu(B_m\cup B_{m+1})\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} |K_{\varepsilon,n+2}(\cos t)|\sin t\, dt\,.$$

It follows that

$$\begin{aligned} F_{\varepsilon,\mathcal{X}}(\zeta) &\leq 2\pi\sum_{m=1}^{M-1}\mu(B_m\cup B_{m+1})\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} |K_{\varepsilon,n+2}(\cos t)|\sin t\, dt \\ &\leq 2\pi\omega_{n-1}\sum_{m=1}^{M-1}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}\sin^{n-1} t\, dt\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} |K_{\varepsilon,n+2}(\cos t)|\sin t\, dt \end{aligned}$$

Assume that we have chosen $k \geq n+2 > \max\{2, n+1\}$. By the bound on $|K_{\varepsilon,n+2}(\cos t)|$ in Theorem 4.5 and because $\sin t \leq t$ on $[0,\pi]$, we have that

$$F_{\varepsilon,\mathcal{X}}(\zeta) \leq 2\pi\omega_{n-1}\beta_{n+2,k,\kappa}\varepsilon^{-n-2}\sum_{m=1}^{M-1}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} t^{n-1}dt\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}\frac{t}{1+(\frac{t}{\varepsilon})^k}dt$$

The mean value theorem for integrals implies that for $2 \le m \le M-1$,

$$\begin{aligned}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} t^{n-1}dt \int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} \frac{t}{1+(\frac{t}{\varepsilon})^k}dt &\le \frac{2\pi}{M}\left(\frac{m+1}{m-1}\right)^{n-1} \int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} \frac{t^n}{1+(\frac{t}{\varepsilon})^k}dt \\ &\le 3^{n-1}\frac{2\pi}{M}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} \frac{t^n}{1+(\frac{t}{\varepsilon})^k}dt\end{aligned}$$

Summing both sides from $m = 2$ to $M-1$, taking account of intervals appearing twice in the sum, and doing some obvious manipulations, we obtain

$$\varepsilon^{-n-2}\sum_{m=2}^{M-1}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} t^{n-1}dt \int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi} \frac{t}{1+(\frac{t}{\varepsilon})^k}dt \le \frac{4\cdot 3^{n-1}\pi}{M\varepsilon}\underbrace{\int_{\frac{\pi}{M\varepsilon}}^{\frac{\pi}{\varepsilon}} \frac{t^n dt}{1+t^k}}_{\le \frac{1}{n+1}+\frac{1}{k-n-1}\le\frac{3}{2}}$$

We have now come to the inequality,

$$\begin{aligned}F_{\varepsilon,\mathcal{X}}(\zeta) &\le 2\pi\omega_{n-1}\beta_{n+2,k,\kappa}\left\{n^{-1}\left(\frac{2\pi}{M\varepsilon}\right)^n \underbrace{\int_0^{\frac{2\pi}{M\varepsilon}}\frac{tdt}{1+t^k}}_{\le\frac{1}{2}+\frac{1}{k}\le 1} + \frac{2\cdot 3^n\pi}{M\varepsilon}\right\} \\ &\le 2\pi\omega_{n-1}\beta_{n+2,k,\kappa}\frac{2\pi}{M\varepsilon}\left\{n^{-1}\left(\frac{2\pi}{M\varepsilon}\right)^{n-1} + 3^n\right\}\end{aligned}$$

To finish up, we want to put our inequalities in terms of the ratio $\|\mathcal{X}\|/\varepsilon$. Since we have assumed that $M \ge 3$, we have that $\pi/M \le \frac{4}{3}\|\mathcal{X}\|$. Using this in the previous inequality and simplifying, we arrive at this:

$$\begin{aligned}F_{\varepsilon,\mathcal{X}}(\zeta) &\le 2\pi\cdot 3^{n-1}\omega_{n-1}\beta_{n+2,k,\kappa}\frac{8\|\mathcal{X}\|}{9\varepsilon}\left\{1+(3n)^{-1}\left(\frac{8\|\mathcal{X}\|}{9\varepsilon}\right)^{n-1}\right\} \\ &< 2\pi\cdot 3^{n-1}\omega_{n-1}\beta_{n+2,k,\kappa}\frac{\|\mathcal{X}\|}{\varepsilon}\left\{1+(3n)^{-1}\left(\frac{\|\mathcal{X}\|}{\varepsilon}\right)^{n-1}\right\}.\end{aligned}$$

We remark that if $\|\mathcal{X}\| \le \varepsilon \le 1$, then the assumption that $M \ge 3$ is automatically fulfilled. In addition, the right side of the inequality above is independent of ζ, so it holds for the left replaced by $\sup_{\zeta\in\mathbb{S}^n} F_{\varepsilon,\mathcal{X}}(\zeta)$. Finally, the inequality itself simplifies considerably. We collect all these observations in the result below.

Proposition 5.1 *Let κ satisfy (14), with $k \geq n+2$, and, for $f \in L^1(\mathbb{S}^n)$, let $g = K_{\varepsilon,n} * f$. If $\mathcal{X}$ is the decomposition of $\mathbb{S}^n$ described above and if $\|\mathcal{X}\| \leq \varepsilon \leq 1$, then*

$$\left| \|g\|_1 - \sum_{\xi \in X} |g(\xi)| \mu(R_\xi) \right| \leq 3^n \pi \omega_{n-1} \beta_{n+2,k,\kappa} \frac{\|\mathcal{X}\|}{\varepsilon} \|f\|_1 \,. \tag{39}$$

This result leads immediately to a version of the Marcinkiewicz-Zygmund inequities for $\mathbb{S}^n$.

Theorem 5.2 ([13, Theorem 3.1]) *Let $L > 0$ be an integer and let $\delta \in (0,1)$. If $\mathcal{X}$ is the decomposition of $\mathbb{S}^n$ described above and $S \in \Pi_L$, then there exists a constant $s_n \geq 1$, which depends only on n, such that*

$$(1-\delta)\|S\|_1 \leq \sum_{\xi \in X} |S(\xi)| \mu(R_\xi) \leq (1+\delta)\|S\|_1 \tag{40}$$

holds whenever $\|\mathcal{X}\| \leq \delta s_n^{-1} (L+\lambda_n)^{-1}$.

Proof: Let κ satisfy (14), with $k \geq n+2$. In addition, require $\kappa(t) \equiv 1$ for $t \in [0,1]$. Choose $\varepsilon = (L+\lambda_n)^{-1}$. By Lemma 4.8, $S = K_{\varepsilon,n} * S$, and so if we take $f = S$ and $\|\mathcal{X}\| \leq \varepsilon = (L+\lambda_n)^{-1} \leq 1$ in Proposition 5.1, then $g = K_{\varepsilon,n} * S = S$ there. Manipulating the resulting expression in (39) then gives us

$$\tilde{s}_n := \sup \frac{\left| \|S\|_1 - \sum_{\xi \in X} |S(\xi)| \mu(R_\xi) \right|}{(L+\lambda_n)\|\mathcal{X}\| \|S\|_1} \leq 3^n \pi \omega_{n-1} \beta_{n+2,k,\kappa} \,,$$

where the supremum is over all $\mathcal{X}$ and $L > 0$ such that $\|\mathcal{X}\| \leq (L+\lambda_n)^{-1}$ and clearly depends only on n. Now, let

$$s_n := \max\{1, \tilde{s}_n\} \leq \max\{1, 3^n \pi \omega_{n-1} \beta_{n+2,k,\kappa}\} \tag{41}$$

If we further restrict $\|\mathcal{X}\|$ so that $\|\mathcal{X}\| \leq \delta s_n^{-1}(L+\lambda_n)^{-1}$ then (40) follows easily. □

We now define an important map associated with Π_L and the decomposition $\mathcal{X}$ and the corresponding finite set X. Let $|X|$ be the cardinality of X. We define the sampling map, $T_X : \Pi_L \to \mathbb{R}^{|X|}$, by

$$T_X S := (S(\xi))_{\xi \in X}. \tag{42}$$

From Theorem 5.2, it follows that if $\|\mathcal{X}\| \leq \delta s_n^{-1}(L+\lambda_n)^{-1}$ holds and if $T_X S = 0$, we have that $\|S\|_1 = 0$ and, hence, $S \equiv 0$. The sampling map, which is linear, is therefore injective. We state this formally below.

Corollary 5.3 *Under the conditions in Theorem 5.2, the sampling map T_X injective.*

5.2 Estimates on Quadrature Weights for $\mathbb{S}^n$

Throughout the discussion below, we will assume that the conditions of Theorem 5.2 hold. Consequently, the inequality (40) holds and T_X is injective; moreover, if we let the subspace $V_L = T_X \Pi_L \subset \mathbb{R}^{|X|}$, then $T_X^{-1} : V_L \to \Pi_L$ is a linear map. Also, we will let $S_X = (S(\xi))_{\xi \in X}$.

Since our interest here is in weights for quadrature, we start with the linear functional $\Phi : \Pi_L \to \mathbb{R}$ given by

$$\Phi(S) := \int_{\mathbb{S}^n} S(\eta) d\mu(\eta), \ S \in \Pi_L \, .$$

Let $\Phi_X(S_X) = \Phi(T_X^{-1}(S_X)) = \Phi(S)$. If $S_X \geq 0$, then $|S(\xi)| = S(\xi)$ for $\xi \in X$, and so from (40) we have that

$$\Big|\Phi(S) - \sum_{\xi \in X} S(\xi)\mu(R_\xi)\Big| \leq \Big|\|S\|_1 - \sum_{\xi \in X} S(\xi)\mu(R_\xi)\Big| \leq \frac{\delta}{1-\delta} \sum_{\xi \in X} S(\xi)\mu(R_\xi),$$

provided only that $\|\mathcal{X}\| \leq \delta s_n^{-1}(L+\lambda_n)^{-1}$. For any $\delta < \frac{1}{2}$, this implies that

$$\frac{1-2\delta}{1-\delta} \sum_{\xi \in X} S(\xi)\mu(R_\xi) \leq \Phi(S) \leq \frac{1}{1-\delta} \sum_{\xi \in X} S(\xi)\mu(R_\xi)$$

From this, we see that the linear functional

$$\Psi_X(S_X) := \Phi_X(S_X) - \frac{1-2\delta}{1-\delta} \sum_{\xi \in X} S(\xi)\mu(R_\xi) \tag{43}$$

is positive on the cone $0 \leq S_X \in V_L$, which itself is contained in the positive cone of $\mathbb{R}^{|X|}$.

There are two facts we will take account of. The first is that the positive cone of V_L is contained in the positive cone of $\mathbb{R}^{|X|}$. The second is that the vector $(1)_{\xi\in X}$, which is in both cones, is an interior point of the positive cone of $\mathbb{R}^{|X|}$. By the Krein-Rutman Theorem [8], there exists a positive linear functional $\widetilde{\Psi}_X$ that extends Ψ_X to all $\mathbb{R}^{|X|}$. Consequently, there exist weights $\alpha_\xi \geq 0$ such that $\widetilde{\Psi}_X(x) = \sum_{\xi\in X} \alpha_\xi x_\xi$. Using this and $\Phi_X(S_X) = \Phi(S)$ in equation (43), we obtain

$$\Phi(S) = \sum_{\xi\in X} c_\xi S(\xi),\ c_\xi := a_\xi + \frac{1-2\delta}{1-\delta}\mu(R_\xi),\ a_\xi \geq 0. \tag{44}$$

This is of a course a positive-weight quadrature formula on $\mathbb{S}^n$, with weights bounded below by $\frac{1-2\delta}{1-\delta}\mu(R_\xi)$.

We want to get upper bounds as well. To do that, we let $L' = \lfloor \frac{L}{2} \rfloor$ and fix $\xi_0 \in X$. If $S \in \Pi_{L'}$, then S^2 is in Π_L. The quadrature formula (44) then implies that

$$\|S\|_2^2 = \Phi(S^2) = \sum_{\xi\in X} c_\xi (S(\xi))^2 \geq c_{\xi_0}(S(\xi_0))^2.$$

Choose $S(\eta) = \sum_{\ell=0}^{L'} \sum_{m=1}^{d_\ell^n} Y_{\ell,m}(\eta)\overline{Y_{\ell,m}(\xi_0)} = \sum_{\ell=0}^{L'} \frac{\ell+\lambda_n}{\omega_n\lambda_n} P_\ell^{(\lambda_n)}(\xi_0\cdot\eta)$, which is real valued. The orthogonality of the $Y_{\ell,m}$'s implies that

$$\|S\|_2^2 = \sum_{\ell=0}^{L'} \sum_{m=1}^{d_\ell^n} Y_{\ell,m}(\xi_0)\overline{Y_{\ell,m}(\xi_0)} = S(\xi_0).$$

From the previous inequality, it follows that

$$c_{\xi_0} \leq \frac{\|S\|_2^2}{S(\xi_0)^2} = \frac{1}{S(\xi_0)} = \frac{1}{\sum_{\ell=0}^{L'} \frac{\ell+\lambda_n}{\omega_n\lambda_n} P_\ell^{(\lambda_n)}(1)}.$$

From (17) and (7), we have

$$\begin{aligned}\sum_{\ell=0}^{L'} \frac{\ell+\lambda_n}{\omega_n\lambda_n} P_\ell^{(\lambda_n)}(1) &= \sum_{\ell=0}^{L'} \frac{\ell+\lambda_n}{\omega_n\lambda_n}\binom{\ell+n-2}{\ell} \\ &= \sum_{\ell=0}^{L'} \frac{d_\ell^n}{\omega_n} = \frac{\dim \Pi_{L'}}{\omega_n}.\end{aligned}$$

Because $\dim \Pi_{L'} = d_{L'}^{n+1}$ [16, p. 4], we finally arrive at the upper bound

$$c_{\xi_0} \leq \frac{\omega_n}{d_{L'}^{n+1}},\ L' := \lfloor L/2 \rfloor.$$

We summarize these results below.

Theorem 5.4 *Adopt the notation of Theorem 5.2. In particular, s_n is given by (41) and depends only on n. For any $0 < \delta < \frac{1}{2}$ and any integer $L > 0$, if $\|\mathcal{X}\| \le \delta s_n^{-1}(L+\lambda_n)^{-1}$, then there exist positive weights c_ξ, $\xi \in X$, such that the quadrature formula,*

$$\int_{\mathbb{S}^n} f(\eta) d\mu(\eta) \doteq \sum_{\xi \in X} c_\xi f(\xi), \tag{45}$$

is exact for spherical harmonics in Π_L. Also, the weights satisfy the bounds

$$\frac{1-2\delta}{1-\delta}\mu(R_\xi) \le c_\xi \le \frac{\omega_n}{d_{L'}^{n+1}}, \; L' = \lfloor L/2 \rfloor. \tag{46}$$

The theorem just proved starts with L and puts conditions on the decomposition $\mathcal{X}$. It's useful to turn this around, starting with $\mathcal{X}$ and putting conditions on L. For a decomposition $\mathcal{X}$, the largest L for which (45) holds is $L := \lfloor \delta s_n^{-1} \|\mathcal{X}\|^{-1} - \lambda_n \rfloor$. Now from (46) and (7), since $L' = \lfloor L/2 \rfloor$, we have

$$\frac{1-2\delta}{1-\delta}\mu(R_\xi) \le c_\xi \le \frac{\omega_n}{d_{L'}^{n+1}} = \mathcal{O}\{L^{-n}\} = \mathcal{O}\{\delta^{-n}\|\mathcal{X}\|^n\}, \tag{47}$$

where the constants involved depend on the dimension n for the sphere.

This has one further application. Fix δ and consider the situation described in section 3.2, where X is a ρ-uniform set of points with separation radius q_X and mesh norm $h = h_X$, and $\mathcal{X}$ is the corresponding Voronoi decomposition. For $\xi \in X$, the corresponding R_ξ contains a spherical cap of radius $q = q_X$, so we have $\mu(R_\xi) \ge Cq^n$. On the other hand, $\|\mathcal{X}\| \le 2h$, because the maximum distance from a point on $\mathbb{S}^n$ to a $\xi \in X$ is h. Since $q \ge \rho^{-1}h$, we have $\mu(R_\xi) \ge Ch^n$. Putting this together with the remarks above, we have $c_\xi = \mathcal{O}\{h^n\}$.

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