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CONVERGENCE OF V-CYCLE AND F-CYCLE MULTIGRID METHODS FOR THE BIHARMONIC PROBLEM USING THE HSIEH-CLOUGH-TOCHER ELEMENT

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ABSTRACT. Multigrid V-cycle and F-cycle algorithms for the biharmonic problem using the H-C-T element are studied in the paper. We show that the contraction numbers can be uniformly improved by increasing the number of smoothing steps.

1. INTRODUCTION

We consider the following variational problem for the biharmonic equation with homogeneous Dirichlet boundary conditions: Find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = F(v) \quad \forall v \in H_0^2(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}$ is a bounded polygonal domain,

$$a(u,v) = \int_\Omega D^2 u : D^2 v\, dx := \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}\, dx,$$

and $F \in H^{-2}(\Omega) = [H_0^2(\Omega)]'$.

By the elliptic regularity of the biharmonic problem (cf. [15]) we know that there exists $\alpha \in (\frac{1}{2}, 1]$ such that the solution u of (1.1) belongs to $H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$ whenever $F \in H^{-2+\alpha}(\Omega)$ and

$$\|u\|_{H^{2+\alpha}(\Omega)} \le C_\Omega \|F\|_{H^{-2+\alpha}(\Omega)}. \tag{1.2}$$

We will use the Hsieh-Clough-Tocher (H-C-T) macro element (cf. [13] and [14]) to obtain the numerical approximation to the solution of (1.1) and study the V-cycle and F-cycle multigrid methods using the H-C-T elements.

W-cycle and variable V-cycle Multigrid methods for (1.1) using macro elements were studied in [24], [3] and [8]. In this paper, we will use the additive theory developed in [9] to study the V-cycle and F-cycle algorithm. The application of the additive theory to multigrid methods for the biharmonic problem using other finite elements can also be found in [25] and [26].

Let $\gamma_{k,m}$ be the contraction number of the k-th level symmetric V-cycle algorithm with m pre-smoothing and m post-smoothing steps. We will prove that there exists

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a constant C, independent of k and m, such that

$$\gamma_{k,m} \leq \frac{C}{m^{\alpha/2}} \quad \text{for } m \geq m_0,$$

where the positive integer m_0 is also independent of k. A similar result also holds for the F-cycle algorithm.

The rest of the paper is organized as follows. We describe the multigrid V-cycle and F-cycle algorithms using the macro element in Section 2. In Section 3 we discuss the properties of the H-C-T interpolation operators and intergrid transfer operators. The convergence of the algorithms will be proved in Section 4 by the additive theory. Numerical results are presented in Section 5.

2. V-cycle and F-cycle algorithms using the Hsieh-Clough-Tocher macro element

The Hsieh-Clough-Tocher macro element is defined on a triangle. The shape functions are those C^1 functions on the triangle whose restrictions to each smaller triangle formed by connecting the centroid and two vertices of the triangle are cubic polynomials. The nodal variables include the evaluations of the shape functions at the vertices of the triangle, the evaluations of the gradients at the vertices and of the normal derivatives at the midpoints of the edges of the triangle (cf. Figure 1).

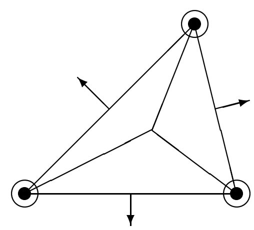


Figure 1. The H-C-T macro element

Let $\{\mathcal{T}_k\}_{k\geq 1}$ be a family of triangulations of Ω, where $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}_k$. We denote the mesh size of $\mathcal{T}_k$ by $h_k = \max\{\text{diam } T : T \in \mathcal{T}_k\}$. Note that

$$h_{k-1} = 2h_k. \tag{2.1}$$

Let V_k be the Hsieh-Clough-Tocher macro element space associated with $\mathcal{T}_k$. Then a function $v \in V_k$ is a function in $C^1(\overline{\Omega})$, whose restriction to each $T \in \mathcal{T}_k$ is a piecewise cubic polynomial, and whose nodal values along $\partial\Omega$ are zero. Note that $V_k \subset H_0^2(\Omega)$. The Hsieh-Clough-Tocher macro element method for the problem (1.1) is as follows. Find $u_k \in V_k$ so that

$$a(u_k, v) = F(v) \quad \forall v \in V_k. \tag{2.2}$$

Let u and u_k be the solutions of (1.1) and (2.2) respectively. Then it is easy to see that

$$\|u - u_k\|_a = \min_{v\in V_k} \|u - v\|_a \leq \|u - \Pi_k u\|_a, \tag{2.3}$$

where the energy norm $\|\cdot\|_a$ is defined by

$$\|v\|_a^2 = a(v, v) \left(= |v|_{H^2(\Omega)}^2\right) \quad \forall v \in V_k, \tag{2.4}$$

and $\Pi_k : C^1(\overline{\Omega}) \cap H_0^2(\Omega) \longrightarrow V_k$ is the nodal interpolation operator from $C^1(\overline{\Omega})$ to V_k. Approximation theory (cf. [12] and [11]) shows that

$$\|\zeta - \Pi_k\zeta\|_{L_2(\Omega)} + h_k^2|\zeta - \Pi_k\zeta|_{H^2(\Omega)} \lesssim h_k^{2+\alpha}|\zeta|_{H^{2+\alpha}(\Omega)} \tag{2.5}$$

for all $v \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$.

Here we use $X \lesssim Y$ to denote that $X \leq CY$, where C is a positive constant, and $X \approx Y$ means $X \lesssim Y$ as well as $Y \lesssim X$. Combining (2.3) and (2.5) we have

$$\|u - u_k\|_a \lesssim h^\alpha \|u\|_{H^{2+\alpha}(\Omega)}. \tag{2.6}$$

We define the discrete inner product $(\cdot,\cdot)_k$ on V_k by

$$\begin{aligned}(v_1, v_2)_k := {}& h_k^2 \sum_{p\in\mathcal{V}_k} n(p)v_1(p)v_2(p) \\ &+h_k^4 \sum_{p\in\mathcal{V}_k} \nabla v_1(p)\cdot\nabla v_2(p) + h_k^4 \sum_{e\in\mathcal{E}_k} \frac{\partial v_1}{\partial n}(m_e)\frac{\partial v_2}{\partial n}(m_e),\end{aligned} \tag{2.7}$$

where $\mathcal{V}_k$ is the set of internal vertices of $\mathcal{T}_k$, $\mathcal{E}_k$ is the set of internal edges of $\mathcal{T}_k$, m_e is the midpoint of the edge e and $n(p) = \frac{1}{6}\times$ the number of triangles sharing the nodes p as a vertex. Note that

$$(v, v)_k \approx \|v\|_{L_2(\Omega)}^2, \quad \forall v \in V_k. \tag{2.8}$$

We can represent the bilinear form $a(\cdot,\cdot)$ by the operator $A_k : V_k \longrightarrow V_k$ defined by

$$(A_k v_1, v_2)_k = a(v_1, v_2) \quad \forall v_1, v_2 \in V_k.$$

The equation (2.2) can then be rewritten as

$$A_k u_k = f_k, \tag{2.9}$$

where $f_k \in V_k$ is defined by $(f_k, v)_k = \phi(v)$ for all $v \in V_k$.

We define the coarse-to-fine intergrid transfer operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ to be $\Pi_k|_{V_{k-1}}$. The fine-to-coarse operator $I_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of I_{k-1}^k with respect to the discrete inner product, i.e.,

$$(I_k^{k-1}v, w)_{k-1} = (v, I_{k-1}^k w)_k \quad \forall v \in V_k, w \in V_{k-1}.$$

Now we are ready to describe the multigrid methods.

Symmetric V-cycle Multigrid Method (cf. [4], [7], [11], [16], [18], [22] and [23]) The symmetric V-cycle multigrid algorithm is an iterative solver for equations of the form of (2.9). Given $g \in V_k$ and initial guess $z_0 \in V_k$, the output $MG_{\mathcal{V}}(k, g, z_0, m)$ of the algorithm is an approximate solution for the equation

$$A_k z = g, \tag{2.10}$$

where m is the number of pre-smoothing and post-smoothing steps.

For $k = 1$, we define

$$MG_{\mathcal{V}}(1, g, z_0, m) = A_1^{-1}g.$$

For $k \geq 2$, we obtain $MG_{\mathcal{V}}(k, g, z_0, m)$ in three steps.

1. (Pre-Smoothing) For $j = 1, 2, \cdots, m$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}),$$

where Λ_k is a constant dominating the spectral radius of A_k.

2. (Coarse Grid Correction) Compute z_{m+1} by

$$z_{m+1} = z_m + I_{k-1}^k MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m).$$

3. (Post-Smoothing) For $j = m+2, \cdots, 2m+1$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

Finally we set $MG_{\mathcal{V}}(k, g, z_0, m)$ to be z_{2m+1}.

F-cycle Multigrid Method (cf. [20], [23], and [22]) The k-th level F-cycle algorithm (associated with the symmetric V-cycle algorithm) produces an approximate solution $MG_{\mathcal{F}}(k, g, z_0, m)$ for (2.10). For $k = 1$, we define

$$MG_{\mathcal{F}}(1, g, z_0, m) = A_1^{-1} g.$$

For $k \geq 2$, we obtain $MG_{\mathcal{F}}(k, g, z_0, m)$ in three steps.

1. (Pre-Smoothing) For $j = 1, 2, \cdots, m$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

2. (Coarse Grid Correction) Compute $z_{m+\frac{1}{2}}$ and z_{m+1} by

$$z_{m+\frac{1}{2}} = MG_{\mathcal{F}}(k-1, I_k^{k-1}(g - A_k z_m), 0, m).$$
$$z_{m+1} = z_m + I_k^{k-1} MG_{\mathcal{V}}(k-1, I_k^{k-1}(g - A_k z_m), z_{m+\frac{1}{2}}, m).$$

3. (Post-Smoothing) For $j = m+2, \cdots, 2m+1$, compute z_j by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

Finally we set $MG_{\mathcal{F}}(k, g, z_0, m)$ to be z_{2m+1}.

In these algorithms we use Richardson relaxation as the smoother for simplicity. Other smoothers can also be used (cf. [1], [5] and [10]).

3. Interpolation operators and Intergrid Transfer operators

In this section we discuss the the properties of interpolation operators and intergrid transfer operators. We begin with an estimate which follows from (2.5) and an interpolation of the operator $Id - \Pi_k$ between the Sobolev spaces, where Id is the identity operator on $L_2(\Omega)$.

$$|\zeta - \Pi_k \zeta|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega). \tag{3.1}$$

Lemma 3.1. *Let $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ and $\zeta_k \in V_k$ be related by*

$$a(\zeta, v) = a(\zeta_k, v) \quad \forall v \in V_k. \tag{3.2}$$

Then we have

$$\|\zeta - \zeta_k\|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)}. \tag{3.3}$$

Proof. Let $F \in H^{-2+\alpha}(\Omega)$ be arbitrary. Then there exists $\xi \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ such that

$$a(\xi, v) = F(v) \quad \forall v \in H_0^2(\Omega). \tag{3.4}$$

From the elliptic regularity estimate (1.2) we have

$$\|\xi\|_{H^{2+\alpha}(\Omega)} \lesssim \|F\|_{H^{-2+\alpha}(\Omega)}. \tag{3.5}$$

Therefore since $\Pi_k\xi \in V_k$, and from (2.6), (3.2), (3.3), (3.4), (3.5) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} F(\zeta - \zeta_k) &= a(\xi, \zeta - \zeta_k) \\ &= a(\xi - \Pi_k\xi, \zeta - \zeta_k) \\ &\le |\xi - \Pi_k\xi|_{H^2(\Omega)}|\zeta - \zeta_k|_{H^2(\Omega)} \\ &\lesssim h_k^{2\alpha}|\xi|_{H^{2+\alpha}(\Omega)}|\zeta|_{H^{2+\alpha}(\Omega)} \\ &\lesssim h_k^{2\alpha}\|F\|_{H^{-2+\alpha}(\Omega)}|\zeta|_{H^{2+\alpha}(\Omega)}. \end{aligned}$$

Then by a duality formula we have

$$\|\zeta - \zeta_k\|_{H^{2-\alpha}(\Omega)} = \sup_{F \in H^{-2+\alpha}(\Omega)} \frac{F(\zeta - \zeta_k)}{\|F\|_{H^{-2+\alpha}(\Omega)}} \lesssim |\zeta|_{H^{2+\alpha}(\Omega)}.$$

□

Now we define the mesh-dependent norms (cf. [2]). For each $v \in V_k$ we define

$$|||v|||_{s,k} = \sqrt{(A_k^{s/2}v, v)_k}. \tag{3.6}$$

From the definition and an inverse estimate we have

$$|||v|||_{0,k}^2 = (v, v)_k \qquad \forall v \in V_k, \tag{3.7}$$

$$|||v|||_{2,k} = \|v\|_a \lesssim h_k^{-2}\|v\|_{L_2(\Omega)} \quad \forall v \in V_k. \tag{3.8}$$

We can also easily see that

$$|||v|||_{s,k} \lesssim h_k^{t-s}|||v|||_{t,k} \quad \forall v \in V_k, 0 < t < s < 4, \tag{3.9}$$

$$|||v|||_{0,k} \approx \|v\|_{L_2(\Omega)} \qquad \forall v \in V_k. \tag{3.10}$$

Moreover, we have the following generalized Cauchy-Schwarz inequality.

$$|||v|||_{2+t,k} = \sup_{w \in V_k \setminus \{0\}} \frac{a(v, w)}{|||w|||_{2-t,k}}. \tag{3.11}$$

The following lemma relates the mesh-dependent norms and the Sobolev norms.

Lemma 3.2. *For $s \in [0, 2]$ but $s \neq \frac{1}{2}, \frac{3}{2}$ it holds that*

$$|||v|||_{s,k} \approx \|v\|_{H^s(\Omega)} \quad \forall v \in V_k. \tag{3.12}$$

Proof. Consider the identity operator Id_k on V_k. From (3.7) and (3.8) we know that it is a bounded operator from $(V_k, |||\cdot|||_{0,k})$ into $L_2(\Omega)$ and from $(V_k, |||\cdot|||_{2,k})$ into $H^2(\Omega)$. By interpolations of Sobolev spaces and Hilbert scales (cf. [21], [19]), we have

$$\|v\|_{H^s(\Omega)} \lesssim |||v|||_{s,k} \quad \forall v \in V_k. \tag{3.13}$$

On the other hand, let $Q_k : L_2(\Omega) \longrightarrow V_k$ be the L_2 projection operator on V_k, i.e., for each $\zeta \in L_2(\Omega)$, the function $Q_k\zeta \in V_k$ satisfies

$$(Q_k\zeta, v)_{L_2(\Omega)} = (\zeta, v)_{L_2(\Omega)} \quad \forall v \in V_k.$$

It is known that (cf. [6])

$$\|Q_k\zeta\|_{L_2(\Omega)} \lesssim \|\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in L_2(\Omega), \tag{3.14}$$

$$|Q_k\zeta|_{H^2(\Omega)} \lesssim |\zeta|_{H^2(\Omega)} \quad \forall \zeta \in H_0^2(\Omega). \tag{3.15}$$

In other words, the operator Q_k is bounded from $L_2(\Omega)$ into $(V_k, \|\!|\cdot|\!\|_{0,k})$ and from $H_0^2(\Omega)$ into $(V_k, \|\!|\cdot|\!\|_{2,k})$. By interpolations of Sobolev spaces and Hilbert scales, we have

$$\|\!|Q_k\zeta|\!\|_{s,k} \lesssim \|\zeta\|_{H^s(\Omega)} \quad \forall \zeta \in H_0^s(\Omega), \tag{3.16}$$

for $s \neq \frac{1}{2}, \frac{3}{2}$. But $Q_k v = v$ for all $v \in V_k$. Therefore

$$\|\!|v|\!\|_{s,k} \lesssim \|v\|_{H^s(\Omega)} \quad \forall v \in V_k. \tag{3.17}$$

□

Lemma 3.3. *For $\zeta_k \in V_k$, let $\zeta \in H_0^2(\Omega)$ be defined by*

$$a(\zeta, \phi) = a(\zeta_k, Q_k\phi) \quad \forall \phi \in H_0^2(\Omega). \tag{3.18}$$

Then

$$a(\zeta, v) = a(\zeta_k, v) \quad \forall v \in V_k, \tag{3.19}$$

$$|\zeta|_{H^2(\Omega)} \lesssim \|\!|\zeta_k|\!\|_{2,k}, \tag{3.20}$$

$$\|\zeta\|_{H^{2+\alpha}(\Omega)} \lesssim \|\!|\zeta_k|\!\|_{2+\alpha,k}, \tag{3.21}$$

$$\|\Pi_k\zeta\|_a \lesssim \|\!|\zeta_k|\!\|_{2,k}. \tag{3.22}$$

Proof. The equality (3.19) follows from (3.18) and the fact that $Q_k v = v$ for all $v \in V_k$.

From (2.4), (3.15), (3.18) and Cauchy-Schwarz inequality we have

$$|\zeta|^2_{H^2(\Omega)} = a(\zeta, \zeta) = a(\zeta_k, Q_k\zeta) \le \|\zeta_k\|_a \|Q_k\zeta\|_a \lesssim \|\!|\zeta_k|\!\|_{2,k} |\zeta|_{H^2(\Omega)},$$

which implies (3.20).

From (3.1), (3.3) and an inverse estimate, we have

$$\begin{aligned}
\|\Pi_k\zeta\|_a &\le \|\Pi_k\zeta - \zeta_k\|_a + \|\zeta_k\|_a \\
&\lesssim h_k^{-\alpha} |\Pi_k\zeta - \zeta_k|_{H^{2-\alpha}(\Omega)} + \|\zeta_k\|_a \\
&\le h_k^{-\alpha} |\Pi_k\zeta - \zeta|_{H^{2-\alpha}(\Omega)} + h_k^{-\alpha} |\zeta|_{H^{2-\alpha}(\Omega)} + \|\zeta_k\|_a \\
&\lesssim \|\zeta_k\|_a,
\end{aligned}$$

which proves (3.22).

From (3.11) and (3.16) we have

$$a(\zeta_k, Q_k\phi) \le \|\!|\zeta_k|\!\|_{2+\alpha,k} \|\!|Q_k\phi|\!\|_{2-\alpha,k} \lesssim \|\!|\zeta_k|\!\|_{2+\alpha,k} |\phi|_{H^{2-\alpha}(\Omega)} \quad \forall \phi \in H_0^2(\Omega).$$

Therefore the right-hand side of (3.18) defines a linear functional F on $H_0^2(\Omega)$ which actually belongs to $H^{-2+\alpha}(\Omega)$ and

$$\|F\|_{H^{-2+\alpha}(\Omega)} \lesssim \|\!|\zeta_k|\!\|_{2+\alpha,k}. \tag{3.23}$$

The estimate (3.21) follows from (1.2) and (3.23). □

Lemma 3.4. *Let $s \in [0,2]$. It holds that*

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} + h_k^s|\Pi_{k-1}v|_{H^s(\Omega)} \lesssim h_k^s|v|_{H^s(\Omega)} \quad \forall v \in V_{k-1} + V_k, \tag{3.24}$$

$$\|\Pi_k v - v\|_{L_2(\Omega)} + h_k^s|\Pi_k v|_{H^s(\Omega)} \lesssim h_k^s|v|_{H^s(\Omega)} \quad \forall v \in V_{k-1} + V_k. \tag{3.25}$$

Proof. Let $T \in \mathcal{T}_{k-1}$ be divided into 4 triangles T_1, T_2, T_3, and T_4 in $\mathcal{T}_k$ and $\tilde{T} = T/h_{k-1}$. Then $|\tilde{T} \approx 1$. (cf. Figure 2).

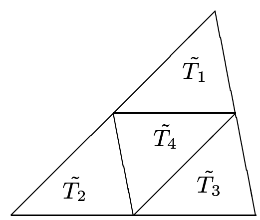


FIGURE 2. A reference triangle $\tilde{T}$ for $\mathcal{T}_{k-1}$

For each $v \in V_k$, define $\tilde{v}(\tilde{x}) = v(h_{k-1}\tilde{x})$ for $\tilde{x} \in \tilde{T}$. If $w = \Pi_{k-1}v$, then we define $\tilde{\Pi}_{k-1}\tilde{v}$ to be $\tilde{w}$.

Let $V(\tilde{T})$ be the H-C-T finite element space associated with $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$ and $\tilde{T}_4$, without boundary restrictions. Note that $V(\tilde{T})$ is a finite dimensional linear space and $|\tilde{v}|_{H^2(\tilde{T})}$ defines a norm on the quotient space $V(\tilde{T})/P_1(\tilde{T})$, where $P_1(\tilde{T})$ is the space of polynomials of degree less then or equal to 1 on $\tilde{T}$. On the other hand,

$$v \longrightarrow \|\tilde{\Pi}_{k-1}\tilde{v} - \tilde{v}\|_{L_2(\tilde{T})}$$

defines a semi-norm on $V(\tilde{T})/P_1(\tilde{T})$. Therefore

$$\|\tilde{\Pi}_{k-1}\tilde{v} - \tilde{v}\|_{L_2(\tilde{T})} \lesssim |\tilde{v}|_{H^2(\tilde{T})}. \tag{3.26}$$

A scaling argument on (3.26) yields

$$\|\Pi_{k-1}v - v\|_{L_2(T)} \lesssim h_k^2|v|_{H^2(T)} \quad \forall v \in V_k,\ T \in \mathcal{T}_{k-1}.$$

Therefore

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} \lesssim h_k^2|v|_{H^2(\Omega)} \lesssim h_k^s|v|_{H^s(\Omega)} \quad \forall v \in V_k. \tag{3.27}$$

From (3.27) and an inverse estimate, we have

$$\begin{aligned} |\Pi_{k-1}v|_{H^s(\Omega)} &\le |\Pi_{k-1}v - v|_{H^s(\Omega)} + |v|_{H^s(\Omega)} \\ &\lesssim h_k^{-s}\|\Pi_{k-1}v - v\|_{L_2(\Omega)} + |v|_{H^s(\Omega)} \\ &\lesssim |v|_{H^s(\Omega)}. \end{aligned}$$

Therefore the estimate (3.24) holds for $v \in V_{k-1}$. The argument above also applies for the functions in a larger space $V_{k-1} + V_k$. This finishes the proof of (3.24). We can similarly prove (3.25). □

Lemma 3.5. *It holds that*

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} \lesssim h_k^{2+\alpha}\|\!|v|\!\|_{2+\alpha,k} \qquad \forall v \in V_k, \tag{3.28}$$

$$\|\Pi_k v - v\|_{L_2(\Omega)} \lesssim h_k^{2+\alpha}\|\!|v|\!\|_{2+\alpha,k-1} \quad \forall v \in V_{k-1}. \tag{3.29}$$

Proof. Let $\zeta_k \in V_k$ be arbitrary. We define $\zeta \in H_0^2(\Omega)$ and $\zeta_{k-1} \in V_{k-1}$ by (3.18) and

$$a(\zeta_{k-1}, v) = a(\zeta, v) \quad \forall v \in V_{k-1}. \tag{3.30}$$

Then from (2.6), Lemma 3.1, Lemma 3.3 and (3.24) we have

$$\begin{aligned}\|\Pi_{k-1}\zeta_k - \zeta_k\|_{L_2(\Omega)} &= \|\Pi_{k-1}(\zeta_k - \zeta_{k-1}) - (\zeta_k - \zeta_{k-1})\|_{L_2(\Omega)} \\ &\lesssim h_k^2|\zeta_k - \zeta_{k-1}|_{H^2(\Omega)} \\ &\lesssim h_k^2(|\zeta_k - \zeta|_{H^2(\Omega)} + |\zeta - \zeta_{k-1}|_{H^2(\Omega)}) \\ &\lesssim h_k^{2+\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)} \lesssim h_k^{2+\alpha}\|\!|\zeta_k|\!\|_{2+\alpha,k},\end{aligned}$$

which proves (3.28). The proof of (3.29) is similar. □

By an inverse estimate, we have the following corollary.

Corollary 3.6. *It holds that*

$$|\Pi_{k-1}v - v|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha}\|\!|v|\!\|_{2+\alpha,k} \qquad \forall v \in V_k, \tag{3.31}$$

$$|\Pi_k v - v|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha}\|\!|v|\!\|_{2+\alpha,k-1} \quad \forall v \in V_{k-1}. \tag{3.32}$$

4. Convergence Analysis

Let $\mathbb{E}_{k,m} : V_k \longrightarrow V_k$ be the error propagation operator of the symmetric V-cycle algorithm applied to the equation (2.10), i.e.,

$$\mathbb{E}_{k,m}(z - z_0) = z - MG_{\mathcal{V}}(k, g, z_0, m),$$

where z is the exact solution of (2.10). The following relations (cf. [4] and [18]) are well-known:

$$\mathbb{E}_{k,m} = R_k^m[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k \mathbb{E}_{k-1,m} P_k^{k-1}]R_k^m \quad \text{for } k \geq 2, \tag{4.1}$$

$$\mathbb{E}_{1,m} = 0, \tag{4.2}$$

where $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is defined by

$$a(P_k^{k-1}v, w) = a(v, I_{k-1}^k w) \quad \forall v \in V_k, w \in V_{k-1}.$$

From (4.1) and (4.2) the following additive expression for $\mathbb{E}_{k,m}$ can be derived that is the starting point of the additive theory (cf. [9]):

$$\begin{aligned}\mathbb{E}_{k,m} = \sum_{j=2}^{k} R_k^m I_{k-1}^k \cdots R_{j+1}^m I_j^{j+1} R_j^m (Id_j - I_{j-1}^j P_j^{j-1}) R_j^m \\ \times P_{j+1}^j R_{j+1}^m \cdots P_k^{k-1} R_k^m.\end{aligned} \tag{4.3}$$

Let $\tilde{\mathbb{E}}_{k,m} : \tilde{V}_k \longrightarrow \tilde{V}_k$ be the error propagation operator of the symmetric F-cycle algorithm applied to the equation (2.10), i.e.,

$$\tilde{\mathbb{E}}_{k,m}(z - z_0) = z - MG_{\mathcal{F}}(k, g, z_0, m),$$

where z is the exact solution of (2.10). The following relations are also well-known (cf. [22]):

$$\tilde{\mathbb{E}}_{1,m} = 0 \tag{4.4}$$

$$\tilde{\mathbb{E}}_{k,m} = R_k^m[(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k \mathbb{E}_{k-1,m}\tilde{\mathbb{E}}_{k-1,m} P_k^{k-1}]R_k^m, \ k \geq 2. \tag{4.5}$$

An additive theory for the convergence analysis of V-cycle and F-cycle multigrid algorithms is developed in [9] based on the expressions (4.3)−(4.5). It is shown there that, to complete the convergence analysis, we only need to verify the following assumptions.

Assumptions on V_k:

$$(v,v)_k \approx \|v\|_{L_2(\Omega)}^2 \qquad \forall v \in V_k. \tag{4.6}$$

$$\|v\|_a \lesssim h_k^{-2}\|v\|_{L_2(\Omega)} \quad \forall v \in V_k. \tag{4.7}$$

Assumptions on I_{k-1}^k and P_k^{k-1}:

$$|||I_{k-1}^k v|||_{2,k}^2 \leq (1+\theta^2)|||v|||_{2,k-1}^2 + C_1\theta^{-2}h_k^{2\alpha}|||v|||_{2+\alpha,k-1}^2 \quad \forall v \in V_{k-1}, \theta \in (0,1). \tag{4.8}$$

$$|||I_{k-1}^k v|||_{2-\alpha,k}^2 \leq (1+\theta^2)|||v|||_{2-\alpha,k-1}^2 + C_2\theta^{-2}h_k^{2\alpha}|||v|||_{2,k-1}^2 \quad \forall v \in V_{k-1}, \theta \in (0,1). \tag{4.9}$$

$$|||P_k^{k-1} v|||_{2-\alpha,k-1}^2 \leq (1+\theta^2)|||v|||_{2-\alpha,k}^2 + C_3\theta^{-2}h_k^{2\alpha}|||v|||_{2,k}^2 \quad \forall v \in V_k, \theta \in (0,1). \tag{4.10}$$

Assumptions on $I_{k-1}^k P_k^{k-1}$ and $P_k^{k-1} I_{k-1}^k$:

$$|||(Id_k - I_{k-1}^k P_k^{k-1})v|||_{2-\alpha,k} \lesssim h_k^{2\alpha}|||v|||_{2+\alpha,k} \quad \forall v \in V_k. \tag{4.11}$$

$$|||(Id_{k-1} - P_k^{k-1} I_{k-1}^k)v|||_{2-\alpha,k-1} \lesssim h_k^{\alpha}|||v|||_{2,k-1} \quad \forall v \in V_k. \tag{4.12}$$

The assumptions (4.6) and (4.7) are (3.10) and (3.8) respectively. We will prove the rest of the assumptions in this section and complete the convergence analysis. We first state an elementary inequality:

$$(a+b)^2 \leq (1+\theta^2)a^2 + (1+\theta^{-2})b^2 \quad \forall a,b \in \mathbb{R}, \theta \in (0,1). \tag{4.13}$$

In the rest of the section, we use C for a mesh-independent constant. The values of C at different appearances are not necessarily identical.

Lemma 4.1. *The estimate* (4.8) *holds. That is*

$$|||I_{k-1}^k v|||_{2,k}^2 \leq (1+\theta^2)|||v|||_{2,k-1}^2 + C_1\theta^{-2}h_k^{2\alpha}|||v|||_{2+\alpha,k-1}^2 \tag{4.14}$$

for all $v \in V_{k-1}$ and $\theta \in (0,1)$.

Proof. Let $v \in V_{k-1}$ be arbitrary. Then from (3.29), (4.13) and an inverse estimate, we have

$$\begin{aligned}
|||I_{k-1}^k v|||_{2,k}^2 &= |\Pi_k v|_{H^2(\Omega)}^2 \\
&\leq (|v|_{H^2(\Omega)} + |\Pi_k v - v|_{H^2(\Omega)})^2 \\
&\leq (1+\theta^2)|v|_{H^2(\Omega)}^2 + C\theta^{-2}|\Pi_k v - v|_{H^2(\Omega)}^2 \\
&\leq (1+\theta^2)|||v|||_{2,k-1}^2 + C\theta^{-2}h_k^{-4}\|\Pi_k v - v\|_{L_2(\Omega)}^2 \\
&\leq (1+\theta^2)|||v|||_{2,k-1}^2 + C\theta^{-2}h_k^{2\alpha}|||v|||_{2+\alpha,k-1}^2.
\end{aligned}$$

□

Lemma 4.2. *It holds that*

$$\|\!|I_{k-1}^k v|\!\|_{0,k}^2 \le (1+\theta^2)\|\!|v|\!\|_{0,k-1}^2 + C\theta^{-2}h_k^{2\alpha}\|\!|v|\!\|_{\alpha,k}^2 \tag{4.15}$$

for all $v \in V_{k-1}$ *and* $\theta \in (0,1)$.

Proof. Let $v \in V_{k-1}$ be arbitrary and $w = I_{k-1}^k v = \Pi_k v$. Then,

$$\begin{aligned} w(p) &= v(p) \qquad \forall p \in \mathcal{T}_k, \\ \nabla w(p) &= \nabla v(p) \qquad \forall p \in \mathcal{T}_k, \\ \frac{\partial w}{\partial n}(m_e) &= \frac{\partial v}{\partial n}(m_e) \quad \forall e \in \mathcal{E}_k, \end{aligned}$$

where m_e is the midpoint of $e \in \mathcal{E}_k$. Therefore from (2.7) and (3.7) we have

$$\begin{aligned} \|\!|v|\!\|_{0,k}^2 = h_{k-1}^2 \sum_{p\in\mathcal{V}_{k-1}} n(p)v(p)^2 \\ + h_{k-1}^4 \sum_{p\in\mathcal{V}_{k-1}} |\nabla v(p)|^2 + h_{k-1}^4 \sum_{e\in\mathcal{E}_{k-1}} \left[\frac{\partial v}{\partial n}(m_e)\right]^2 \end{aligned} \tag{4.16}$$

and

$$\|\!|w|\!\|_{0,k}^2 = h_k^2 \sum_{p\in\mathcal{V}_k} n(p)v(p)^2 + h_k^4 \sum_{p\in\mathcal{V}_k} |\nabla v(p)|^2 + h_k^4 \sum_{e\in\mathcal{E}_k} \left[\frac{\partial v}{\partial n}(m_e)\right]^2. \tag{4.17}$$

If $p \in \mathcal{V}_{k-1} \setminus \mathcal{V}_{k-1}$, then p is the midpoint of some edge $e \in \mathcal{E}_{k-1}$, which is the common edge of two triangles $T, T' \in \mathcal{V}_{k-1}$. Therefore p is the common vertex of 6 triangles in $\mathcal{T}_k$ and $n(p) = 1$. (cf. Figure 3). The first part of (4.17) can be expressed as

$$\sum_{p\in\mathcal{V}_k} n(p)v(p)^2 = \sum_{p\in\mathcal{V}_{k-1}} v(p)^2 + \sum_{p\in\mathcal{V}_k\setminus\mathcal{V}_{k-1}} n(p)v(p)^2. \tag{4.18}$$

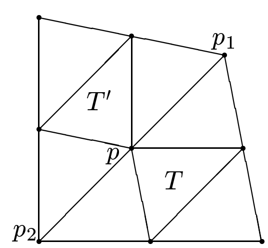


FIGURE 3. A vertex $p \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}$

Suppose p_1 and p_2 are the endpoints of e (cf. Figure 3). Then from (4.13) we have

$$\begin{aligned} v(p) &= [v(p_1) + (v(p) - v(p_1))]^2 \\ &\le (1+\theta^2)v(p_1)^2 + C\theta^{-2}[v(p) - v(p_1)]^2. \end{aligned}$$

By the Mean-Value Theorem and a standard inverse estimate we have

$$[v(p) - v(p_1)]^2 \le |p - p_1|^2 \|\nabla v\|^2_{L_\infty(T)} \le C|v|^2_{H^1(T)}.$$

Then

$$v(p)^2 \le (1 + \theta^2) v(p_1)^2 + C\theta^{-2}|v|^2_{H^1(T)},$$

and similarly

$$v(p)^2 \le (1 + \theta^2) v(p_2)^2 + C\theta^{-2}|v|^2_{H^1(T')}.$$

Therefore

$$v(p)^2 \le \frac{1}{2}(1 + \theta^2)[v(p_1)^2 + v(p_2)^2] + C\theta^{-2}[|v|^2_{H^1(T)} + |v|^2_{H^1(T')}]. \tag{4.19}$$

Taking summation of the inequality above over all $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$ gives

$$\begin{aligned} \sum_{p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} v(p)^2 &\le \frac{1}{2}(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} |S_p| v(p)^2 + C\theta^{-2} \sum_{T \in \mathcal{T}_{k-1}} |v|^2_{H^1(T)} \\ &= 3(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 + C\theta^{-2}|v|^2_{H^1(\Omega)}, \end{aligned}$$

where $|S_p|$ is the number of triangles sharing p as a node. From (4.18) we then have

$$\sum_{p \in \mathcal{V}_k} n(p) v(p)^2 \le 4(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 + C\theta^{-2}|v|^2_{H^1(\Omega)}. \tag{4.20}$$

Let $T \in \mathcal{T}_k$ and e be an edge of T. By a standard inverse estimate we have

$$\left[\frac{\partial v}{\partial n}(m_e)\right]^2 \lesssim \|\nabla v\|^2_{L_\infty(T)} \lesssim h_k^{-2}|v|^2_{H^1(T)}.$$

Therefore

$$h_k^4 \sum_{e \in \mathcal{E}_k} \left[\frac{\partial v}{\partial n}(m_e)\right]^2 \le C h_k^2 |v|^2_{H^1(\Omega)}. \tag{4.21}$$

Similarly

$$h_k^4 \sum_{p \in \mathcal{V}_k} |\nabla v(p)|^2 \le C h_k^2 |v|^2_{H^1(\Omega)}. \tag{4.22}$$

From (2.1), (3.9), (3.12), (4.17), (4.20), (4.21) and (4.22) we have

$$\begin{aligned} \|w\|^2_{0,k} &\le h_k^2 \left[4(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 + C\theta^{-2}|v|^2_{H^1(\Omega)}\right] \\ &\le (1 + \theta^2) h_{k-1}^2 \sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 + C\theta^{-2} h_k^2 \|v\|^2_{1,k-1} \\ &\le (1 + \theta^2)\|v\|^2_{0,k-1} + C\theta^{-2} h_k^{2\alpha} \|v\|^2_{\alpha,k-1}. \end{aligned}$$

□

Lemma 4.3. *The estimate* (4.9) *holds.*

Proof. We use the approach in the proof of Lemma 6.4 in [9].

Let C_* be a constant dominating the C's in (4.14) and (4.15). We define

$$\langle v_1, v_2\rangle_{k-1,\theta} = (1+\theta^2)(v_1, v_2)_{k-1} + C_*\theta^{-2}h_k^{2\alpha}(A_{k-1}^{\alpha/2}v_1, v_2)_{k-1} \tag{4.23}$$

for all $v_1, v_2 \in V_{k-1}$ and $\theta \in (0,1)$. Note that A_{k-1} is symmetric positive definite with respect to the inner product $\langle\cdot,\cdot\rangle_{k-1,\theta}$.

It follows from (4.14), (4.15) and (4.23) that

$$\|\!|I_{k-1}^k v\|\!|_{0,k}^2 \le \left\langle A_{k-1}^0 v, v\right\rangle_{k-1,\theta} \quad \forall v \in V_{k-1},$$
$$\|\!|I_{k-1}^k v\|\!|_{2,k}^2 \le \left\langle A_{k-1}^1 v, v\right\rangle_{k-1,\theta} \quad \forall v \in V_{k-1},$$

By interpolation between Hilbert scales,

$$\|\!|I_{k-1}^k v\|\!|_{2-\alpha}^2 \le \left\langle A_{k-1}^{1-\alpha/2} v, v\right\rangle_{k-1,\theta} = (1+\theta^2)\|\!|v\|\!|_{2-\alpha}^2 + C_*\theta^{-2}h_k^{2\alpha}\|\!|v\|\!|_{2,k-1}^2.$$

□

Lemma 4.4. *It holds that*

$$\|\!|\Pi_{k-1}v\|\!|_{2,k-1}^2 \le (1+\theta^2)\|\!|v\|\!|_{2,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\!|v\|\!|_{2+\alpha,k}^2 \tag{4.24}$$

for all $v \in V_k$ *and* $\theta \in (0,1)$.

Proof. Let $v \in V_k$ be arbitrary. From (3.8), an inverse estimate, (3.28) and (4.13) we have

$$\begin{aligned}
\|\!|\Pi_{k-1}v\|\!|_{2,k-1}^2 &= |\Pi_{k-1}v|_{H^2(\Omega)}^2 \\
&\le \left(|v|_{H^2(\Omega)} + |\Pi_{k-1}v - v|_{H^2(\Omega)}\right)^2 \\
&\le \left(\|\!|v\|\!|_{2,k} + Ch_k^{-2}\|\Pi_{k-1}v - v\|_{L_2(\Omega)}\right)^2 \\
&\le \left(\|\!|v\|\!|_{2,k} + Ch_k^{\alpha}\|\!|v\|\!|_{2+\alpha,k}\right)^2 \\
&\le (1+\theta^2)\|\!|v\|\!|_{2,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\!|v\|\!|_{2+\alpha,k}^2.
\end{aligned}$$

□

Lemma 4.5. *It holds that*

$$\|\!|\Pi_{k-1}v\|\!|_{0,k-1}^2 \le (1+\theta^2)\|\!|v\|\!|_{0,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\!|v\|\!|_{\alpha,k}^2 \tag{4.25}$$

for all $v \in V_k$ *and* $\theta \in (0,1)$.

Proof. Let $v \in V_k$ be arbitrary. From (2.7) we have

$$\|\!|v\|\!|_{0,k}^2 = \frac{1}{6}h_k^2\sum_{T\in\mathcal{T}_k}\sum_{p\in\mathcal{V}_T} v(p)^2 + h_k^4\sum_{p\in\mathcal{V}_k}|\nabla v(p)|^2 + h_k^4\sum_{e\in\mathcal{E}_k}\left[\frac{\partial v}{\partial n}(m_e)\right]^2, \tag{4.26}$$

where $\mathcal{V}_T$ is the set of the vertices of the triangle T.

Let $w = \Pi_{k-1}v$. Then the nodal values of w and v on $\mathcal{T}_{k-1}$ are the same. Therefore

$$\|\!|w\|\!|_{0,k-1}^2 = \frac{1}{6}h_{k-1}^2 \sum_{T\in\mathcal{T}_{k-1}} \sum_{p\in\mathcal{V}_T} v(p)^2 + h_{k-1}^4 \sum_{p\in\mathcal{V}_{k-1}} |\nabla v(p)|^2 + h_{k-1}^4 \sum_{e\in\mathcal{E}_{k-1}} \left[\frac{\partial v}{\partial n}(m_e)\right]^2. \tag{4.27}$$

Let $T \in \mathcal{T}_{k-1}$ be divided into four triangles T_1, T_2, T_3 and T_4 in $\mathcal{T}_k$, whose vertices are labeled as in Figure 4. Then we have

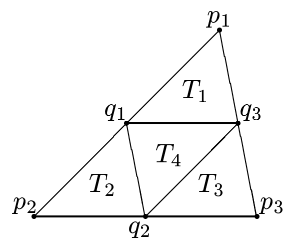


FIGURE 4. A Triangle $T \in \mathcal{T}_{k-1}$ divided into four triangles in $\mathcal{T}_k$

$$\begin{aligned} 4\sum_{p\in\mathcal{V}_T} v(p)^2 &= \sum_{i=1}^3 v(p_i)^2 + 3\sum_{i=1}^3 [v(q_i) + (v(p_i) - v(q_i))]^2 \\ &\le \sum_{i=1}^3 v(p_i)^2 + 3\sum_{i=1}^3 [(1+\theta^2)v(q_i)^2 + (1+\theta^{-2})(v(p_i) - v(q_i))^2] \\ &\le (1+\theta^2)\sum_{i=1}^4 \sum_{p\in\mathcal{V}_{T_i}} v(p)^2 + C\theta^{-2}\sum_{i=1}^4 |v|_{H^1(T_i)}^2. \end{aligned} \tag{4.28}$$

From (2.1) and (4.28) we have

$$h_{k-1}^2 \sum_{p\in\mathcal{V}_T} v(p)^2 \le (1+\theta^2)h_k^2 \sum_{i=1}^4 \sum_{p\in\mathcal{V}_{T_i}} v(p)^2 + C\theta^{-2}h_k^2 \sum_{i=1}^4 |v|_{H^1(T_i)}^2. \tag{4.29}$$

Summing up over all $T \in \mathcal{T}_{k-1}$ gives

$$\begin{aligned} h_{k-1}^2 &\sum_{T\in\mathcal{T}_{k-1}} \sum_{p\in\mathcal{V}_T} v(p)^2 \\ &\le h_k^2(1+\theta^2) \sum_{T\in\mathcal{T}_k} \sum_{p\in\mathcal{V}_T} v(p)^2 + C\theta^{-2}h_k^2 \sum_{T\in\mathcal{T}_k} |v|_{H^1(T)}^2. \end{aligned} \tag{4.30}$$

Similar to (4.21) and (4.22), we have

$$h_{k-1}^4 \sum_{p\in\mathcal{V}_{k-1}} |\nabla v(p)|^2 + h_{k-1}^4 \sum_{e\in\mathcal{E}_{k-1}} \left[\frac{\partial v}{\partial n}(m_e)\right]^2 \le Ch_k^2 |v|_{H^1(\Omega)}^2. \tag{4.31}$$

It follows from (3.9), Lemma 3.2, (4.27), (4.30) and (4.31) that

$$
\begin{aligned}
\|\Pi_{k-1}v\|_{0,k-1}^2 &\le \frac{h_k^2}{6}(1+\theta^2)\sum_{T\in\mathcal{T}_k}\sum_{p\in\mathcal{V}_T} v(p)^2 + C\theta^{-2}h_k^2\sum_{T\in\mathcal{T}_k}|v|_{H^1(T)}^2\\
&\le (1+\theta^2)\|v\|_{0,k}^2 + C\theta^{-2}h_k^2\|v\|_{1,k}^2\\
&\le (1+\theta^2)\|v\|_{0,k}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{\alpha,k}^2.
\end{aligned}
$$

□

Again, from Lemma 4.4 and Lemma 4.5, and interpolation between Hilbert scales, we have the following Lemma.

Lemma 4.6. *It holds that*

$$\|\Pi_{k-1}v\|_{2-\alpha,k-1}^2 \le (1+\theta^2)\|v\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{2,k}^2 \tag{4.32}$$

for all $v\in V_k$ *and* $\theta\in(0,1)$.

Lemma 4.7. *Let* $\zeta_k\in V_k$. *Define* $\zeta\in H_0^2(\Omega)$ *and* $\zeta_{k-1}\in V_{k-1}$ *by* (3.18) *and* (3.30). *Then*

$$\|\zeta_{k-1}-P_k^{k-1}\zeta_k\|_{2-\alpha,k-1} \lesssim h_k^{2\alpha}\|\zeta_k\|_{2+\alpha,k}. \tag{4.33}$$

Proof. Let $w\in V_{k-1}$ be arbitrary. Then from Lemma 3.3 we have

$$
\begin{aligned}
a(\zeta_{k-1}-P_k^{k-1}\zeta_k,w) &= a(\zeta_{k-1},w)-a(P_k^{k-1}\zeta_k,w)\\
&= a(\zeta,w)-a(\zeta_k,\Pi_k w)\\
&= a(\zeta,w)-a(\zeta,\Pi_k w)\\
&= a(\zeta,w-\Pi_k w).
\end{aligned}
$$

A duality estimate for the bilinear form $a(\cdot,\cdot)$ (cf. [8]) states that

$$a(u,v)\lesssim |u|_{H^{2+\alpha}(\Omega)}|v|_{H^{2-\alpha}(\Omega)} \quad \forall u\in H^{2+\alpha}(\Omega)\cap H_0^2(\Omega), v\in H_0^2(\Omega). \tag{4.34}$$

From (3.32), Lemma 3.3 and (4.34) we have

$$
\begin{aligned}
a(\zeta,w-\Pi_k w) &\lesssim |\zeta|_{H^{2+\alpha}(\Omega)}|w-\Pi_k w|_{H^{2-\alpha}(\Omega)}\\
&\lesssim h_k^{2\alpha}\|\zeta_k\|_{2+\alpha,k}\|w\|_{2+\alpha,k-1}.
\end{aligned}
$$

Therefore

$$a(\zeta_{k-1}-P_k^{k-1}\zeta_k,w)\lesssim h_k^{2\alpha}\|\zeta_k\|_{2+\alpha,k}\|w\|_{2+\alpha,k-1}.$$

Since $w\in V_{k-1}$ is arbitrary, it follows from (3.11) that

$$\|\zeta_{k-1}-P_k^{k-1}\zeta_k\|_{2-\alpha,k-1}\lesssim h_k^{2\alpha}\|\zeta_k\|_{2+\alpha,k}.$$

□

Lemma 4.8. *The estimate* (4.10) *holds. That is*

$$\|P_k^{k-1}v\|_{2-\alpha,k-1}^2 \le (1+\theta^2)\|v\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{2,k}^2 \tag{4.35}$$

for all $v\in V_k$ *and* $\theta\in(0,1)$.

Proof. Let $\zeta_k \in V_k$ be arbitrary, and define $\zeta \in H_0^2(\Omega)$ and $\zeta_{k-1} \in V_{k-1}$ by (3.18) and (3.30). From (4.13) and Lemma 4.7 we have

$$\begin{aligned}\|P_k^{k-1}\zeta_k\|_{2-\alpha,k-1}^2 &\le (\|\zeta_{k-1}\|_{2-\alpha,k-1} + \|P_k^{k-1}\zeta_{k-1} - \zeta_k\|_{2-\alpha,k-1})^2 \\ &\le (1+\theta^2)\|\zeta_{k-1}\|_{2-\alpha,k-1}^2 + C\theta^{-2}\|P_k^{k-1}\zeta_{k-1} - \zeta_k\|_{2-\alpha,k-1}^2 \\ &\le (1+\theta^2)\|\zeta_{k-1}\|_{2-\alpha,k-1}^2 + C\theta^{-2}h_k^{4\alpha}\|\zeta_k\|_{2+\alpha,k}^2.\end{aligned}$$

But

$$\begin{aligned}\|\zeta_{k-1}\|_{2-\alpha,k-1}^2 &\le (\|\Pi_{k-1}\zeta_k\|_{2-\alpha,k-1} + \|\zeta_{k-1} - \Pi_{k-1}\zeta_k\|_{2-\alpha,k-1})^2 \\ &\le (1+\theta^2)\|\Pi_{k-1}\zeta_k\|_{2-\alpha,k-1}^2 + C\theta^{-2}\|\zeta_{k-1} - \Pi_{k-1}\zeta_k\|_{2-\alpha,k-1}^2.\end{aligned}$$

Moreover, let $w = \zeta_{k-1} - \zeta_k \in V_{k-1} + V_k$. Then from Lemma 3.1, Lemma 3.2, Lemma 3.3, and (3.24) we have

$$\begin{aligned}\|\zeta_{k-1} - \Pi_{k-1}\zeta_k\|_{2-\alpha,k-1} &= \|\Pi_{k-1}w\|_{2-\alpha,k-1} \\ &\approx |\Pi_{k-1}w|_{H^{2-\alpha}(\Omega)} \\ &\lesssim |w|_{H^{2-\alpha}(\Omega)} \\ &= |\zeta_{k-1} - \zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\le |\zeta_{k-1} - \zeta|_{H^{2-\alpha}(\Omega)} + |\zeta - \zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\lesssim h_k^{2\alpha}|\zeta|_{H^{2+\alpha}(\Omega)} \lesssim h_k^{4\alpha}\|\zeta_k\|_{2+\alpha,k}.\end{aligned}$$

Putting these estimates together, and by Lemma 4.4 and an inverse estimate, we have

$$\begin{aligned}\|P_k^{k-1}\zeta_k\|_{2-\alpha,k-1}^2 &\le (1+\theta^2)^2\|\Pi_{k-1}\zeta_k\|_{2-\alpha,k-1}^2 + C\theta^{-2}h_k^{4\alpha}\|\zeta_k\|_{2+\alpha,k}^2 \\ &\le (1+\theta^2)^3\|\zeta_k\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{2,k}^2 + C\theta^{-2}h_k^{4\alpha}\|\zeta_k\|_{2+\alpha,k}^2 \\ &\le (1+\theta^2)^3\|\zeta_k\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{2,k}^2.\end{aligned}$$

The lemma follows since $\theta \in (0,1)$ is arbitrary. □

Lemma 4.9. *The estimate* (4.11) *holds.*

Proof. Let $\zeta_k \in V_k$ be arbitrary. Define $\zeta \in H_0^2(\Omega)$ and $\zeta_{k-1} \in V_{k-1}$ by (3.18) and (3.30). Then from Lemma 3.1, Lemma 3.2, Lemma 3.3, (3.25) and Lemma 4.7,

$$\begin{aligned}&\|\zeta_k - I_{k-1}^k P_k^{k-1}\zeta_k\|_{2-\alpha,k} \\ &\quad\approx |\zeta_k - \Pi_k P_k^{k-1}\zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\quad= |\Pi_k(\zeta_k - P_k^{k-1}\zeta_k)|_{H^{2-\alpha}(\Omega)} \\ &\quad\lesssim |\zeta_k - P_k^{k-1}\zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\quad\le |\zeta_k - \zeta|_{H^{2-\alpha}(\Omega)} + |\zeta - \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} + |\zeta_{k-1} - P_k^{k-1}\zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\quad\lesssim h_k^{2\alpha}\|\zeta_k\|_{2+\alpha,k}.\end{aligned}$$

□

Before proving the last assumption, we note from (3.9), Lemma 3.2 and Lemma 4.8 that

$$|P_k^{k-1}v|_{H^{2-\alpha}(\Omega)} \lesssim |v|_{H^{2-\alpha}(\Omega)} \quad \forall v \in V_k. \tag{4.36}$$

Lemma 4.10. *The estimate* (4.12) *holds.*

Proof. Let $\zeta_{k-1} \in V_{k-1}$ and define $\zeta \in H_0^2(\Omega)$ by

$$a(\zeta, \phi) = a(\zeta_{k-1}, Q_{k-1}\phi) \quad \forall \phi \in V_{k-1},$$
$$a(\zeta_k, v) = a(\zeta, v) \quad \forall v \in V_k.$$

Then from Lemma 3.1, Lemma 3.2, Lemma 3.3, Corollary 3.6, Lemma 4.7 and (4.36),

$$\begin{aligned}
&\|\!|\zeta_{k-1} - P_k^{k-1} I_{k-1}^k \zeta_{k-1}\|\!|_{2-\alpha,k-1} \\
&\quad\approx |\zeta_{k-1} - P_k^{k-1} I_{k-1}^k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} \\
&\quad= |\zeta_{k-1} - P_k^{k-1} \Pi_k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} \\
&\quad\leq |\zeta_{k-1} - P_k^{k-1} \zeta_k|_{H^{2-\alpha}(\Omega)} + |P_k^{k-1}(\zeta_k - \Pi_k \zeta_{k-1})|_{H^{2-\alpha}(\Omega)} \\
&\quad\lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} + |\zeta_k - \Pi_k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} \\
&\quad\lesssim h^{2\alpha} \|\!|\zeta_{k-1}\|\!|_{2+\alpha,k-1} + |\zeta_k - \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} + |\zeta_{k-1} - \Pi_k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} \\
&\quad\lesssim h_k^{2\alpha} \|\!|\zeta_{k-1}\|\!|_{2+\alpha,k-1}.
\end{aligned}$$

□

We have proved all of the assumptions. The following theorems are then established.

Theorem 4.11. *There exist a positive constant C and a positive integer m_0, both independent of k, such that for all $m \geq m_0$ and $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{V}}(k, g, z_0, m)\|_a \leq C m^{-\alpha/2} \|z - z_0\|_a, \tag{4.37}$$

where z is the exact solution of (2.10).

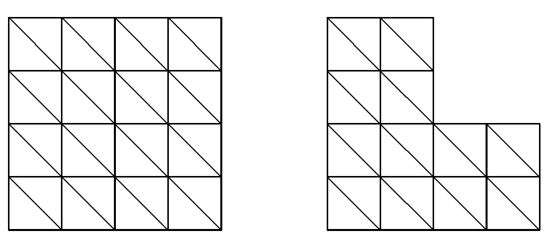
Theorem 4.12. *There exist a positive constant C and a positive integer m_0, both independent of k, such that for all $m \geq m_0$ and $z_0 \in V_k$,*

$$\|z - MG_{\mathcal{F}}(k, g, z_0, m)\|_a \leq C m^{-\alpha/2} \|z - z_0\|_a, \tag{4.38}$$

where z is the exact solution of (2.10).

5. Numerical Results

In this section we present some numerical results to illustrate the theorems in the previous section.



(a) Square domain (b) L-shaped domain

Figure 5. The triangulation $\mathcal{T}_k$ for $k = 2$.

Our first experiment is done on the unit square domain (cf. Figure 5 (a)). A family of triangulations $\{\mathcal{T}_k\}_{k\geq 1}$ for the domain is obtained by regular subdivision.

On Figure 5, The second level triangulation $\mathcal{T}_2$ is shown. We compute the contraction numbers for V-cycle, F-cycle and W-cycle algorithms at different levels. The results are shown in Table 1, 2 and 3.

$\gamma_{m,k,v}$	m=1	m=4	m=7	m=10	m=13	m=16	m=19	m=22
k=3	0.96	0.88	0.82	0.78	0.74	0.71	0.67	0.66
k=4	0.95	0.88	0.82	0.78	0.74	0.71	0.68	0.66
k=5	0.95	0.88	0.82	0.77	0.74	0.71	0.68	0.65
k=6	0.95	0.88	0.82	0.77	0.74	0.71	0.68	0.66
k=7	0.95	0.88	0.82	0.77	0.74	0.71	0.68	0.66
k=8	0.95	0.88	0.82	0.77	0.74	0.71	0.68	0.66

TABLE 1. Contraction numbers for V-cycle algorithms on the unit square

$\gamma_{m,k,f}$	m=1	m=4	m=7	m=10	m=13	m=16	m=19	m=22
k=3	0.96	0.88	0.82	0.76	0.73	0.70	0.68	0.65
k=4	0.96	0.88	0.82	0.77	0.74	0.71	0.68	0.65
k=5	0.96	0.88	0.81	0.77	0.74	0.71	0.68	0.65
k=6	0.96	0.88	0.81	0.77	0.74	0.70	0.68	0.65
k=7	0.96	0.88	0.81	0.77	0.74	0.70	0.68	0.65
k=8	0.96	0.88	0.81	0.77	0.74	0.71	0.68	0.65

TABLE 2. Contraction numbers for F-cycle algorithms on the unit square

$\gamma_{m,k,w}$	m=1	m=4	m=7	m=10	m=13	m=16	m=19	m=22
k=3	0.95	0.88	0.82	0.76	0.71	0.69	0.66	0.65
k=4	0.95	0.88	0.81	0.76	0.72	0.70	0.67	0.64
k=5	0.95	0.87	0.81	0.76	0.73	0.70	0.67	0.65
k=6	0.95	0.87	0.80	0.76	0.73	0.70	0.67	0.65
k=7	0.95	0.87	0.80	0.76	0.73	0.70	0.67	0.65
k=8	0.95	0.87	0.80	0.76	0.73	0.70	0.67	0.65

TABLE 3. Contraction numbers for W-cycle algorithms on the unit square

The results show that the algorithms converge for m as small as 1. In contrast with [25], where V-cycle algorithm using the nonconforming Morley element needs m greater than 50 for convergence, we can see the advantage of conforming methods.

On the other hand, we can also see from the tables that the convergence rates of the three algorithms are almost the same. We do not observe the superior performance of the F-cycle algorithm as we saw in [25].

Since the domain is convex, we have full elliptic regularity, i.e., the elliptic regularity index $\alpha = 1$. According to Theorems 4.11 and 4.12, there exists a constant C, independent of k and m, such that

$$m^{1/2}\gamma_{k,m,v} \leq C.$$

$m^{1/2}\gamma_{m,k,v}$	m=10	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	2.42	2.97	3.24	3.28	3.40	3.35	3.49	3.44
k=4	2.45	3.02	3.29	3.27	3.29	3.27	3.39	3.29
k=5	2.45	3.01	3.26	3.26	3.31	3.30	3.21	3.13
k=6	2.44	3.00	3.25	3.25	3.31	3.26	3.15	3.09
k=7	2.45	3.00	3.25	3.25	3.31	3.24	3.14	3.06
k=8	2.44	3.01	3.24	3.24	3.31	3.23	3.13	3.05

TABLE 4. V-cycle results on the unit square

We can see these properties from Table 4. It turns out that the constant C could be just 4.

Next, we use an L-shaped domain (cf. Figure 5 (b)). For this domain, the index of elliptic regularity is $\alpha_* = 0.5444837368$. (cf. [17]). The numerical results are shown in Table 5 and Table 6. They are also consistent with the theoretical results.

$m^{\alpha_*/2}\gamma_{m,k,v}$	m=10	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	1.45	1.49	1.46	1.41	1.36	1.30	1.25	1.19
k=4	1.45	1.52	1.47	1.45	1.39	1.31	1.24	1.21
k=5	1.45	1.51	1.49	1.43	1.36	1.28	1.24	1.23
k=6	1.44	1.52	1.49	1.43	1.36	1.28	1.20	1.21
k=7	1.44	1.52	1.49	1.44	1.36	1.27	1.20	1.14
k=8	1.45	1.52	1.49	1.43	1.35	1.27	1.19	1.13

TABLE 5. V-cycle results on the L-shaped domain

$m^{\alpha_*/2}\gamma_{m,k,f}$	m=10	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	1.43	1.50	1.47	1.44	1.36	1.32	1.23	1.18
k=4	1.44	1.50	1.49	1.45	1.37	1.30	1.21	1.13
k=5	1.45	1.51	1.48	1.44	1.36	1.29	1.21	1.15
k=6	1.44	1.51	1.49	1.45	1.36	1.28	1.20	1.13
k=7	1.44	1.51	1.49	1.44	1.36	1.28	1.20	1.12
k=8	1.44	1.51	1.49	1.44	1.36	1.28	1.19	1.11

TABLE 6. F-cycle results on the L-shaped domain

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