Two-level additive Schwarz preconditioners for $C^0$ interior penalty methods

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TWO-LEVEL ADDITIVE SCHWARZ PRECONDITIONERS FOR 
$C^0$ INTERIOR PENALTY METHODS

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Abstract. We study two-level additive Schwarz preconditioners that can be used in the iterative solution of the discrete problems resulting from $C^0$ interior penalty methods for fourth order elliptic boundary value problems. We show that the condition number of the preconditioned system is bounded by $C(1 + (H^3/\delta^3))$, where $H$ is the typical diameter of a subdomain, $\delta$ measures the overlap among the subdomains and the positive constant $C$ is independent of the mesh sizes and the number of subdomains.

1. Introduction

$C^0$ interior penalty methods for fourth order elliptic boundary value problems have recently been analyzed in [19, 12]. The idea behind this approach can be explained in terms of the following model problem:

Find $u \in H^2_0(\Omega)$ such that

\begin{equation}
\sum_{i,j=1}^{2} \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H^2_0(\Omega),
\end{equation}

where $\Omega$ is a bounded polygonal domain in $\mathbb{R}^2$ and $f \in L^2(\Omega)$.

Let $T_h$ be a (simplicial or convex quadrilateral) triangulation of $\Omega$ and $V_h \subset H^1_0(\Omega)$ be a Lagrange (triangular or tensor product) finite element space associated with $T_h$. By a careful integration by parts argument [12], it can be shown that the solution $u$ of (1.1), which by elliptic regularity [24, 15, 26, 3] belongs to $H^{2+\alpha}(\Omega)$ for some $\alpha > 1/2$, satisfies

\begin{equation}
A_h(u, v) = \int_{\Omega} f v \, dx \quad \forall v \in V_h,
\end{equation}

where

\begin{equation}
A_h(w, v) = a_h(w, v) + b_h(w, v) + \eta c_h(w, v),
\end{equation}

\begin{equation}
a_h(w, v) = \sum_{D \in T_h} \sum_{i,j=1}^{2} \int_{D} \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, dx,
\end{equation}

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\begin{align*}
&b_h(w, v) = \sum_{e \in \mathcal{E}_h} \int_e \left( \left\{ \frac{\partial^2 w}{\partial n^2} \right\} \left[ \frac{\partial v}{\partial n} \right] + \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \left[ \frac{\partial w}{\partial n} \right] \right) \, ds, \\
&c_h(w, v) = \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e \left[ \frac{\partial w}{\partial n} \right] \left[ \frac{\partial v}{\partial n} \right] \, ds.
\end{align*}

\(\mathcal{E}_h\) in (1.5) and (1.6) is the set of all the edges of \(T_h\), and \(\eta\) in (1.3) is a penalty parameter. The jumps \([\cdot]\) and averages \(\{\cdot\}\) in (1.5) and (1.6) are defined as follows.

Let \(e\) be an interior edge of \(T_h\) and \(n_e\) be a unit normal vector of \(e\). Then \(e\) is shared by two elements \(D_+\) and \(D_-\) in \(T_h\) where the normal vector \(n_e\) points from \(D_-\) to \(D_+\). We define on \(e\), for any function \(v\) that is piecewise \(H^s\) with respect to the triangulation \(T_h\) for some \(s > \frac{5}{2}\),

\begin{equation}
\left[ \frac{\partial v}{\partial n} \right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \quad \text{and} \quad \left\{ \frac{\partial^2 v}{\partial n^2} \right\} = \frac{1}{2} \left( \frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2} \right),
\end{equation}

where \(v_\pm = v\big|_{D_\pm}\). Note that \([\partial v/\partial n]\) and \(\{\partial^2 v/\partial n^2\}\) are independent of the choice of \(n_e\). For an edge \(e\) that is a subset of \(\partial \Omega\), we take \(n_e\) to be the outward pointing unit normal vector and define

\begin{equation}
\left[ \frac{\partial v}{\partial n} \right] = - \frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{ \frac{\partial^2 v}{\partial n^2} \right\} = \frac{\partial^2 v}{\partial n_e^2}.
\end{equation}

In the \(C^0\) interior penalty approach, the discrete problem for (1.1) is: Find \(u_h \in V_h\) such that

\begin{equation}
\mathcal{A}_h(u_h, v) = \int_\Omega f v \, dx \quad \forall v \in V_h.
\end{equation}

In view of (1.2), the \(C^0\) interior penalty method defined by (1.9) is consistent and, for a sufficiently large \(\eta\), it is also stable. Therefore the discretization error \(u - u_h\) is quasi-optimal with respect to appropriate norms [19, 12].

The \(C^0\) interior approach has certain advantages: (i) The simplest \(C^0\) interior penalty methods for (1.1), i.e., those based on the \(P_2\) Lagrange triangular element or the \(Q_2\) Lagrange tensor product element, are as simple as classical nonconforming finite elements. (ii) Unlike nonconforming finite elements, the \(C^0\) interior penalty methods come in arbitrary orders. For smooth solutions, the higher order \(C^0\) interior penalty methods have the same convergence rate as higher order \(C^1\) finite elements for smooth solutions and at the same time are much simpler. (iii) Because the finite element spaces in the \(C^0\) interior penalty approach are the standard spaces for second order problems, Poisson solves can be used naturally as preconditioners [13] in iterative methods for (1.9). (iv) Unlike mixed methods, this approach can be extended in a straight-forward way to more complicated fourth order problems, such as the fourth order elliptic systems that appear in strain-gradient elasticity and plasticity theory [22, 29].

Multigrid methods for (1.9) have been analyzed in [13]. In this paper we construct two-level additive Schwarz preconditioners [16, 17, 27] for the discrete problem (1.9). We show
that the classical results for this overlapping domain decomposition method can be extended to (1.9). More precisely, we prove that the condition number of the preconditioned system is bounded by 
$$C(1 + (H^3/\delta^3)),$$
where $H$ is the typical diameter of a subdomain, $\delta$ measures the overlap among the subdomains and the positive constant $C$ is independent of the mesh sizes and the number of subdomains.

The rest of the paper is organized as follows. We recall two-level additive Schwarz preconditioners in Section 2. Then we define the coarse spaces and derive some preliminary estimates in Section 3. Condition number estimates are established in Sections 4 and 5, followed by numerical results in Section 6. We conclude the paper with some remarks in Section 7.

We note in passing that domain decomposition methods for discontinuous Galerkin methods for second order problems were studied in [20, 25], and two-level additive Schwarz preconditioners for the discontinuous interior penalty method [4] for fourth order problems was investigated in [21].

2. Two-level Additive Schwarz Preconditioners

We will use quadrilateral meshes in this paper in view of the potential of their three dimensional counterparts for future investigation. For simplicity we will also focus on the case where $T_h$ is a rectangular mesh. The extension to general convex quadrilateral meshes will be discussed in Section 7.

Let $V_h \subset H^1_0(\Omega)$ be the $Q_2$ finite element space [14, 11] associated with $T_h$ and the operator $A_h : V_h \rightarrow V'_h$ be defined by

\begin{equation}
\langle A_h v_1, v_2 \rangle = A_h(v_1, v_2) \quad \forall v_1, v_2 \in V_h,
\end{equation}

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form between a vector space and its dual, and $A_h$ is the variational form defined by (1.3). We can then rewrite (1.9) as $A_h u_h = \phi_h$, where $\langle \phi_h, v \rangle = \int_\Omega f v \, dx$ for all $v \in V_h$. Therefore $A_h$ is the operator that needs to be preconditioned.

Note that, for $\eta$ sufficiently large (which is assumed to be the case), the following relation [12] holds:

\begin{equation}
C_1 |v|^2_{H^2(\Omega, T_h)} \leq \langle A_h v, v \rangle = A_h(v, v) \leq C_2 |v|^2_{H^2(\Omega, T_h)} \quad \forall v \in V_h,
\end{equation}

where

\begin{equation}
|v|^2_{H^2(\Omega, T_h)} = \sum_{D \in T_h} |v|^2_{H^2(D)} + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \|\partial v/\partial n\|_{L^2(e)}^2
\end{equation}

and the constants $C_1$ and $C_2$ depend only on the shape regularity of $T_h$. Here and throughout this paper we follow the standard notation for $L^2$-based Sobolev spaces [1, 14, 11].

The two-level additive Schwarz preconditioner is constructed in terms of subdomain solves and a coarse grid solve.

Let $T_H$ be a coarse rectangular mesh for $\Omega$ and $V_H \subset H^1_0(\Omega)$ be a finite element space associated with $T_H$. (The choice of the finite element for the coarse space will be discussed
in Section 3.) We can then define $A_0 : V_H \rightarrow V'_H$ by
\[
\langle A_0 v_1, v_2 \rangle = A_H(v_1, v_2) \quad \forall v_1, v_2 \in V_H,
\]
where $A_H$ is the analog of $A_h$ for the coarse grid $T_H$.

Let $\Omega_j$, $1 \leq j \leq J$, be overlapping subdomains of $\Omega$ such that $\Omega = \cup_{j=1}^{J} \Omega_j$ and the boundaries of $\Omega_j$ are aligned with the edges of $T_h$. We assume that there exist $\theta_j \in C^\infty(\bar{\Omega})$ for $1 \leq j \leq J$ such that
\[
\theta_j = 0 \quad \text{on } \Omega \setminus \Omega_j,
\]
\[
\sum_{j=1}^{J} \theta_j = 1 \quad \text{on } \bar{\Omega},
\]
\[
\|\nabla \theta_j\|_{L^\infty(\Omega)} \leq \frac{C}{\delta}, \quad \|\nabla^2 \theta_j\|_{L^\infty(\Omega)} \leq \frac{C}{\delta^2},
\]
where $\nabla^2 \theta_j$ is the Hessian of $\theta_j$, $\delta > 0$ is a parameter that measures the overlap among the subdomains and $C$ is a positive constant independent of $h$, $H$ and $J$.

**Remark 2.1.** Suppose $T_h$ is a refinement of $T_H$. We can construct $\Omega_j$ by enlarging the subdomains of $T_H$ by the amount $\delta$ so that each $\Omega_j$ is the union of rectangles in $T_h$ (cf. Figure 1). The construction of $\theta_j$ satisfying (2.5)–(2.7) is then standard [28].

**Figure 1.** $T_h$, $T_H$ and $\Omega_j$

Let $V_j \subset H_0^1(\Omega_j)$ ($\subset H_0^1(\Omega)$) be the $Q_2$ finite element space associated with the fine grid $T_h$ on $\Omega_j$. The following variational form is the analog of (1.3):
\[
A_j(w, v) = \sum_{D \in T_h} \sum_{i,j=1}^{2} \int_D \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx
\]
\[
+ \sum_{e \in \xi_h} \int_e \left( \left\{ \frac{\partial^2 w}{\partial n^2} \right\} \left[ \frac{\partial v}{\partial n} \right] + \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \left[ \frac{\partial w}{\partial n} \right] \right) ds
\]
(2.8)
where the jumps \([\cdot]\) and averages \([\{\cdot\}]\) are defined by (1.7) if \(e \subset \Omega_j\) and (1.8) if \(e \subset \partial \Omega_j\).

The bilinear form \(A_j(\cdot, \cdot)\) can be represented by the operator \(A_j : V_j \to V_j'\) defined by

\[
\langle A_j v_1, v_2 \rangle = A_j(v_1, v_2) \quad \forall v_1, v_2 \in V_j,
\]

and we have the following analog of (2.2):

\[
C_1 |v|_{H^2(\Omega_j, T_h)}^2 \leq \langle A_j v, v \rangle = A_j(v, v) \leq C_2 |v|_{H^2(\Omega_j, T_h)}^2 \quad \forall v \in V_j,
\]

where

\[
|v|_{H^2(\Omega_j, T_h)}^2 = \sum_{D \subset T_h \subset \Omega_j} |v|_{H^2(D)}^2 + \sum_{e \subset \partial \Omega_j} \frac{1}{|e|} \|\partial v / \partial n\|_{L^2(e)}^2.
\]

Let \(I_j : V_j \to V_h\) be the natural injection for \(1 \leq j \leq J\). Note that (2.3) and (2.11) imply immediately

\[
|I_j v|_{H^2(\Omega, T_h)}^2 \leq C_3 |v|_{H^2(\Omega_j, T_h)}^2 \quad \forall v \in V_j,
\]

where the positive constant \(C_3\) is independent of \(h, H\) and \(J\). Furthermore, under the condition \(\delta \leq H\) (which is assumed to be the case),

\[
\text{the relation } A_h(v_j, v_\ell) = 0 \forall v_j \in V_j \text{ and } v_\ell \in V_\ell \text{ holds for all but } N_c \text{ many } \ell \text{'s,}
\]

where the positive integer \(N_c\) is independent of \(h, H, J\) and \(\delta\), and in particular,

\[
\text{each point of } \Omega \text{ belongs to less than } N_c \text{ many subdomains.}
\]

Suppose the space \(V_H\) is connected to \(V_h\) by an operator \(I_0 : V_H \to V_h\). (The construction of \(I_0\) will be given in Section 3.) We can now define the two-level additive Schwarz preconditioner \(B : V_h' \to V_h\) by

\[
B = \sum_{j=0}^J I_j A_j^{-1} I_j',
\]

where \(I_j : V_h' \to V_j'\) is the transpose of \(I_j : V_j \to V_h\), i.e.,

\[
\langle I_j' \psi, v \rangle = \langle \psi, I_j v \rangle \quad \forall v \in V_j.
\]

(We take \(V_0\) to be \(V_H\).)

Remark 2.2. The operators \(A_j\) (\(0 \leq j \leq J\)), \(I_j\) (\(0 \leq j \leq J\)) and \(A_h\) can be represented by matrices with respect to the natural nodal bases of \(V_j'\) and \(V_h\) and the canonical dual bases of \(V_j'\) and \(V_h'\). The matrix for \(I_j'\) is then the transpose of the matrix for \(I_j\). In other words, the matrix form for the preconditioner \(B\) is also given by (2.15).
Let $\Pi_h : C(\bar{\Omega}) \longrightarrow V_h$ be the nodal interpolation operator. Then every $v \in V_h$ can be written as $v = \sum_{j=1}^J v_j$ where $v_j = \Pi_h(\theta_j v)$ and the $\theta_j$’s are the partition of unity functions that appear in (2.5)–(2.7). From (2.5) we see that $v_j \in V_j$ and therefore the condition

$$V_h = \sum_{j=0}^J I_j V_j$$

is satisfied. It follows from the additive Schwarz theory [16, 17, 32, 5, 31, 23, 30, 11] that $B$ is symmetric positive definite and therefore the eigenvalues of $BA_h$ are positive. Furthermore, the maximum and minimum eigenvalues of $BA_h$ are characterized by the following formulas:

$$\lambda_{\text{max}}(BA_h) = \max_{v \in V_h \setminus \{0\}} \frac{\langle A_h v, v \rangle}{\min_{v = \sum_{j=0}^J I_j v_j \setminus v_j \in V_j} \sum_{j=0}^J \langle A_j v_j, v_j \rangle},$$

$$\lambda_{\text{min}}(BA_h) = \min_{v \in V_h \setminus \{0\}} \frac{\langle A_h v, v \rangle}{\min_{v = \sum_{j=0}^J I_j v_j \setminus v_j \in V_j} \sum_{j=0}^J \langle A_j v_j, v_j \rangle}.$$  

Remark 2.3. It is clear from (2.18) and (2.19) that the results of this paper would not be affected if the exact solve $A^{-1}_j$ is replaced by an inexact solve $B^{-1}_j$ as long as $\langle B_j v, v \rangle \approx \langle A_j v, v \rangle$ for all $v \in V_j$.

3. Coarse Spaces and Preliminary Estimates

In this section, we define the coarse spaces and derive some preliminary estimates which will be used in the analysis of the condition number of $BA_h$.

Our first choice of the coarse space $V_H \subset H^1_0(\Omega)$ is the $Q_1$ Lagrange tensor product finite element space associated with $T_H$. The $Q_1$ Lagrange element is depicted in Figure 2 together with the $Q_3$ Bogner-Fox-Schmit element [6], where we use the solid dot $\bullet$ to denote pointwise evaluation of the shape functions, the circle $\circ$ to denote pointwise evaluation of all the first order derivatives of the shape functions and the arrow $\nabla$ to denote pointwise evaluation of the mixed second order derivative.

Let $\tilde{V}_H \subset H^2_0(\Omega)$ be the Bogner-Fox-Schmit finite element space associated with $T_H$. The two spaces $V_H$ and $\tilde{V}_H$ are connected by a map $E_H : V_H \longrightarrow \tilde{V}_H$ defined by

$$\langle E_H v \rangle(p) = v(p),$$

$$\nabla \langle E_H v \rangle(p) = \frac{1}{|T_p|} \sum_{D \in T_p} \nabla v_D(p),$$
Figure 2. $Q_1$ element and $Q_3$ Bogner-Fox-Schmit element

\[
\frac{\partial^2 (E_H v)}{\partial x_1 \partial x_2}(p) = \frac{1}{|T_p|} \sum_{D \in T_p} \frac{\partial^2 v_p}{\partial x_1 \partial x_2}(p),
\]

where $p$ is any interior vertex of the rectangular mesh, $T_p$ is the set of the rectangles sharing $p$ as a vertex, $|T_p|$ is the number of elements in $T_p$ and $v_D = v|_D$. The operator $I_0 : V_H \rightarrow V_h$ appearing in (2.15) is then defined by

\[
I_0 = \Pi_h \circ E_H,
\]

where $\Pi_h : C^0(\bar{\Omega}) \rightarrow V_h$ is the nodal interpolation operator.

**Remark 3.1.** The $Q_3$ Bogner-Fox-Schmit element is a $C^1$ relative of the $Q_1$ Lagrange tensor product element. The *enriching* map $E_H$ that connects a $C^0$ element to a $C^1$ element enables us to derive estimates that are independent of the mesh sizes (cf. (3.22) below).

The following lemma gives the properties of the operator $E_H$. To avoid the proliferation of constants, we henceforth use the notation $A \lesssim B$ to represent the statement $A \leq \text{constant} \times B$, where the positive constant depends only on the shape regularity of the meshes and not $h$, $H$, $J$ nor $\delta$. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

**Lemma 3.2.** The following estimates hold:

\[
\|v - E_H v\|_{L_2(\Omega)} \lesssim H^2 |v|_{H^2(\Omega,T_H)} \quad \forall v \in V_H,
\]

\[
|v - E_H v|_{H^1(\Omega)} \lesssim H |v|_{H^2(\Omega,T_H)} \quad \forall v \in V_H,
\]

\[
|E_H v|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega,T_H)} \quad \forall v \in V_H,
\]

where $| \cdot |_{H^2(\Omega,T_H)}$ is the analog of $| \cdot |_{H^2(\Omega,T_h)}$ (cf. (2.3)) for $v \in V_H$.

**Proof.** Let $v \in V_H$ be arbitrary. Let $D$ be a subdomain in $T_H$ and $p$ be a vertex of $D$ that is interior to $\Omega$. By (3.2), the Cauchy-Schwarz inequality and a standard inverse estimate \cite{14,11}, we have

\[
|\nabla v_D(p) - \nabla(E_H v)(p)|^2 = \left| \frac{1}{|T_p|} \sum_{D' \in T_p} [\nabla v_D(p) - \nabla v_{D'}(p)] \right|^2
\]

\[
\lesssim \sum_{D' \in T_p} |\nabla v_D(p) - \nabla v_{D'}(p)|^2
\]
\[
\leq \sum_{e \in \mathcal{E}_p} \frac{1}{|e|} \left\| \nabla v \right\|_{L^2(e)}^2 \leq \sum_{e \in \mathcal{E}_p} \frac{1}{|e|} \left\| \nabla v / \partial n \right\|_{L^2(e)}^2,
\]

where \( \mathcal{E}_p \) is the set of edges sharing \( p \) as a common vertex. Note that in the derivation of (3.8) we have used the fact that for any two subdomains \( D_1 \) and \( D_2 \) sharing \( e \) as a common edge, the tangential derivatives of \( v_{D_1} \) and \( v_{D_2} \) agree on \( e \).

For a vertex \( p \) of \( D \) that belongs to \( \partial \Omega \), we also have, by (1.8),

\[
|\nabla v_p(p) - \nabla (E_h v)(p)|^2 = |\nabla v_{D}(p)|^2 \lesssim \frac{1}{|e|} \left\| \nabla v / \partial n \right\|_{L^2(e)}^2,
\]

where \( e \subseteq \partial \Omega \) is an edge of \( T_h \) with \( p \) as an endpoint and we have used the fact that the tangential derivative of \( v \) along \( \partial \Omega \) vanishes.

Furthermore, the second order nodal values of \( E_h v \) can be easily estimated through a standard inverse estimate:

\[
\frac{\partial^2(E_h v)}{\partial x_1 \partial x_2}(p) \lesssim \sum_{D \in T_p} \left\| \frac{\partial^2 v_D}{\partial x_1 \partial x_2}(p) \right\|^2 = \sum_{D \in T_p} (\text{diam } D)^{-2} |v_D|_{H^2(D)}^2.
\]

Finally we observe that, by scaling,

\[
\|w\|_{L^2(D)}^2 \approx (\text{diam } D)^2 \sum_{p \in V_D} (w(p))^2 + (\text{diam } D)^4 \sum_{p \in V_D} (\nabla w(p))^2
\]

\[
+ (\text{diam } D)^6 \sum_{p \in V_D} \left\| \frac{\partial^2 w}{\partial x_1 \partial x_2}(p) \right\|^2 \quad \forall \, w \in \tilde{V}_H,
\]

where \( V_D \) is the set of the vertices of \( D \).

Since \( v - E_h v \in \tilde{V}_H \), it follows from (3.1) and (3.8)–(3.11) that

\[
\|v - E_h v\|_{L^2(D)}^2 \lesssim (\text{diam } D)^4 \left( \sum_{p \in V_D} \sum_{e \in \mathcal{E}_p} \frac{1}{|e|} \left\| \nabla v / \partial n \right\|_{L^2(e)}^2 + \sum_{p \in V_D} \sum_{D' \in T_p} |v_{D'}|_{H^2(D')}^2 \right),
\]

which together with standard inverse estimates also implies

\[
|v - E_h v|_{H^1(D)}^2 \lesssim (\text{diam } D)^2 \left( \sum_{p \in V_D} \sum_{e \in \mathcal{E}_p} \frac{1}{|e|} \left\| \nabla v / \partial n \right\|_{L^2(e)}^2 + \sum_{p \in V_D} \sum_{D' \in T_p} |v_{D'}|_{H^2(D')}^2 \right),
\]

\[
|v - E_h v|_{H^2(D)}^2 \lesssim \sum_{p \in V_D} \sum_{e \in \mathcal{E}_p} \frac{1}{|e|} \left\| \nabla v / \partial n \right\|_{L^2(e)}^2 + \sum_{p \in V_D} \sum_{D' \in T_p} |v_{D'}|_{H^2(D')}^2.
\]

Summing up (3.12)–(3.13) over all the subdomains of \( T_H \), we obtain (3.5) and (3.6).

Finally, from (3.14) we have

\[
|E_h v|_{H^2(\Omega)}^2 \lesssim \sum_{D \in T_H} \left( |v - E_h v|_{H^2(\Omega)}^2 + |v|_{H^2(\Omega)}^2 \right) \lesssim |v|_{H^2(\Omega, T_H)}^2.
\]

\( \square \)

The next lemma gives the relevant properties of \( \Pi_h \).
Lemma 3.3. The following estimates on $\Pi_h$ are valid:

\[ \| \zeta - \Pi_h \zeta \|_{L^2(\Omega)} + h \| \zeta - \Pi_h \zeta \|_{H^1(\Omega)} \lesssim h^2 \| \zeta \|_{H^2(\Omega)} \quad \forall \zeta \in H^2_0(\Omega), \]

\[ \| \Pi_h \zeta \|_{H^2(\Omega, T_h)} \lesssim \| \zeta \|_{H^2(\Omega)} \quad \forall \zeta \in H^2_0(\Omega). \]

**Proof.** Let $\zeta \in H^2_0(\Omega)$ be arbitrary. On each $D \in T_h$ we have the standard estimates [14, 11]:

\[ \| \zeta - \Pi_h \zeta \|_{L^2(D)} + (\text{diam } D) \| \zeta - \Pi_h \zeta \|_{H^1(D)} \lesssim (\text{diam } D)^2 \| \zeta \|_{H^2(D)}, \]

\[ \| \Pi_h \zeta \|_{H^2(D)} \lesssim \| \zeta \|_{H^2(D)}, \]

which implies (3.15) immediately.

For any $e \in E_h$, by the trace theorem (with scaling), (3.17) and (3.18), we have

\[ \frac{1}{|e|} \left\| \left[ \frac{\partial (\Pi_h \zeta)}{\partial n} \right] \right\|_{L^2(e)}^2 \leq \frac{1}{|e|} \left( \left\| \left[ \frac{\partial (\zeta - \Pi_h \zeta)}{\partial n} \right] \right\|_{L^2(e)}^2 \right) \]

\[ \lesssim \frac{1}{|e|} \sum_{D \in T_e} \left\| \frac{\partial (\zeta - \Pi_h \zeta)}{\partial n_e} \right\|_{L^2(e)}^2 \]

\[ \lesssim \sum_{D \in T_e} (\text{diam } D)^{-2} \left\| \frac{\partial (\zeta - \Pi_h \zeta)}{\partial n_e} \right\|_{L^2(D)}^2 + \left\| \frac{\partial (\zeta - \Pi_h \zeta)}{\partial n_e} \right\|_{H^1(D)}^2 \]

\[ \lesssim \sum_{D \in T_e} |\zeta|_{H^2(D)}^2, \]

where $T_e$ is the set of the subdomains in $T_h$ sharing $e$ as a common edge.

The estimate (3.16) follows by summing up (3.18) and (3.19) over all the subdomains in $T_h$ and all the edges in $E_h$. \qed

We can now derive the key estimates for the operator $I_0$.

**Lemma 3.4.** The following estimates on $I_0$ hold:

\[ \| v - I_0 v \|_{L^2(\Omega)} \lesssim H^2 \| v \|_{H^2(\Omega, T_h)} \quad \forall v \in V_H, \]

\[ \| v - I_0 v \|_{H^1(\Omega)} \lesssim H \| v \|_{H^2(\Omega, T_h)} \quad \forall v \in V_H, \]

\[ \| I_0 v \|_{H^2(\Omega, T_h)} \lesssim \| v \|_{H^2(\Omega, T_h)} \quad \forall v \in V_H. \]

**Proof.** Let $v \in V_H$ be arbitrary. Since $E_H v \in H^2_0(\Omega)$, using (3.4), the triangle inequality, Lemma 3.2, Lemma 3.3 and the fact that $h \leq H$, we have

\[ \| v - I_0 v \|_{L^2(\Omega)} = \| v - I_0 v \|_{L^2(\Omega)} \]

\[ \lesssim \| v - E_H v \|_{L^2(\Omega)} + \| E_H v - \Pi_h E_H v \|_{L^2(\Omega)} \]

\[ \lesssim H^2 \| v \|_{H^2(\Omega, T_h)} + h^2 \| E_H v \|_{H^2(\Omega)} \]

\[ \lesssim H^2 \| v \|_{H^2(\Omega, T_h)}. \]
Similarly we can obtain (3.21). Finally, it follows from (3.4), (3.7) and (3.16) that
\[ |I_0 v|_{H^2(\Omega)} = |\Pi_h E_H v|_{H^2(\Omega)} \lesssim |E_H v|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega)}. \]

\(\square\)

**Remark 3.5.** If we replace the operator \(I_0\) defined by (3.4) by the natural injection from \(V_H\) to \(V_h\), then the estimate (3.22) is not valid.

Recall that \(V_h\) is the \(Q^2\) Lagrange tensor product finite element space associated with \(T_h\). The \(Q_2\) element and the \(Q^4\) Bogner-Fox-Schmit element are depicted in Figure 3, where we use the arrow \(\uparrow\) to denote pointwise evaluation of the normal derivative of the shape functions.

**Figure 3.** \(Q^2\) element and \(Q^4\) Bogner-Fox-Schmit element

Let \(\tilde{V}_h(\subset H^2_0(\Omega))\) be the \(Q^4\) Bogner-Fox-Schmit finite element space associated with \(T_h\). We can define a map \(E_h : V_h \rightarrow \tilde{V}_h\) analogous to \(E_H\) by
\[
(3.23) \quad (E_h v)(p) = v(p),
\]
\[
(3.24) \quad \nabla (E_h v)(p) = \frac{1}{|T_p|} \sum_{D \in T_p} \nabla v_D(p),
\]
\[
(3.25) \quad \frac{\partial (E_h v)}{\partial n_e}(m_e) = \frac{1}{|T_e|} \sum_{D \in T_e} \frac{\partial v_D}{\partial n_e}(m_e),
\]
\[
(3.26) \quad \frac{\partial^2 (E_h v)}{\partial x_1 \partial x_2}(p) = \frac{1}{|T_p|} \sum_{D \in T_p} \frac{\partial^2 v_D}{\partial x_1 \partial x_2}(p),
\]
where (3.23) is defined for any interior node \(p\) associated with \(T_h\), (3.24) and (3.26) are defined for any interior vertex \(p\) of \(T_h\), and (3.25) is defined for any interior edge \(e\) of \(T_h\) with midpoint \(m_e\). Note that (3.23) implies
\[
(3.27) \quad \Pi_h E_h v = v \quad \forall v \in V_h.
\]

**Remark 3.6.** The \(Q^4\) Bogner-Fox-Schmit element is a \(C^1\) relative of the \(Q^2\) Lagrange tensor product element. The map \(E_h\) was introduced in [12] for the post-processing of the solutions obtained by the \(C^0\) interior penalty methods.

The following lemma is the analog of Lemma 3.2.
Lemma 3.7. The following estimates hold:

\[(3.28) \quad \|v - E_h v\|_{L_2(\Omega)} \lesssim h^2 |v|_{H^2(\Omega, \tau_h)} \quad \forall v \in V_h,\]
\[(3.29) \quad |v - E_h v|_{H^1(\Omega)} \lesssim h |v|_{H^2(\Omega, \tau_h)} \quad \forall v \in V_h,\]
\[(3.30) \quad |E_h v|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega, \tau_h)} \quad \forall v \in V_h.\]

Proof. Let \(v \in V_h\) be arbitrary. First of all the analogs of (3.8)–(3.10) are valid for \(E_h\) and we also have the following analog of (3.11) on each \(D \in \mathcal{T}_h:\)

\[(3.31) \quad \|w\|_{L_2(D)}^2 \approx (\text{diam } D)^2 \sum_{p \in V_D} (w(p))^2 + (\text{diam } D)^4 \sum_{p \in E_D} (\nabla w(p))^2 \]
\[+ (\text{diam } D)^4 \sum_{e \in \mathcal{E}(D)} \left[ \frac{\partial w}{\partial n_e}(m_e) \right]^2 + (\text{diam } D)^6 \sum_{p \in V_D} \left[ \frac{\partial^2 w}{\partial x_1 \partial x_2}(p) \right]^2 \quad \forall w \in \tilde{V}_h,\]

where \(\mathcal{E}(D)\) is the set of the edges of \(D\).

Let \(e \subset \partial \Omega\) be an edge of \(D \in \mathcal{T}_h\). It follows from (3.25) and a standard inverse estimate that

\[(3.32) \quad \left| \frac{\partial v_D}{\partial n_e}(m_e) - \frac{\partial (E_h v)}{\partial n_e}(m_e) \right|^2 = \frac{1}{2} \left| \sum_{p \in E} \left( \frac{\partial v_D}{\partial n_e}(m_e) - \frac{\partial v_{\tau'}^E}{\partial n_e}(m_e) \right) \right|^2 \]
\[\lesssim \frac{1}{|e|} \|\partial v/\partial n\|_{L_2(e)}^2 .\]

On the other hand, if \(e \subset \partial \Omega\) is an edge of \(\mathcal{T}_h\), then we have, by (1.8),

\[(3.33) \quad \left| \frac{\partial v}{\partial n_e}(m_e) - \frac{\partial (E_h v)}{\partial n_e}(m_e) \right|^2 = \left| \frac{\partial v}{\partial n_e}(m_e) \right|^2 \lesssim \frac{1}{|e|} \|\partial v/\partial n\|_{L_2(e)}^2 .\]

The estimates (3.28)–(3.30) can then be obtained from (3.31)–(3.33) and the analogs of (3.8)–(3.10), as in the proof of Lemma 3.2. \(\square\)

Let \(\Pi_H : C^0(\Omega) \longrightarrow V_H\) be the nodal interpolation operator and \(J^H_h : V_h \longrightarrow V_H\) be the restriction of \(\Pi_H\) to \(V_h\). The operator \(J^H_h\) will play a role in the analysis of the two-level additive Schwarz preconditioner. Note that (3.23) implies

\[J^H_h v = \Pi_H E_h v \quad \forall v \in V_h,\]

and the lemma below is an analog of Lemma 3.4 that follows from Lemma 3.7 and the analog of Lemma 3.3 for \(\Pi_H\).

Lemma 3.8. The following estimates on \(J^H_h\) are valid:

\[(3.34) \quad \|v - J^H_h v\|_{L_2(\Omega)} \lesssim H^2 |v|_{H^2(\Omega, \tau_h)} \quad \forall v \in V_h,\]
\[(3.35) \quad |v - J^H_h v|_{H^1(\Omega)} \lesssim H |v|_{H^2(\Omega, \tau_h)} \quad \forall v \in V_h,\]
\[(3.36) \quad |J^H_h v|_{H^2(\Omega, \tau_h)} \lesssim |v|_{H^2(\Omega, \tau_h)} \quad \forall v \in V_h.\]
Our second choice for the coarse space \( V_H \subset H^1_0(\Omega) \) is the \( Q_2 \) Lagrange tensor product finite element space associated with \( T_H \). In this case we take \( \tilde{V}_H \) to be the \( Q_4 \) Bogner-Fox-Schmit space associated with \( T_H \) and define \( E_H : V_H \rightarrow \tilde{V}_H \) by the analogs of (3.23)–(3.26). We can then define \( I_0 : V_H \rightarrow V_h \) by (3.4). In view of Lemma 3.7 (which also holds for \( E_H \)), the results in Lemma 3.4 remain valid for this \( I_0 \).

4. A Condition Number Estimate

In this section we derive an estimate for the condition number of \( BA_h \). We begin with an upper bound for the eigenvalues of \( BA_h \).

**Lemma 4.1.** The following upper bound for the eigenvalues of \( BA_h \) holds:

\[
\lambda_{\text{max}}(BA_h) \lesssim 1. \tag{4.1}
\]

**Proof.** Let \( v \in V_h \) be arbitrary. For any \( v_j, v \in V_j \) such that \( v = \sum_{j=0}^J I_j v_j \), we have, by (2.2) (and its analog for \( A_H \), (2.10), (2.12), (2.13), (3.22) and the Cauchy-Schwarz inequality,

\[
\langle A_h v, v \rangle \approx |v|_{H^2(\Omega,T_h)}^2 \lesssim |I_0 v_0|_{H^2(\Omega,T_h)}^2 + \left| \sum_{j=1}^J I_j v_j \right|_{H^2(\Omega,T_h)}^2
\]

\[
\lesssim |v_0|_{H^2(\Omega,T_h)}^2 + \sum_{j=1}^J |v_j|_{H^2(\Omega,T_h)}^2 \approx \sum_{j=0}^J \langle A_j v_j, v_j \rangle,
\]

which implies

\[
\langle A_h v, v \rangle \lesssim \min_{\substack{v = \sum_{j=0}^J I_j v_j \atop v_j \in V_j}} \sum_{j=0}^J \langle A_j v_j, v_j \rangle. \tag{4.2}
\]

The estimate (4.1) follows from (2.18) and (4.2). \( \square \)

We now turn our attention to a lower bound for the eigenvalues of \( BA_h \).

**Lemma 4.2.** The following lower bound for the eigenvalues of \( BA_h \) holds:

\[
\lambda_{\text{min}}(BA_h) \gtrsim \left( 1 + \frac{H^4}{\delta^4} \right)^{-1}. \tag{4.3}
\]

**Proof.** Let \( v \in V_h \) be arbitrary,

\[
v_0 = J_h^H v, \tag{4.4}
\]

\[
v_j = \Pi_h(\theta_j(v - I_0 v_0)) \quad \text{for} \quad 1 \leq j \leq J. \tag{4.5}
\]

From (2.6) and the fact that \( v - I_0 v_0 \in V_h \), it is clear that

\[
\sum_{j=0}^J I_j v_j = v. \tag{4.6}
\]
Below we will carefully estimate the terms $\langle A_jv_j, v_j \rangle$ for $0 \leq j \leq J$.

First we consider $v_0$. From (3.36), (4.4) and the analogs of (2.2) and (2.3) for $|\cdot|_{H^2(\Omega,T_h)}$, we have

\begin{equation}
\langle A_0v_0, v_0 \rangle = \langle A_Hv_0, v_0 \rangle \approx |J^Hv|_{H^2(\Omega,T_h)}^2 \lesssim |v|_{H^2(\Omega,T_h)}^2 \approx \langle A_hv, v \rangle.
\end{equation}

Next we consider $v_j$ for $1 \leq j \leq J$. Let

\begin{equation}
w = v - I_0v_0 \quad \text{and} \quad w_j = \theta_jw.
\end{equation}

From Lemma 3.4, Lemma 3.8, (4.4) and (4.8) we have

\begin{equation}
\|w\|_{L^2(\Omega)} = \|v - I_0J^Hv\|_{L^2(\Omega)} \\
\lesssim \|v - J^Hv\|_{L^2(\Omega)} + \|I_0J^Hv - J^Hv\|_{L^2(\Omega)} \\
\lesssim H^2|v|_{H^2(\Omega,T_h)} + H^2|J^Hv|_{H^2(\Omega,T_h)} \lesssim H^2|v|_{H^2(\Omega,T_h)},
\end{equation}

and similarly,

\begin{equation}
|w|_{H^1(\Omega)} \lesssim H|v|_{H^2(\Omega,T_h)},
\end{equation}

\begin{equation}
|w|_{H^2(\Omega,T_h)} \lesssim |v|_{H^2(\Omega,T_h)}.
\end{equation}

We can also rewrite (4.5) as

\begin{equation}
v_j = \Pi_h(\theta_jw) = \Pi_hw_j.
\end{equation}

Let $D \subset \Omega_j$ be an arbitrary subdomain in $\mathcal{T}_h$ and $\tilde{\theta}_{j,D}$ be the bilinear interpolant of $\theta_j$ on $D$, i.e., $\tilde{\theta}_{j,D} \in Q_1(D)$ and $\tilde{\theta}_{j,D} = \theta_j$ at the vertices of $D$. We have the following standard interpolation error estimates [14, 11]:

\begin{equation}
\|\tilde{\theta}_{j,D}\|_{L^\infty(D)} \leq \|\theta_j\|_{L^\infty(D)},
\end{equation}

\begin{equation}
\|\nabla\tilde{\theta}_{j,D}\|_{L^\infty(D)} \lesssim \|\nabla\theta_j\|_{L^\infty(D)},
\end{equation}

\begin{equation}
\|\nabla^2\tilde{\theta}_{j,D}\|_{L^\infty(D)} \lesssim \|\nabla^2\theta_j\|_{L^\infty(D)},
\end{equation}

\begin{equation}
\|\tilde{\theta}_{j,D} - \theta_j\|_{L^\infty(D)} \lesssim (\text{diam } D)^2\|\nabla^2\theta_j\|_{L^\infty(D)}.
\end{equation}

It follows from (2.6), (2.7), (3.18), (4.12)–(4.16), a standard inverse estimate and the chain rule that

\begin{equation}
|v_j|_{H^2(D)}^2 \lesssim |\Pi_h(\tilde{\theta}_{j,D}w)|_{H^2(D)}^2 + |\Pi_h((\theta_j - \tilde{\theta}_{j,D})w)|_{H^2(D)}^2 \\
\lesssim |\tilde{\theta}_{j,D}w|_{H^2(D)}^2 + (\text{diam } D)^{-4}\|\Pi_h((\theta_j - \tilde{\theta}_{j,D})w)\|_{L^2(D)}^2 \\
\lesssim \|\tilde{\theta}_{j,D}\|_{L^\infty(D)}^2|w|_{H^2(D)}^2 + \|\nabla\tilde{\theta}_{j,D}\|_{L^\infty(D)}^2|w|_{H^1(D)}^2 \\
+ \|\nabla^2\tilde{\theta}_{j,D}\|_{L^\infty(D)}^2|w|_{L^2(D)}^2 + (\text{diam } D)^{-4}\|\theta_j - \tilde{\theta}_{j,D}\|_{L^\infty(D)}^2|w|_{L^2(D)}^2 \\
\lesssim |w|_{H^2(D)}^2 + \frac{1}{\delta^2}|w|_{H^1(D)}^2 + \frac{1}{\delta^4}|w|_{L^2(D)}^2.
\end{equation}
Let \( e \in \mathcal{E}_h \) be arbitrary. We have, from (4.12),

\[
\frac{1}{|e|} ||[\partial w_j / \partial n]||^2_{L^2(e)} = \frac{1}{|e|} ||[\partial (\Pi_h w_j) / \partial n]||^2_{L^2(e)} \\
\lesssim \frac{1}{|e|} ||[\partial w_j / \partial n]||^2_{L^2(e)} + \frac{1}{|e|} ||[\partial (\Pi_h w_j - w_j) / \partial n]||^2_{L^2(e)}.
\]

Using (4.8) we can estimate the first term on the right-hand side of (4.18) as follows:

\[
\frac{1}{|e|} ||[\partial w_j / \partial n]||^2_{L^2(e)} = \frac{1}{|e|} ||[\partial (\theta_j w) / \partial n]||^2_{L^2(e)} \\
= \frac{1}{|e|} ||[\theta_j (\partial w / \partial n)]||^2_{L^2(e)} \\
\leq \frac{1}{|e|} ||[\partial w / \partial n]||^2_{L^2(e)} \\
\lesssim \frac{1}{|e|} ||[\partial v / \partial n]||^2_{L^2(e)} + \frac{1}{|e|} ||[\partial (I_0 v_0) / \partial n]||^2_{L^2(e)}.
\]

For the second term on the right-hand side of (4.18), we find from (2.6), (2.7), (3.17), (3.18), (4.8), the trace theorem (with scaling) and the chain rule that

\[
\frac{1}{|e|} ||[\partial (\Pi_h w_j - w_j) / \partial n]||^2_{L^2(e)} \lesssim \sum_{D \in \mathcal{T}_e} \frac{1}{|e|} ||\partial (\Pi_h (w_j)_D - (w_j)_D) / \partial n_e||^2_{L^2(e)} \\
\lesssim \sum_{D \in \mathcal{T}_e} (\text{diam } D)^{-2} ||\Pi_h w_j - w_j||^2_{H^1(D)} + \sum_{D \in \mathcal{T}_e} ||\Pi_h w_j - w_j||^2_{H^2(D)} \\
\lesssim \sum_{D \in \mathcal{T}_e} ||\theta_j||^2_{L^\infty(D)} ||w||^2_{H^2(D)} + \sum_{D \in \mathcal{T}_e} ||\nabla \theta_j||^2_{L^\infty(D)} ||w||^2_{H^1(D)} \\
+ \sum_{D \in \mathcal{T}_e} ||\nabla^2 \theta_j||^2_{L^\infty(D)} ||w||^2_{L^2(D)} \\
\lesssim \sum_{D \in \mathcal{T}_e} \left[ ||w||^2_{H^2(D)} + \frac{1}{\delta^2} ||w||^2_{H^1(D)} + \frac{1}{\delta^4} ||w||^2_{L^2(D)} \right].
\]

Combining (4.18)–(4.20), we have

\[
\frac{1}{|e|} ||[\partial v_j / \partial n]||^2_{L^2(e)} \lesssim \frac{1}{|e|} ||[\partial v / \partial n]||^2_{L^2(e)} + \frac{1}{|e|} ||[\partial (I_0 v_0) / \partial n]||^2_{L^2(e)} \\
+ \sum_{D \in \mathcal{T}_e} \left[ ||w||^2_{H^2(D)} + \frac{1}{\delta^2} ||w||^2_{H^1(D)} + \frac{1}{\delta^4} ||w||^2_{L^2(D)} \right].
\]
We can now conclude from (2.3), (2.11), (2.14), (3.20), (4.7), (4.9)–(4.11), (4.17) and (4.21) that

$$\sum_{j=1}^{J} |v_j|_{H^2(\Omega_j)}^2 \approx \sum_{j=1}^{J} \left( \sum_{D \in T_h} |v_j|_{H^2(D)}^2 + \sum_{e \in E_h} \frac{1}{|e|} \|\partial v_j / \partial n\|_{L^2(e)}^2 \right)$$

(4.22)

$$\lesssim |v|_{H^2(\Omega)}^2 + |I_0 v_0|_{H^2(\Omega)}^2 + \sum_{D \in T_h} \left[ |w|_{H^2(D)}^2 + \frac{1}{\delta^2} |w|_{H^1(D)}^2 + \frac{1}{\delta^4} \|w\|_{L^2(D)}^2 \right]$$

$$\lesssim |v|_{H^2(\Omega)}^2 + \frac{H^2}{\delta^2} |v|_{H^2(\Omega)}^2 + \frac{H^4}{\delta^4} |v|_{H^2(\Omega)}^2$$

$$\lesssim \left( 1 + \frac{H^4}{\delta^4} \right) |v|_{H^2(\Omega)}^2.$$

Combining (2.2), (2.10), (4.7) and (4.22) we arrive at the estimate

$$\langle A_0 v_0, v_0 \rangle + \sum_{j=1}^{J} \langle A_j v_j, v_j \rangle \lesssim \left( 1 + \frac{H^4}{\delta^4} \right) \langle A_h v, v \rangle,$$

which together with (4.6) implies

(4.23)

$$\min_{v = \sum_{j=0}^{J} v_j} \sum_{j=0}^{J} \langle A_j v_j, v_j \rangle \lesssim \left( 1 + \frac{H^4}{\delta^4} \right) \langle A_h v, v \rangle.$$

Since $v \in V_h$ is arbitrary, the estimate (4.3) follows from (2.19) and (4.23).

From Lemma 4.1 and Lemma 4.2 we have the following condition number estimate for the two-level additive Schwarz preconditioner.

**Theorem 4.3.** The condition number of $BA_h$ satisfies the estimate

$$\kappa(BA_h) = \frac{\lambda_{\text{max}}(BA_h)}{\lambda_{\text{min}}(BA_h)} \leq C \left( 1 + \frac{H^4}{\delta^4} \right),$$

(4.24)

where the positive constant $C$ depends on the shape regularity of $T_h$ and $T_H$ but not $h$, $H$, $\delta$ nor $J$.

5. The Case of Small Overlap

Theorem 4.3 implies in particular that the two-level additive Schwarz preconditioner is an optimal preconditioner when $\delta$ is comparable to $H$ (the case of generous overlap). In the case of a small overlap, i.e. when $\delta << H$, the factor $[1 + (H/\delta)^4]$ in Theorem 4.3 becomes significant. In this section we show that it can be improved to $[1 + (H/\delta)^3]$ provided that we have more information on the subdomains $\Omega_j$. The arguments we use follow those in [18] for conforming finite elements.
More precisely, we assume, in addition to (2.5)–(2.7), that

\[(5.1)\] the subdomains are rectangles,

\[(5.2)\] \( h \leq \delta \ll H \) and in particular \( 2\delta \) is less than the length of any edge of the subdomains,

\[(5.3)\] \( \Omega_i \cap \Omega_j \subset \{(x_1, x_2) \in \Omega_i : \text{dist}((x_1, x_2), \partial \Omega_i) < 2\delta\} \cap \{(x_1, x_2) \in \Omega_j : \text{dist}((x_1, x_2), \partial \Omega_j) < 2\delta\}. \]

**Remark 5.1.** These additional assumptions on the subdomains are valid if the subdomains are constructed according to Remark 2.1.

The proof of the following lemma can be found in [9].

**Lemma 5.2.** Let \( d < l \) be two positive numbers, \( G = \{(x_1, x_2) : 0 < x_1 < l, 0 < x_2 < d\} \) and \( Q = \{(x_1, x_2) : 0 < x_1 < l, 0 < x_2 < l\} \). Then there exists a positive constant \( C \) independent of \( d \) and \( l \) such that

\[(5.4)\] \[ \|\zeta\|_{L^2(G)}^2 \leq C \frac{d}{l} \left( \|\zeta\|_{L^2(Q)}^2 + l^4 |\zeta|^2_{H^2(Q)} \right) \quad \forall \zeta \in H^2(Q). \]

We now return to the proof of Lemma 4.2.

Let \( \Omega_{j,\varepsilon} \) and \( \Omega_{j,\varepsilon,h} \) be defined by

\[ \Omega_{j,\varepsilon} = \{(x_1, x_2) \in \Omega_j : \text{dist}((x_1, x_2), \partial \Omega_j) < \varepsilon\}, \]

\[ \Omega_{j,\varepsilon,h} = \text{the union of all } D \in T_h \text{ such that } D \subset \Omega_{j,\varepsilon}. \]

Conditions (2.5), (2.6) and (5.3) imply that \( \theta_j \) is identically one in the rectangle \( \{(x_1, x_2) \in \Omega_j : \text{dist}((x_1, x_2), \partial \Omega_j) \geq 2\delta\} \). Thus the terms \( \Pi_h((\theta_j - \tilde{\theta}_j,D)w) \), \( \nabla \tilde{\theta}_j,D, \nabla^2 \tilde{\theta}_j,D, \nabla \theta_j \) and \( \nabla^2 \theta_j \) in (4.17) vanish except for those \( D \) which are subsets of \( \Omega_{j,2\delta} \).

Therefore, derivatives similar to (4.17) and (4.20) yield

\[(5.5)\] \[
\sum_{D \subset \Omega_j} \frac{1}{|D|} \left\| \frac{\partial (\Pi_h w_j - w_j)}{\partial n} \right\|_{L^2(H)}^2 \lesssim \sum_{D \subset \Omega_j} \frac{1}{|D|} \left\| \frac{\partial (\Pi_h w_j - w_j)}{\partial n} \right\|_{L^2(H)}^2
\]

and

\[(5.6)\] \[
\lesssim \sum_{\frac{1}{|\omega|} \frac{1}{\delta}} \left\| \frac{\partial (\Pi_h w_j - w_j)}{\partial n} \right\|_{L^2(H)}^2
\]
Furthermore, from Lemma 3.7, (4.9), (4.11), (5.2) and a standard inverse estimate, we have
\[ (5.11) \]
\[ \lambda \]
Combining (5.8) and (5.9) we find
\[ (5.10) \]
\[ \min \]
Recall that \( E_h w \in \tilde V_h \subseteq H^2(\Omega_j) \). Hence it follows from Lemma 3.3, (3.27), (5.1), (5.2), Lemma 5.2 (with scaling) and a standard inverse estimate that
\[ (5.7) \]
\[ \lambda \]
Finally (2.19) and (5.10) yield a new lower bound for \( \lambda_{\text{min}}(BA_h) : \)
\[ (5.11) \]
\[ \lambda_{\text{min}}(BA_h) \gtrsim \left( 1 + \frac{H^3}{\delta^3} \right)^{-1}. \]

The second condition number estimate for the two-level additive Schwarz preconditioner now follows from (4.1) and (5.11).
Theorem 5.3. Under the additional assumptions (5.1)–(5.3), we have the following improved bound on the condition number of $BA_h$:

$$\kappa(BA_h) \leq C \left(1 + \frac{H^3}{\delta^3}\right),$$

where the positive constant $C$ depends on the shape regularity of $T_h$ and $T_H$ but not $h$, $H$, $\delta$ nor $J$.

6. Numerical Experiments

In this section we present the results of some numerical experiments for the biharmonic problem on the unit square. The penalty parameter $\eta$ in $A_h$, $A_H$ and $A_j$ is taken to be 5.

In the first set of experiments we take the coarse space $V_H$ to be the $Q_1$ finite element space associated with $T_H$, for $H = 2^{-1}$, $2^{-2}$ and $2^{-3}$. The corresponding overlapping domain decomposition has $J = 4$, 16 and 64 subdomains (cf. Remark 2.1 for the construction of the domain decomposition). For each choice of $H$ and $h$, we generate a vector $v_h \in V_h$ randomly, compute the right-hand side vector $g = A_h v_h$ and apply the preconditioned conjugate gradient algorithm to the equation $A_h z = g$ using the two-level additive Schwarz preconditioner. The number of iterations needed for reducing the energy norm error by a factor of $10^{-2}$ is computed for 5 random choices of $v_h$ and then averaged. The results are reported in Table 1, Table 2 and Table 3. They demonstrate that the condition number $\kappa(BA_h)$ is independent of $h$.

<table>
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<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
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<td>$2^{-2}$</td>
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<td>4</td>
<td>4</td>
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<td>4</td>
</tr>
<tr>
<td>$2^{-3}$</td>
<td></td>
<td>—</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td></td>
<td>—</td>
<td>—</td>
<td>5.2</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td></td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 1. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_1$ coarse space, $J = 4$, $H = 1/2$)

In Table 4 we collect the number of iterations for $h = 2^{-6}$ according to $J$ and $H/\delta$. They demonstrate the independence of $\kappa(BA_h)$ on $J$ and at the same time the adverse effect of the increase in $H/\delta$. 
ADDITIVE SCHWARZ PRECONDITIONERS FOR $C^0$ INTERIOR PENALTY METHODS

<table>
<thead>
<tr>
<th>$h$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
<th>$2^{-7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-3}$</td>
<td>6</td>
<td>5.6</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>—</td>
<td>5.4</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>—</td>
<td>—</td>
<td>5.2</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 2. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_1$ coarse space, $J = 16$, $H = 1/4$)

<table>
<thead>
<tr>
<th>$h$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
<th>$2^{-7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-4}$</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>—</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_1$ coarse space, $J = 64$, $H = 1/16$)

In the second set of experiments we take the coarse space $V_H$ to be the $Q_2$ finite element space associated with $T_H$, and carry out similar computations. The results are reported in Table 5, Table 6, Table 7 and Table 8. They are very similar to the results for the first set of experiments.

In the last experiment we take $H$ to be $1/4$, $V_H$ to be the $Q_1$ finite element space associated with $T_H$ and replace the operator $I_0$ by the natural injection operator from $V_H$ into $V_h$. The results are reported in Table 9. They show that the performance of the two-level preconditioner suffers from the absence of the enriching operator (cf. Remark 3.5).

7. Concluding Remarks

We have demonstrated that the two-level additive Schwarz preconditioner can be extended to $C^0$ interior penalty methods for fourth order elliptic boundary value problems. The preconditioned system behaves in the same way as the preconditioned system for classical conforming and nonconforming finite element methods [8, 9, 10, 7]. The novelty of the
Table 4. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_1$ coarse space, $h = 2^{-6}$)

<table>
<thead>
<tr>
<th>$\delta / h$</th>
<th>2</th>
<th>4</th>
<th>16</th>
<th>64</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>4</td>
<td>5.4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_2$ coarse space, $J = 4$, $H = 1/2$)

<table>
<thead>
<tr>
<th>$\delta / h$</th>
<th>$2^{-2}$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
<th>$2^{-7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-2}$</td>
<td>5.2</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$2^{-3}$</td>
<td>—</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>—</td>
<td>—</td>
<td>4.6</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

The analysis is in the role played by the jumps of the normal derivatives across the edges (see for example (3.12)–(3.14), (3.19), (4.18)–(4.21) and (5.6)), and the enriching map developed in [12] for the post-processing of the solutions of $C^0$ interior penalty methods is useful for both the construction and the analysis of the preconditioners.

The results of this paper can be extended to $C^0$ interior penalty methods that are based on triangular or convex quadrilateral finite elements [12]. The key again is to use the enriching maps that connect $C^0$ finite elements to their $C^1$ relatives, which are the triangular Argyris elements [2] or the quadrilateral generalized Bogner-Fox-Schmit elements [12].
Table 6. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_2$ coarse space, $J = 16$, $H = 1/4$)

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>2$^{-3}$</th>
<th>2$^{-4}$</th>
<th>2$^{-5}$</th>
<th>2$^{-6}$</th>
<th>2$^{-7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2$^{-3}$</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>2$^{-4}$</td>
<td></td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2$^{-5}$</td>
<td></td>
<td></td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 7. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_2$ coarse space, $J = 64$, $H = 1/16$)

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>2$^{-4}$</th>
<th>2$^{-5}$</th>
<th>2$^{-6}$</th>
<th>2$^{-7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2$^{-4}$</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>2$^{-5}$</td>
<td></td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

References

Table 8. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_2$ coarse space, $h = 2^{-6}$)

<table>
<thead>
<tr>
<th>$h/\delta$</th>
<th>4</th>
<th>16</th>
<th>64</th>
<th>256</th>
</tr>
</thead>
<tbody>
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<td>2</td>
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<td>5</td>
<td>5</td>
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<tr>
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<td>5</td>
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</tr>
<tr>
<td>8</td>
<td>4</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9. Average number of iterations for reducing the energy norm error by a factor of $10^{-2}$ ($Q_1$ coarse space without enriching operator, $J = 16$, $H = 1/4$)

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$h$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
<th>$2^{-7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-3}$</td>
<td>8</td>
<td>9</td>
<td>9.8</td>
<td>8</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td></td>
<td>10.4</td>
<td>10.4</td>
<td>9.6</td>
<td>7.8</td>
<td></td>
</tr>
<tr>
<td>$2^{-5}$</td>
<td></td>
<td></td>
<td>11.8</td>
<td>11.2</td>
<td>7.8</td>
<td></td>
</tr>
</tbody>
</table>


