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CONVERGENCE OF MULTIGRID ALGORITHMS FOR INTERIOR PENALTY METHODS

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ABSTRACT. V-cycle, F-cycle and W-cycle multigrid algorithms for interior penalty methods for second order elliptic boundary value problems are studied in this paper. It is shown that these algorithms converge uniformly with respect to all grid levels if the number of smoothing steps is sufficiently large, and that the contraction numbers decrease as the number of smoothing steps increases, at a rate determined by the elliptic regularity of the problem.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded open polygonal domain and $f \in L_2(\Omega)$. For simplicity we consider the following model variational problem for the Poisson equation with homogeneous Dirichlet boundary condition: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx \qquad \forall\, v \in H_0^1(\Omega). \tag{1.1}$$

Here and throughout the paper we use the standard notation [1, 18, 17] for L_2-based Sobolev spaces.

Note that there exists a number $\alpha \in (1/2, 1]$ such that [22, 20, 27]

$$\|u\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|\phi\|_{H^{-1+\alpha}(\Omega)} \tag{1.2}$$

whenever $u \in H_0^1(\Omega)$ satisfies

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \phi(v) \qquad \forall\, v \in H_0^1(\Omega). \tag{1.3}$$

From here on we use C (with or without subscript) to denote generic positive constants that can take different values at different occurrences. We shall refer to α as the index of elliptic regularity. In particular, the regularity estimate implies that the solution u of (1.1) belongs to $H^{1+\alpha}(\Omega)$ and $\|u\|\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|f\|_{L_2(\Omega)}$.

Let $\mathcal{T}_h$ be a (simplicial or quadrilateral) triangulation of Ω and V_h be a finite dimensional vector space of piecewise polynomial functions. The interior penalty approach for (1.1) is based on the observation that, using integration by parts, the solution u of (1.1) can be shown to satisfy

$$\mathcal{A}_h(u, v) = \int_\Omega f v\, dx \quad \forall\, v \in V_h, \tag{1.4}$$

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where

$$\mathcal{A}_h(w,v) = \sum_{D\in\mathcal{T}_h}\int_D \nabla w\cdot\nabla v\,dx + \eta\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e [\![w]\!]\,[\![v]\!]\,ds + \sum_{e\in\mathcal{E}_h}\int_e\left(\left\{\!\!\left\{\frac{\partial w}{\partial n}\right\}\!\!\right\}[\![v]\!] + \left\{\!\!\left\{\frac{\partial v}{\partial n}\right\}\!\!\right\}[\![w]\!]\right)ds, \tag{1.5}$$

$\mathcal{E}_h$ is the set of the edges of $\mathcal{T}_h$, $|e|$ is the length of the edge e and η is any positive number. The averages $\{\!\{\cdot\}\!\}$ and jumps $[\![\cdot]\!]$ in (1.5) are defined as follows.

Let e be an interior edge of $\mathcal{T}_h$ and n_e be a unit vector normal to e. Then e is shared by two elements $D_\pm$, where n_e points from $D_-$to D_+. We define on e

$$[\![v]\!] = v_+ - v_- \quad\text{and}\quad \left\{\!\!\left\{\frac{\partial v}{\partial n}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial v_+}{\partial n_e} + \frac{\partial v_-}{\partial n_e}\right),$$

where $v_\pm = v|_{D_\pm}$. Note that the bilinear form $\mathcal{A}_h(\cdot,\cdot)$ is independent of the choice of n_e. For an edge $e\subset\partial\Omega$, we take n_e to be the outer unit normal vector and define

$$[\![v]\!] = -v \quad\text{and}\quad \left\{\!\!\left\{\frac{\partial v}{\partial n}\right\}\!\!\right\} = \frac{\partial v}{\partial n_e}.$$

The interior penalty method [30, 2] for (1.1) is to find $u_h\in V_h$ such that

$$\mathcal{A}_h(u_h,v) = \int_\Omega fv\,dx \qquad \forall\, v\in V_h. \tag{1.6}$$

From (1.4) and (1.6) we see that the interior penalty method is consistent. If the penalty parameter η is sufficiently large (which is assumed to be the case from here on), the variational form $\mathcal{A}_h(\cdot,\cdot)$ is both bounded and coercive [2] with respect to the norm $\|\cdot\|_h$ defined by

$$\|v\|_h^2 = \sum_{D\in\mathcal{T}_h}|v|_{H^1(D)}^2 + \sum_{e\in\mathcal{E}_h}|e|^{-1}\,\|[\![v]\!]\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}_h}|e|\left\|\left\{\!\!\left\{\frac{\partial v}{\partial n_e}\right\}\!\!\right\}\right\|^2. \tag{1.7}$$

More precisely, we have

$$|\mathcal{A}_h(\zeta_1,\zeta_2)| \le C_1\|\zeta_1\|_h\|\zeta_2\|_h \qquad \forall\,\zeta_1,\zeta_2\in H^{1+\alpha}(\Omega)+V_h, \tag{1.8}$$

$$\mathcal{A}_h(v,v) \ge C_2\|v\|_h^2 \qquad \forall\, v\in V_h, \tag{1.9}$$

where α is the index of elliptic regularity and C_1 and C_2 are positive constants depending only on η and the shape regularity of $\mathcal{T}_h$. It follows that the solution u_h of the interior penalty method satisfies the quasi-optimal error estimate

$$\|u-u_h\|_h \le C\inf_{v\in V_h}\|u-v\|_h, \tag{1.10}$$

from which we can deduce the error estimate [21]

$$\|u-u_h\|_h \le Ch^\alpha|u|_{H^{1+\alpha}(\Omega)}. \tag{1.11}$$

It follows from (1.11) and a standard duality argument that we also have

$$\|u-u_h\|_{H^{1-\alpha}(\Omega)} \le Ch^{2\alpha}|u|_{H^{1+\alpha}(\Omega)}. \tag{1.12}$$

The positive constant C in (1.10)–(1.12) depends only on η and the shape regularity of $\mathcal{T}_h$.

The variable V-cycle multigrid preconditioner for the interior penalty method (1.6) was investigated in [21], where it was shown to be an optimal preconditioner and then applied to a discontinuous Galerkin method for advection-diffusion problems. The aim of this paper is to complete the analysis of multigrid algorithms for (1.6), which is an indispensable step towards the analysis of multigrid algorithms for other discontinuous Galerkin methods [19, 3].

Let $\gamma_{k,m}$ be the norm of the error propagation operator (with respect to the energy norm defined in terms of the bilinear form $\mathcal{A}_h(\cdot,\cdot)$) for the k-th level V-cycle, F-cycle or W-cycle algorithm with m pre-smoothing and m post-smoothing steps. Our main result states that

$$\gamma_{k,m} \leq \frac{C}{m^\alpha} \qquad \text{for} \quad k \geq 1 \quad \text{and} \quad m \geq m_0, \tag{1.13}$$

where m_0 is a positive integer independent of k. It follows that the V-cycle, F-cycle or W-cycle algorithms are contractions if m is sufficiently large and the contraction numbers decrease at a rate determined by the index of elliptic regularity.

The rest of the paper is organized as follows. We describe the multigrid algorithms in Section 2. Section 3 is devoted to a discussion of mesh dependent norms, which is one of the main tools for the convergence analysis. The estimate (1.13) is established for the W-cycle algorithm in Section 4. In Section 5 we derive certain two-level estimates that are crucial for the convergence analysis of the V-cycle algorithm and the F-cycle algorithm in Section 6. We conclude the paper by presenting the results of some numerical experiments in Section 7.

Finally we remark that the results in this paper can be extended to more general elliptic boundary value problems [2].

2. Multigrid Algorithms

For simplicity we consider a triangulation $\mathcal{T}_1$ of Ω consisting of rectangles and use uniform subdivision to obtain the triangulations $\mathcal{T}_2, \mathcal{T}_3, \cdots$, and define the (discontinuous) finite element space V_k by

$$V_k = \{v \in L_2(\Omega) : v|_D \in Q_1(D) \quad \forall\, D \in \mathcal{T}_k\},$$

where $Q_1(D)$ is the space of bilinear polynomials on D.[1] It follows that the finite element spaces are nested, i.e. $V_1 \subset V_2 \subset \cdots$, and the mesh sizes are related by

$$h_k = \frac{1}{2} h_{k-1}. \tag{2.1}$$

We assign four interior nodes to each rectangular element corresponding to the nodes $(\pm\frac{1}{2}, \pm\frac{1}{2})$ in the reference biunit square $(-1,1) \times (-1,1)$ (cf. Figure 1) and denote by $\mathcal{V}_k$ the set of the interior nodes of the elements in $\mathcal{T}_k$. We can then introduce a discrete inner product

$$(v_1, v_2)_k = h_k^2 \sum_{p \in \mathcal{V}_k} v_1(p) v_2(p). \tag{2.2}$$

Let $\mathcal{A}_k(\cdot,\cdot)$ be the bilinear form on V_k corresponding to $\mathcal{A}_h(\cdot,\cdot)$ defined in (1.5). The discrete equation

$$\mathcal{A}_k(u_k, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_k$$

can be written as $A_k u_k = f_k$, where $A_k : V_k \longrightarrow V_k$ and $f_k \in V_k$ are defined by

$$(A_k v_1, v_2)_k = \mathcal{A}_k(v_1, v_2) \qquad \forall v_1, v_2 \in V_k, \tag{2.3}$$

[1]The results in this paper can be extended to simplicial meshes and general convex quadrilateral meshes.

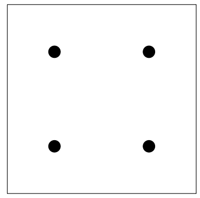

Figure 1. Interior nodes for the Q_1 element on the reference square

and

$$(f_k, v)_k = \int_\Omega f v\, dx \qquad \forall\, v \in V_k.$$

Note that (1.7), (1.8) and scaling imply

$$\mathcal{A}_k(v,v) \lesssim h_k^{2s-2} |v|_{H^s(\Omega)}^2 \qquad \forall\, v \in V_k,\ 0 \le s < \frac{1}{2}. \tag{2.4}$$

To avoid the proliferation of constants, from here on we use the notation $A \lesssim B$ to represent the statement that A is bounded by B multiplied by a constant which is independent of mesh sizes, mesh levels and all the variables in A and B. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

Multigrid algorithms are iterative schemes for equations of the form

$$A_k z = g, \tag{2.5}$$

where $g \in V_k$. They are defined in terms of intergrid transfer operators and a smoothing scheme. Since the finite element spaces are nested, we can take the coarse-to-fine operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ to be the natural injection and define the fine-to-coarse operator $I_k^{k-1} : V_k \longrightarrow V_{k-1}$ by

$$(I_k^{k-1} v, w)_{k-1} = (v, I_{k-1}^k w)_k \qquad \forall\, v \in V_k, w \in V_{k-1}. \tag{2.6}$$

For smoothing we shall use the Richardson relaxation scheme[2]

$$z_j = z_{j-1} + \Lambda_k^{-1}(g - A_k z_{j-1}) \tag{2.7}$$

where $\Lambda_k = C h_k^{-2}$ is a positive number dominating the spectral radius of A_k.

Below we describe the symmetric V-cycle, F-cycle and W-cycle algorithms [23, 26, 7, 11, 29] for (2.5).

The Symmetric *V*-cycle Multigrid Algorithm Given $g \in V_k$ and an initial guess $z_0 \in V_k$, the output $MG_{\mathcal{V}}(k, g, z_0, m)$ of the V-cycle algorithm is an approximate solution of (2.5) obtained recursively as follows. For $k = 1$, we take $MG_{\mathcal{V}}(k, g, z_0, m)$ to be $A_1^{-1} g$. For $k \ge 2$, we obtain $MG_{\mathcal{V}}(k, g, z_0, m)$ in three steps.

1. (Pre-Smoothing) Apply the Richardson scheme (2.7) m times to compute z_m.

2. (Coarse Grid Correction) Compute the residual of z_m, transfer it to the coarse grid, solve the coarse grid equation using the $(k-1)$-st level V-cycle algorithm with 0 as the initial guess, transfer the solution back to the k-th level and make the correction. In other words, compute z_{m+1} by

$$z_{m+1} = z_m + I_{k-1}^k MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m). \tag{2.8}$$

[2]Other smoothers can of course also be used [4, 8, 13].

3. (Post-Smoothing) Apply the Richardson scheme (2.7) m times to compute z_{2m+1}.

Finally we set $MG_{\mathcal{V}}(k, g, z_0, m) = z_{2m+1}$.

The Symmetric *W*-cycle Multigrid Algorithm The output $MG_{\mathcal{W}}(k, g, z_0, m)$ of the W-cycle algorithm is obtained by replacing (2.8) in the symmetric V-cycle algorithm with

$$\begin{aligned} z_{m+\frac{1}{2}} &= MG_{\mathcal{W}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m), \\ z_{m+1} &= z_m + I_{k-1}^k MG_{\mathcal{W}}(k-1, I_k^{k-1}(g - A_k z_m), z_{m+\frac{1}{2}}, m). \end{aligned} \tag{2.9}$$

In other words, the $(k-1)$-st level algorithm is used twice in the coarse grid correction step.

The *F*-cycle Multigrid Algorithm The output $MG_{\mathcal{F}}(k, g, z_0, m)$ of the F-cycle algorithm is obtained by replacing (2.8) in the symmetric V-cycle algorithm with

$$\begin{aligned} z_{m+\frac{1}{2}} &= MG_{\mathcal{F}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m), \\ z_{m+1} &= z_m + I_{k-1}^k MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_m), z_{m+\frac{1}{2}}, m). \end{aligned} \tag{2.10}$$

In other words, in the coarse grid correction step we apply the $(k-1)$-st level F-cycle algorithm once and then the $(k-1)$-st level V-cycle algorithm once.

3. Mesh Dependent Norms

Since the operator A_k defined in (2.3) is symmetric positive-definite with respect to the discrete inner product $(\cdot, \cdot)_k$, we can define for each $s \in \mathbb{R}$ the mesh-dependent norm

$$|||v|||_{s,k} = \sqrt{(A_k^s v, v)_k} \qquad \forall\, v \in V_k. \tag{3.1}$$

The spaces $(V_k, |||\cdot|||_{s,k})$ form a Hilbert scale [24, 7].

From (2.2), (2.4) and (3.1) we see that

$$|||v|||_{0,k}^2 = (v, v)_k \approx \|v\|_{L_2(\Omega)}^2 \qquad \forall\, v \in V_k, \tag{3.2}$$

$$|||v|||_{1,k} = \|v\|_{\mathcal{A}_k} \lesssim h_k^{-1} \|v\|_{L_2(\Omega)} \qquad \forall\, v \in V_k, \tag{3.3}$$

where the energy norm $\|\cdot\|_{\mathcal{A}_k}$ is defined by

$$\|v\|_{\mathcal{A}_k} = \sqrt{\mathcal{A}_k(v, v)} \qquad \forall\, v \in V_k. \tag{3.4}$$

It is clear from (1.5) that

$$\|v\|_{\mathcal{A}_{k-1}} \le \|v\|_{\mathcal{A}_k} \qquad \forall\, v \in V_{k-1}, \tag{3.5}$$

and (1.7)–(1.9) imply the stability estimate

$$\|I_{k-1}^k v\|_{\mathcal{A}_k} \lesssim \|v\|_{\mathcal{A}_{k-1}} \qquad \forall\, v \in V_{k-1}. \tag{3.6}$$

The following estimates for mesh-dependent norms are standard [5, 17]:

$$|||v|||_{s,k} \lesssim h_k^{t-s} |||v|||_{t,k} \qquad \forall\, v \in V_k \text{ and } 0 \le t \le s \le 2, \tag{3.7}$$

$$|||v|||_{1+s,k} = \sup_{w \in V_k \setminus \{0\}} \frac{\mathcal{A}_k(v, w)}{|||w|||_{1-s,k}} \qquad \forall\, v \in V_k \text{ and } s \in \mathbb{R}. \tag{3.8}$$

The convergence analysis of multigrid methods rely on the *smoothing property* that measures the effect of smoothing and the *approximation property* that measures the effect of coarse grid correction. Both of these properties are described in terms of the mesh-dependent norms. The derivation of the approximation property involves the elliptic regularity estimate

(1.2) and therefore we need to relate the mesh-dependent norms and the Sobolev norms. To this end we first introduce a conforming finite element space

$$\tilde{V}_k = \{v \in H_0^1(\Omega) :\, v|_D \in Q_4(D) \quad \forall\, D \in \mathcal{T}_h\}.$$

The continuous Q_4 tensor product element is a *relative* of the discontinuous Q_1 element (cf. Figure 2) in the sense that the shape functions of the Q_1 element are shape functions of the Q_4 element and the nodal variables (degree of freedoms) of the Q_1 element are also nodal variables of the Q_4 element.

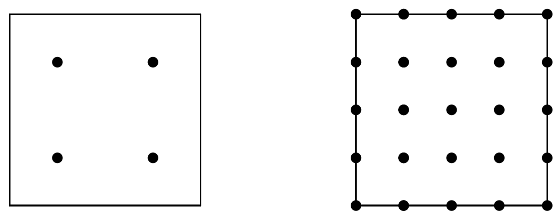

FIGURE 2. Discontinuous $\mathbb{Q}_1$ element and continuous $\mathbb{Q}_4$ element

We can connect V_k to $\tilde{V}_k$ by a linear map $E_k : V_k \longrightarrow \tilde{V}_k$ constructed by averaging. Let $p \in \Omega$ be a node for the Q_4 element and $\mathcal{T}_p = \{D \in \mathcal{T}_k :\, p \in \bar{D}\}$. Then we define, for $v \in V_k$,

$$(E_k v)(p) = \frac{1}{|\mathcal{T}_p|} \sum_{D \in \mathcal{T}_k} v_D(p), \tag{3.9}$$

where $|\mathcal{T}_p| = 1, 2$ or 4 is the number of subdomains in $\mathcal{T}_p$ and $v_D = v|_D$. Note that

$$(E_k)(p) = v(p) \qquad \text{when} \quad |\mathcal{T}_p| = 1. \tag{3.10}$$

Since $E_k v$ and v belong to $Q_4(D)$ for each $D \in \mathcal{T}_k$, the following discrete estimate can be established in a straight-forward manner (cf. [14, 16] for similar calculations):

$$\|E_k v - v\|_{L_2(\Omega)}^2 \lesssim h_k^2 \sum_{e \in \mathcal{E}_k} |e|^{-1} \|[\![v]\!]\|_{L_2(e)}^2 \lesssim h_k \mathcal{A}_k(v, v) \qquad \forall\, v \in V_k, \tag{3.11}$$

where $\mathcal{E}_k$ is the set of the edges of $\mathcal{T}_k$. In view of (3.10), E_k is a right inverse of the nodal interpolation operator Π_k for the discontinuous finite element space V_k, i.e.,

$$\Pi_k E_k = Id_k, \tag{3.12}$$

where Id_k is the identity operator on V_k.

We have the following standard interpolation error estimate [18, 17]

$$\|\tilde{v} - \Pi_k \tilde{v}\|_{L_2(D)} \lesssim (\operatorname{diam} D) |\tilde{v}|_{H^1(D)} \qquad \forall\, \tilde{v} \in \tilde{V}_k,\ D \in \mathcal{T}_k, \tag{3.13}$$

which implies by a standard inverse estimate [18, 17]

$$\|\Pi_k \tilde{v}\|_{L_2(\Omega)} \lesssim \|\tilde{v}\|_{L_2(\Omega)} \quad \text{and} \quad \sum_{D \in \mathcal{T}_k} |\Pi_k \tilde{v}|_{H^1(D)}^2 \lesssim |\tilde{v}|_{H^1(\Omega)}^2 \qquad \forall\, v \in V_k. \tag{3.14}$$

Furthermore, it follows from (1.7), (3.13), (3.14) and scaling that

$$\|\Pi_k \tilde{v}\|_{h_k}^2 \lesssim |\tilde{v}|_{H^1(\Omega)}^2 + \sum_{e \in \mathcal{E}_k} |e|^{-1} \|[\![\Pi_k \tilde{v} - \tilde{v}]\!]\|_{L_2(e)}^2 \tag{3.15}$$

$$\lesssim |\tilde{v}|^2_{H^1(\Omega)} + \sum_{D\in\mathcal{T}_k} (\operatorname{diam} D)^{-2} \|\Pi_k\tilde{v} - \tilde{v}\|^2_{L_2(D)} \lesssim |\tilde{v}|^2_{H^1(\Omega)} \qquad \forall\, \tilde{v} \in \tilde{V}_k.$$

Similarly we have

$$\|\Pi_{k-1}\Pi_k\tilde{v}\|_{L_2(\Omega)} \lesssim \|\tilde{v}\|_{L_2(\Omega)} \quad \text{and} \quad \|\Pi_{k-1}\Pi_k\tilde{v}\|_{h_k} \lesssim |\tilde{v}|_{H^1(\Omega)} \qquad \forall\, \tilde{v} \in \tilde{V}_k. \tag{3.16}$$

Let $Q_k : L_2(\Omega) \longrightarrow \tilde{V}_k$ be the $L_2(\Omega)$-orthogonal projection operator. It is known [10] that

$$\|Q_k\zeta\|_{L_2(\Omega)} \lesssim \|\zeta\|_{L_2(\Omega)} \qquad \forall\, \zeta \in L_2(\Omega), \tag{3.17}$$

$$|Q_k\zeta|_{H^1(\Omega)} \lesssim |\zeta|_{H^1(\Omega)} \qquad \forall\, \zeta \in H^1_0(\Omega). \tag{3.18}$$

Finally we introduce the operator $J_k : H^1_0(\Omega) \longrightarrow V_k$ defined by

$$J_k = \Pi_k \circ Q_k. \tag{3.19}$$

Lemma 3.1. *It holds that*

$$\|E_k v\|_{H^s(\Omega)} \approx \|\!|v|\!\|_{s,k} \qquad \forall\, v \in V_k \tag{3.20}$$

for $s \in [0,1]$ *and* $s \neq 1/2$.

Proof. It follows from (1.7), (1.9), (3.3), (3.11) and a standard inverse estimate that

$$\|E_k v\|_{L_2(\Omega)} \lesssim \|\!|v|\!\|_{0,k} \quad \text{and} \quad \|E_k v\|_{H^1(\Omega)} \lesssim \|\!|v|\!\|_{1,k} \qquad \forall\, v \in V_k,$$

which imply, by the operator interpolation theory of Sobolev spaces and Hilbert scales [28, 24, 7],

$$\|E_k v\|_{H^s(\Omega)} \lesssim \|\!|v|\!\|_{s,k} \quad \forall\, v \in V_k,\ 0 \le s \le 1. \tag{3.21}$$

On the other hand, combining (1.8), (3.14)–(3.18), we find

$$\|\!|J_k\zeta|\!\|_{0,k} \approx \|\Pi_k Q_k\zeta\|_{L_2(\Omega)} \lesssim \|Q_k\zeta\|_{L_2(\Omega)} \lesssim \|\zeta\|_{L_2(\Omega)} \quad \forall\, \zeta \in L_2(\Omega),$$
$$\|\!|J_k\zeta|\!\|_{1,k} = \|\Pi_k Q_k\zeta\|_{\mathcal{A}_k} \lesssim |Q_k\zeta|_{H^1(\Omega)} \lesssim \|\zeta\|_{H^1(\Omega)} \qquad \forall\, \zeta \in H^1_0(\Omega),$$

which imply, by the operator interpolation theory of Sobolev spaces and Hilbert scales,

$$\|\!|J_k\zeta|\!\|_{s,k} \lesssim \|\zeta\|_{H^s(\Omega)} \qquad \forall\, \zeta \in H^s_0(\Omega),\ 0 \le s \le 1,\ s \neq \frac{1}{2}. \tag{3.22}$$

Finally, since $E_k v \in \tilde{v}_k$ for $v \in V_k$, it follows from (3.12) and (3.19) that

$$J_k E_k v = \Pi_k Q_k E_k v = \Pi_k E_k v = v \qquad \forall\, v \in V_k. \tag{3.23}$$

Therefore we conclude from (3.22) and (3.23) that

$$\|\!|v|\!\|_{s,k} = \|\!|J_k E_k v|\!\|_{s,k} \lesssim \|E_k v\|_{H^s(\Omega)} \qquad \forall\, v \in V_k,\ 0 \le s \le 1,\ s \neq \frac{1}{2}.$$

□

Remark 3.2. The estimate is also valid for $s = 1/2$ if the $H^{1/2}(\Omega)$ norm is replaced by the $\tilde{H}^{1/2}(\Omega)$ ($H^{1/2}_{00}(\Omega)$) norm [25, 28].

Lemma 3.3. *It holds that*

$$\|v\|_{H^s(\Omega)} \approx \|\!|v|\!\|_{s,k} \qquad \forall\, v \in V_k,\ 0 \le s < \frac{1}{2}. \tag{3.24}$$

Proof. Let $v \in V_k$ be arbitrary. Then $v \in H^s(\Omega)$ for $0 \le s < 1/2$. From (3.3), (3.7), (3.11), (3.20) and an inverse estimate [6] we have

$$\begin{aligned} \|v\|_{H^s(\Omega)} &\le \|v - E_k v\|_{H^s(\Omega)} + \|E_k v\|_{H^s(\Omega)} \\ &\lesssim h_k^{-s}\|v - E_k v\|_{L_2(\Omega)} + \|E_k v\|_{H^s(\Omega)} \lesssim h_k^{1-s}|||v|||_{1,k} + |||v|||_{s,k} \lesssim |||v|||_{s,k}. \end{aligned}$$

Similarly, from (2.4), (3.4), (3.11), (3.20) and an inverse estimate, we have

$$|||v|||_{s,k} \lesssim \|E_k v\|_{H^s(\Omega)} \le \|v - E_k v\|_{H^s(\Omega)} + \|v\|_{H^s(\Omega)} \lesssim h_k^{1-s}\|v\|_{\mathcal{A}_k} + \|v\|_{H^s(\Omega)} \lesssim \|v\|_{H^s(\Omega)}.$$

□

4. Convergence Analysis for the W-cycle algorithm

We only need to establish the smoothing property and approximation property.

Let $R_k : V_k \longrightarrow V_k$ be defined by

$$R_k = Id_k - \Lambda_k^{-1} A_k, \tag{4.1}$$

i.e., R_k is the error propagation operator for one step of the Richardson relation scheme (2.7). The proof of the following lemma on the smoothing property, which involves only calculus, can be found in [5, 23, 17].

Lemma 4.1. *It holds that*

$$|||R_k^m v|||_{s,k} \lesssim h_k^{t-s} m^{(t-s)/2} |||v|||_{t,k} \qquad \forall\, v \in V_k,\ 0 \le t \le s \le 2. \tag{4.2}$$

Let $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ be defined by

$$\mathcal{A}_{k-1}(P_k^{k-1} v, w) = \mathcal{A}_k(v, I_{k-1}^k w) = \mathcal{A}_k(v, w) \qquad \forall\, v \in V_k,\ w \in V_{k-1}. \tag{4.3}$$

Lemma 4.2. *It holds that*

$$|||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{1-\alpha,k} \lesssim h_k^{2\alpha} |||v|||_{1+\alpha,k} \qquad \forall\, v \in V_k, \tag{4.4}$$

where $\alpha \in (1/2, 1]$ is the index of elliptic regularity.

Proof. Let $v \in V_k$ be arbitrary. By Lemma 3.3 and a standard duality formula we have

$$\begin{aligned} |||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{1-\alpha,k} &\approx \|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{H^{1-\alpha}(\Omega)} \\ &= \sup_{\phi \in H^{-1+\alpha}(\Omega)\setminus\{0\}} \frac{\phi((Id_k - I_{k-1}^k P_k^{k-1})v)}{\|\phi\|_{H^{-1+\alpha}(\Omega)}}. \end{aligned} \tag{4.5}$$

Let $\phi \in H^{-1+\alpha}(\Omega)$ be arbitrary and define $\zeta \in H_0^1(\Omega), \zeta_k \in V_k$ and $\zeta_{k-1} \in V_{k-1}$ by

$$\int_\Omega \nabla\zeta \cdot \nabla v\, dx = \phi(v) \qquad \forall\, v \in H_0^1(\Omega), \tag{4.6}$$

$$\mathcal{A}_k(\zeta_k, v) = \phi(v) \qquad \forall\, v \in V_k, \tag{4.7}$$

$$\mathcal{A}_{k-1}(\zeta_{k-1}, v) = \phi(v) \qquad \forall\, v \in V_{k-1}. \tag{4.8}$$

In other words, ζ_k and ζ_{k-1} are the approximations of ζ obtained by the interior penalty method. In view of (1.2), (1.12) and (2.1), we have

$$\|\zeta - \zeta_k\|_{H^{1-\alpha}(\Omega)} \lesssim h_k^{2\alpha} \|\phi\|_{H^{-1+\alpha}(\Omega)}, \tag{4.9}$$

$$\|\zeta - \zeta_{k-1}\|_{H^{1-\alpha}(\Omega)} \lesssim h_k^{2\alpha} \|\phi\|_{H^{-1+\alpha}(\Omega)}. \tag{4.10}$$

Observe that (4.7) and (4.8) yield

$$\mathcal{A}_{k-1}(\zeta_{k-1}, v) = \mathcal{A}_k(\zeta_k, v) \quad \forall\, v \in V_{k-1},$$

which implies

$$\zeta_{k-1} = P_k^{k-1}\zeta_k. \tag{4.11}$$

Combing (3.8), (3.24), (4.3), (4.7), and (4.9)–(4.11) we find

$$\begin{aligned}
\phi((Id_k - I_{k-1}^k P_k^{k-1})v) &= \mathcal{A}_k(\zeta_k, v) - \mathcal{A}_k(\zeta_k, I_{k-1}^k P_k^{k-1} v) \\
&= \mathcal{A}_k(\zeta_k, v) - \mathcal{A}_{k-1}(P_k^{k-1}\zeta_k, P_k^{k-1} v) \\
&= \mathcal{A}_k(\zeta_k, v) - \mathcal{A}_{k-1}(\zeta_{k-1}, P_k^{k-1} v) \\
&= \mathcal{A}_k(\zeta_k - I_{k-1}^k \zeta_{k-1}, v) \\
&\le |||\zeta_k - \zeta_{k-1}|||_{1-\alpha,k} |||v|||_{1+\alpha,k} \\
&\lesssim \|\zeta_k - \zeta_{k-1}\|_{H^{1-\alpha}(\Omega)} |||v|||_{1+\alpha,k} \\
&\le (\|\zeta_k - \zeta\|_{H^{1-\alpha}(\Omega)} + \|\zeta - \zeta_{k-1}\|_{H^{1-\alpha}(\Omega)}) |||v|||_{1+\alpha,k} \\
&\lesssim h_k^{2\alpha} \|\phi\|_{H^{-1+\alpha}(\Omega)} |||v|||_{1+\alpha,k}.
\end{aligned} \tag{4.12}$$

The lemma follows from (4.5) and (4.12). □

In view of the inverse estimate (3.7) and (4.4), the following corollary is immediate.

Corollary 4.3. *It holds that*

$$|||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{1,k} \lesssim h_k^{\alpha} |||v|||_{1+\alpha,k} \qquad \forall\, v \in V_k, \tag{4.13}$$

$$|||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{1-\alpha,k} \lesssim h_k^{\alpha} |||v|||_{1,k} \qquad \forall\, v \in V_k. \tag{4.14}$$

With Lemma 4.1 (the smoothing property), Lemma 4.2 (the approximation property) and the stability estimate (3.6) in hand, the convergence of W-cycle algorithm can be established by a standard argument [5, 23, 17].

Theorem 4.4. *There exists a positive constant C and a positive integer m_0, both independent of k, such that for all $m \ge m_0$ and initial guess $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{W}}(k, g, z_0, m)\|_{\mathcal{A}_k} \le C m^{-\alpha} \|z - z_0\|_{\mathcal{A}_k},$$

where z is the exact solution of (2.5).

Remark 4.5. It follows from Lemma 4.2 and the Bramble-Pasciak-Xu theory [9] for variable V-cycle algorithm that the variable V-cycle preconditioner is an optimal preconditioner [21].

5. Two-Level Estimates

In this section we derive certain two-level estimates that are needed for the analysis of the V-cycle algorithm and the F-cycle algorithm in Section 6. We use C to denote a generic mesh-independent positive constant, and for convenience, we state here an elementary inequality:

$$(a+b)^2 \le (1+\theta^2)a^2 + (1+\theta^{-2})b^2 \qquad \forall\, a, b \in \mathbb{R},\ \theta \in (0,1). \tag{5.1}$$

Lemma 5.1. *There exists a positive constant C such that*

$$|||I_{k-1}^k v|||_{0,k}^2 \le (1+\theta^2)|||v|||_{0,k-1}^2 + C\theta^{-2} h_k^{2\alpha} |||v|||_{\alpha,k-1}^2 \qquad \forall\, v \in V_{k-1},\ k \ge 2. \tag{5.2}$$

Proof. Let $v \in V_{k-1}$and $\theta \in (0,1)$ be arbitrary. From (2.2) and (3.2) we have

$$\|I_{k-1}^k v\|_{0,k}^2 = h_k^2 \sum_{p\in\mathcal{V}_k} v(p)^2. \tag{5.3}$$

Let Q be a rectangle in $\mathcal{T}_k$, $p_1,p_2,p_3,p_4 \in Q$ be the nodes from $\mathcal{V}_k$, and $p \in Q$ be the node from $\mathcal{V}_{k-1}$ (cf. Figure 3).

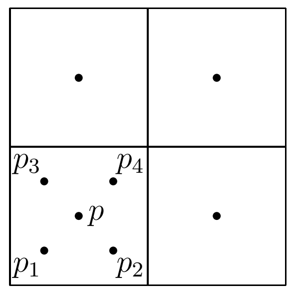

FIGURE 3. A rectangle in $\mathcal{T}_{k-1}$ subdivided into four rectangles in $\mathcal{T}_k$

It follows from (2.1), (5.1), the Mean-Value Theorem and an inverse estimate that

$$\begin{aligned} h_k^2\sum_{j=1}^4 v(p_j)^2 &= h_k^2\sum_{j=1}^4\left[v(p)+(v(p_j)-v(p))\right]^2 \\ &\le h_k^2\sum_{j=1}^4\left[(1+\theta^2)v(p)^2 + C\theta^{-2}\left(v(p_j)-v(p)\right)^2\right] \\ &\le h_k^2\sum_{j=1}^4\left[(1+\theta^2)v(p)^2 + C\theta^{-2}|p_j-p|^2\|\nabla v\|_{L_\infty(Q)}^2\right] \\ &\le h_{k-1}^2(1+\theta^2)v(p)^2 + C\theta^{-2}h_k^2|v|_{H^1(Q)}^2. \end{aligned} \tag{5.4}$$

Summing up (5.4) over all $Q \in \mathcal{T}_k$ yields, through (1.7), (1.9), (3.3), (3.4), (3.7) and (5.3),

$$\begin{aligned} \|I_{k-1}^k v\|_{0,k}^2 &\le (1+\theta^2)\|v\|_{0,k-1}^2 + C\theta^{-2}h_k^2\sum_{Q\in T_k}|v|_{H^1(Q)}^2 \\ &= (1+\theta^2)\|v\|_{0,k-1}^2 + C\theta^{-2}h_k^2\sum_{Q\in T_{k-1}}|v|_{H^1(Q)}^2 \\ &\le (1+\theta^2)\|v\|_{0,k-1}^2 + C\theta^{-2}h_k^2\|v\|_{1,k-1}^2 \\ &\le (1+\theta^2)\|v\|_{0,k-1}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{\alpha,k-1}^2. \end{aligned}$$

□

The following lemma can be derived by similar arguments.

Lemma 5.2. *There exists a positive constant C such that*

$$\|\Pi_{k-1}v\|_{0,k-1}^2 \le (1+\theta^2)\|v\|_{0,k}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{1,k}^2 \qquad \forall\, v\in V_k,\ k\ge 2. \tag{5.5}$$

Before deriving the next set of two-level estimates, we first consider the error estimate for a modified interior penalty method. Let $\phi \in H^{-1+\alpha}(\Omega)$ and define $u'_k \in V_k$ by

$$\mathcal{A}_k(u'_k, v) = \phi(E_k v) \qquad \forall\, v \in V_k, \tag{5.6}$$

where $E_k : V_k \longrightarrow \tilde{V}_k$ is the connection operator defined in (3.9).

Lemma 5.3. *Let $u \in H_0^1(\Omega)$ be the solution of (1.3) and $u'_k \in V_k$ be defined by (5.6). We have the following error estimate:*

$$\|u - u'_k\|_{h_k} \lesssim h_k^\alpha \|\phi\|_{H^{-1+\alpha}(\Omega)}, \tag{5.7}$$

where $\|\cdot\|_{h_k}$ is defined in (1.7) and α is the index of elliptic regularity.

Proof. Observe that, by (3.3), (3.11), and an inverse estimate [6],

$$\|w - E_k w\|_{H^{-1+\alpha}(\Omega)} \lesssim h_k^{\alpha-1}\|w - E_k w\|_{L_2(\Omega)} \lesssim h_k^\alpha |\!|\!| w |\!|\!|_{1,k} \qquad \forall\, w \in V_k. \tag{5.8}$$

Let $v \in V_k$ be arbitrary. From (1.3), (1.8), (1.9), (3.3), (5.6) and (5.8) we have

$$\begin{aligned}
\|u - u'_k\|_{h_k} &\le \|u - v\|_{h_k} + \|v - u'_k\|_{h_k} \\
&\lesssim \|u - v\|_{h_k} + \max_{w \in V_k \setminus \{0\}} \frac{\mathcal{A}_k(v - u'_k, w)}{\|w\|_{h_k}} \\
&\lesssim \|u - v\|_{h_k} + \max_{w \in V_k \setminus \{0\}} \frac{\mathcal{A}_k(u - u'_k, w)}{\|w\|_{h_k}} \\
&= \|u - v\|_{h_k} + \max_{w \in V_k \setminus \{0\}} \frac{\phi(w - E_k w)}{\|w\|_{h_k}} \\
&\lesssim \|u - v\|_{h_k} + h_k^\alpha \|\phi\|_{H^{-1+\alpha}(\Omega)},
\end{aligned}$$

which implies

$$\|u - u'_k\|_{h_k} \lesssim \inf_{v \in V_k} \|u - v\|_{h_k} + h_k^\alpha \|\phi\|_{H^{-1+\alpha}(\Omega)}. \tag{5.9}$$

The error estimate (5.7) follows from (1.2), (5.9) and the estimate [21]

$$\inf_{v \in V_k} \|u - v\|_{h_k} \lesssim h_k^\alpha |u|_{H^{1+\alpha}(\Omega)}.$$

□

Next we consider the operator $J_k^* : V_k \longrightarrow H_0^1(\Omega)$ defined by

$$\int_\Omega \nabla(J_k^* v) \cdot \nabla \zeta = \mathcal{A}_k(v, J_k \zeta) \qquad \forall\, \zeta \in H_0^1(\Omega), \tag{5.10}$$

where J_k is defined in (3.19).

Lemma 5.4. *The following properties hold for J_k^* :*

$$\|J_k^* v\|_{H^{1+\alpha}(\Omega)} \lesssim |\!|\!| v |\!|\!|_{1+\alpha,k} \qquad \forall\, v \in V_k, \tag{5.11}$$

$$\|v - J_k^* v\|_{h_k} \lesssim h_k^\alpha |\!|\!| v |\!|\!|_{1+\alpha,k} \qquad \forall\, v \in V_k. \tag{5.12}$$

Proof. Let $v \in V_k$ be arbitrary. Observe that (3.8) and (3.22) imply

$$|\mathcal{A}_k(v, J_k \zeta)| \le |\!|\!| v |\!|\!|_{1+\alpha,k} |\!|\!| J_k \zeta |\!|\!|_{1-\alpha,k} \lesssim |\!|\!| v |\!|\!|_{1+\alpha,k} \|\zeta\|_{H^{1-\alpha}(\Omega)} \qquad \forall\, \zeta \in H_0^1(\Omega). \tag{5.13}$$

Let ϕ be the linear functional defined by

$$\phi(\zeta) = \mathcal{A}_k(v, J_k \zeta) \qquad \forall\, \zeta \in H_0^1(\Omega). \tag{5.14}$$

In view of (5.13), we have $\phi \in H^{-1+\alpha}(\Omega)$ and

$$\|\phi\|_{H^{-1+\alpha}(\Omega)} \lesssim \|v\|_{1+\alpha,k}. \tag{5.15}$$

Furthermore, we can rewrite (5.10) as

$$\int_\Omega \nabla(J_k^* v)\cdot\nabla\zeta = \phi(\zeta) \qquad \forall\,\zeta \in H_0^1(\Omega). \tag{5.16}$$

It then follows from (1.2) that $J_k^*\zeta \in H^{1+\alpha}(\Omega)$ and (5.11) is valid.

From (3.23) and (5.14), we have

$$\mathcal{A}_k(v,w) = \mathcal{A}_k(v, J_k E_k w) = \phi(E_k w) \qquad \forall\, v \in V_k. \tag{5.17}$$

The estimate (5.12) now follows from Lemma 5.3 and (5.15)–(5.17). □

We are now ready to derive another set of two-level estimates.

Lemma 5.5. *There exists a positive constant C such that*

$$\|I_{k-1}^k v\|_{1,k}^2 \le \|v\|_{1,k-1}^2 + C h_k^{2\alpha}\|v\|_{1+\alpha,k-1}^2 \qquad \forall\, v \in V_k,\ k \ge 2. \tag{5.18}$$

Proof. Let $v \in V_{k-1}$ be arbitrary. From (1.5) and (3.3) we see that

$$\begin{aligned}\|I_{k-1}^k v\|_{1,k}^2 = \mathcal{A}_k(v,v) &= \mathcal{A}_{k-1}(v,v) + \frac{1}{2}\sum_{e\in\mathcal{E}_k}|e|^{-1}\|[\![v]\!]\|_{L_2(e)}^2 \\ &= \|v\|_{1,k-1}^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_k}|e|^{-1}\frac{1}{2}\|[\![v - J_{k-1}^* v]\!]\|_{L_2(e)}^2.\end{aligned} \tag{5.19}$$

Moreover, we have, from (1.7),

$$\sum_{e\in\mathcal{E}_k}|e|^{-1}\frac{1}{2}\|[\![v - J_{k-1}^* v]\!]\|_{L_2(e)}^2 \lesssim \|v - J_{k-1}^* v\|_{h_k}^2 \lesssim \|v - J_{k-1}^* v\|_{h_{k-1}}^2. \tag{5.20}$$

The estimate (5.18) follows from (2.1), Lemma 5.4, (5.19) and (5.20). □

Lemma 5.6. *There exists a positive constant C such that*

$$\|\Pi_{k-1} v\|_{1,k-1}^2 \le (1+\theta^2)\|v\|_{1,k}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{1+\alpha,k}^2 \qquad \forall\, v\in V_k,\ \theta\in(0,1),\ k\ge 2. \tag{5.21}$$

Proof. First we observe that, from (3.12), (3.16) and Lemma 3.1,

$$\|\Pi_{k-1} v\|_{1,k-1} = \|\Pi_{k-1}\Pi_k E_k v\|_{1,k-1} \lesssim |E_k v|_{H^1(\Omega)} \lesssim \|v\|_{1,k} \qquad \forall\, v \in V_k. \tag{5.22}$$

Let $v \in V_k$ and $\theta \in (0,1)$ be arbitrary. It follows from (3.5), (4.13), (5.1) and (5.22) that

$$\begin{aligned}\|\Pi_{k-1} v\|_{1,k-1}^2 &\le (1+\theta^2)\|P_k^{k-1} v\|_{1,k-1}^2 + C\theta^{-2}\|\Pi_{k-1}(v - P_k^{k-1} v)\|_{1,k-1}^2 \\ &\le (1+\theta^2)\|P_k^{k-1} v\|_{1,k}^2 + C\theta^{-2}\|v - P_k^{k-1} v\|_{1,k}^2 \\ &\le (1+\theta^2)^2\|v\|_{1,k}^2 + C\theta^{-2}\|v - P_k^{k-1} v\|_{1,k}^2 \\ &\le (1+\theta^2)^2\|v\|_{1,k}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{1+\alpha,k}^2,\end{aligned}$$

which implies (5.21) because $\theta \in (0,1)$ is arbitrary. □

6. Convergence Analysis for the V-cycle Algorithm and the F-cycle Algorithm

According to the additive multigrid theory developed in [12, 15, 31, 32], the convergence of V-cycle and F-cycle multigrid algorithms can be established using (4.2), (4.4), (5.18) and the following two estimates:

$$\|I_{k-1}^k v\|_{1-\alpha,k}^2 \leq (1+\theta^2)\|v\|_{1-\alpha,k-1}^2 + C_1\theta^{-2}h_k^{2\alpha}\|v\|_{1,k-1}^2 \qquad \forall\, v \in V_{k-1}, \tag{6.1}$$

$$\|P_k^{k-1} v\|_{1-\alpha,k-1}^2 \leq (1+\theta^2)\|v\|_{1-\alpha,k}^2 + C_2\theta^{-2}h_k^{2\alpha}\|v\|_{1,k}^2 \qquad \forall\, v \in V_k, \tag{6.2}$$

where α is the index of elliptic regularity, $\theta \in (0,1)$ is arbitrary and the constants C_1 and C_2 are independent of θ and k.

Lemma 6.1. *The estimate* (6.1) *holds.*

Proof. Let C_1 be a number dominating the constants in (5.2) and (5.18). For $\theta \in (0,1)$, we define the inner product

$$\langle v_1, v_2\rangle_{k-1,\theta} = (1+\theta^2)(v_1,v_2)_{k-1} + C_1\theta^{-2}h_k^{2\alpha}(A_{k-1}^\alpha v_1, v_2)_{k-1} \qquad \forall\, v_1, v_2 \in V_{k-1}. \tag{6.3}$$

Note that A_{k-1} is symmetric positive-definite with respect to the inner product $\langle\cdot,\cdot\rangle_{k-1,\theta}$, and it follows from (3.1), (5.2), (5.18) and (6.3) that

$$\|I_{k-1}^k v\|_{0,k}^2 \leq \langle A_{k-1}^0 v, v\rangle_{k-1,\theta} \quad \forall v \in V_{k-1},$$
$$\|I_{k-1}^k v\|_{1,k}^2 \leq \langle A_{k-1}^1 v, v\rangle_{k-1,\theta} \quad \forall v \in V_{k-1}.$$

Therefore, we have, by (3.1) and interpolation between Hilbert scales,

$$\|I_{k-1}^k v\|_{1-\alpha,k}^2 \leq \langle A_{k-1}^{1-\alpha} v, v\rangle_{k-1,\theta} = (1+\theta^2)\|v\|_{1-\alpha,k-1}^2 + C_1\theta^{-2}h_k^{2\alpha}\|v\|_{1,k-1}^2 \qquad \forall\, v \in V_{k-1}.$$

□

Similarly, we obtain from (5.5) and (5.21) the following lemma.

Lemma 6.2. *There exists a positive constant C_3 such that*

$$\|\Pi_{k-1} v\|_{1-\alpha,k-1}^2 \leq (1+\theta^2)\|v\|_{1-\alpha,k}^2 + C_3\theta^{-2}h_k^{2\alpha}\|v\|_{1,k}^2 \qquad \forall\, v \in V_k,\ \theta \in (0,1),\ k \geq 2. \tag{6.4}$$

Lemma 6.3. *The estimate* (6.2) *holds.*

Proof. First we observe that, by (3.2), (3.12), (3.16) and (3.20),

$$\|\Pi_{k-1} v\|_{0,k-1} \lesssim \|\Pi_{k-1}\Pi_k E_k v\|_{L_2(\Omega)} \lesssim \|E_k v\|_{L_2(\Omega)} \lesssim \|v\|_{0,k} \qquad \forall\, v \in V_k. \tag{6.5}$$

It then follows from (5.22), (6.5) and interpolation between Hilbert scales that

$$\|\Pi_{k-1} v\|_{1-\alpha,k-1} \lesssim \|v\|_{1-\alpha,k} \qquad \forall\, v \in V_k. \tag{6.6}$$

Let $v \in V_k$ and $\theta \in (0,1)$ be arbitrary. Combining (4.14), (5.1), (6.4), (6.6), we find

$$\begin{aligned}
\|P_k^{k-1} v\|_{1-\alpha,k-1}^2 &\leq (1+\theta^2)\|\Pi_{k-1} v\|_{1-\alpha,k-1}^2 + C\theta^{-2}\|\Pi_{k-1}(P_k^{k-1} v - v)\|_{1-\alpha,k-1}^2 \\
&\leq (1+\theta^2)^2\|v\|_{1-\alpha,k}^2 + C\theta^{-2}\Big(h_k^{2\alpha}\|v\|_{1,k}^2 + \|P_k^{k-1} v - v\|_{1-\alpha,k}^2\Big) \\
&\leq (1+\theta^2)^2\|v\|_{1-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{1,k}^2,
\end{aligned}$$

which implies (6.2) because $\theta \in (0,1)$ is arbitrary. □

The theorems below on the convergence of the V-cycle and F-cycle multigrid algorithms now follow from the additive multigrid theory.

Theorem 6.4. *There exists a positive constant C and a positive integer m_0, both independent of k, such that for all $m \geq m_0$ and $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{V}}(k, g, z_0, m)\|_{\mathcal{A}_k} \leq Cm^{-\alpha}\|z - z_0\|_{\mathcal{A}_k},$$

where z is the exact solution of (2.5)*.*

Theorem 6.5. *There exists a positive constant C and a positive integer m_0, both independent of k, such that for all $m \geq m_0$ and $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{F}}(k, g, z_0, m)\|_{\mathcal{A}_k} \leq Cm^{-\alpha}\|z - z_0\|_{\mathcal{A}_k},$$

where z is the exact solution of (2.5)*.*

7. Numerical Experiments

In this section we present some numerical results for multigrid algorithms for the interior penalty method based on the discontinuous Q_1 element. The penalty parameter η is taken to be 2 in all of the experiments.

In the first set of experiments we apply the multigrid algorithms to the model problem on the unit square, where the first triangulation $\mathcal{T}_1$ has four elements. The contraction numbers for the V-cycle, F-cycle and W-cycle algorithms are recorded in Tables 1–3.

Convergence for the V-cycle, F-cycle and W-cycle algorithms is observed for $m = 8$, $m = 6$, and $m = 3$ respectively. We also observe that the performance of the F-cycle algorithm and the W-cycle algorithm are almost identical for $m \geq 8$.

In the second set of experiments we apply the multigrid algorithms to the model problem on the L-shaped domain with vertices $(0, 0)$, $(1, 0)$, $(1, 1/2)$, $(1/2, 1/2)$, $(1/2, 1)$ and $(0, 1)$, where the first triangulation $\mathcal{T}_1$ has three elements. The contraction numbers of the algorithms are reported in Tables 4–6. They exhibit similar behaviors as those for the unit square.

$\gamma_{m,k,v}$	m=8	m=9	m=10	m=11	m=12	m=13	m=14
k=2	0.14	0.11	0.09	0.07	0.05	0.04	0.03
k=3	0.32	0.25	0.20	0.16	0.13	0.10	0.09
k=4	0.51	0.39	0.30	0.25	0.20	0.16	0.13
k=5	0.67	0.50	0.39	0.31	0.25	0.20	0.16
k=6	0.78	0.58	0.44	0.35	0.27	0.22	0.18
k=7	0.86	0.63	0.47	0.36	0.29	0.22	0.18
k=8	0.93	0.66	0.49	0.37	0.29	0.23	0.18

Table 1. V-cycle contraction numbers for the unit square

References

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[3] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39:1749–1779, 2001/02.

$\gamma_{m,k,f}$	m=6	m=7	m=8	m=9	m=10	m=11	m=12
k=2	0.26	0.20	0.15	0.11	0.09	0.07	0.05
k=3	0.36	0.30	0.25	0.20	0.17	0.13	0.11
k=4	0.37	0.33	0.28	0.25	0.21	0.18	0.15
k=5	0.39	0.33	0.28	0.25	0.21	0.18	0.16
k=6	0.39	0.33	0.28	0.25	0.21	0.18	0.16
k=7	0.39	0.33	0.28	0.25	0.21	0.18	0.16
k=8	0.38	0.33	0.28	0.25	0.21	0.18	0.16

Table 2. F-cycle contraction numbers for the unit square

$\gamma_{m,k,w}$	m=3	m=4	m=5	m=6	m=7	m=8	m=9
k=2	0.66	0.47	0.35	0.26	0.18	0.14	0.11
k=3	0.71	0.57	0.43	0.37	0.30	0.25	0.20
k=4	0.77	0.55	0.45	0.40	0.34	0.29	0.25
k=5	0.80	0.59	0.46	0.40	0.34	0.30	0.25
k=6	0.85	0.57	0.46	0.40	0.34	0.29	0.25
k=7	0.88	0.58	0.46	0.40	0.35	0.29	0.25
k=8	0.90	0.55	0.46	0.40	0.34	0.29	0.26

Table 3. W-cycle contraction numbers for the unit square

$\gamma_{m,k,v}$	m=8	m=9	m=10	m=11	m=12	m=13	m=14
k=2	0.15	0.12	0.10	0.09	0.06	0.05	0.04
k=3	0.29	0.25	0.18	0.17	0.14	0.12	0.10
k=4	0.50	0.39	0.31	0.25	0.20	0.17	0.14
k=5	0.66	0.50	0.39	0.31	0.25	0.22	0.20
k=6	0.78	0.58	0.44	0.35	0.33	0.30	0.28
k=7	0.87	0.63	0.49	0.45	0.41	0.38	0.35
k=8	0.92	0.65	0.60	0.55	0.50	0.45	0.42

Table 4. V-cycle contraction numbers for an L-shaped domain

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$\gamma_{m,k,f}$	m=6	m=7	m=8	m=9	m=10	m=11	m=12
k=2	0.25	0.19	0.15	0.12	0.10	0.08	0.07
k=3	0.36	0.30	0.25	0.20	0.17	0.13	0.11
k=4	0.37	0.33	0.29	0.25	0.21	0.17	0.15
k=5	0.39	0.33	0.29	0.25	0.21	0.18	0.16
k=6	0.39	0.33	0.28	0.25	0.21	0.18	0.15
k=7	0.39	0.33	0.28	0.25	0.21	0.18	0.16
k=8	0.39	0.33	0.28	0.25	0.21	0.18	0.16

TABLE 5. F-cycle contraction numbers for an L-shaped domain

$\gamma_{m,k,w}$	m=3	m=4	m=5	m=6	m=7	m=8	m=9
k=2	0.58	0.43	0.29	0.25	0.19	0.15	0.12
k=3	0.65	0.53	0.43	0.36	0.30	0.25	0.20
k=4	0.72	0.54	0.45	0.38	0.34	0.29	0.25
k=5	0.73	0.53	0.45	0.39	0.34	0.29	0.24
k=6	0.78	0.54	0.46	0.40	0.34	0.30	0.25
k=7	0.81	0.54	0.46	0.40	0.34	0.30	0.26
k=8	0.87	0.56	0.46	0.40	0.35	0.30	0.25

TABLE 6. W-cycle contraction numbers for an L-shaped domain

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