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trigonometric series

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Abstract

The divergence set of the trigonometric series

$$S(x) := \sum_{n=1}^{\infty} \frac{\{nx\}}{n}; \quad T(x) := \sum_{n=1}^{\infty} d(n) \frac{\sin 2\pi nx}{\pi n}; \quad U(x) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin 2\pi mn x}{\pi mn}$$

is studied, where $\{\cdot\} := \frac{1}{2} - \text{frac}(\cdot)$, and $\text{frac}(x)$ denotes the fractional part of the real number x ; $d(n) := \sum_{d|n} 1$ – the divisor function. The complete characterization of this set is established in terms of the continued fractions.

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This article is a development of the recent works of the author [8] and M. Garaev [2], concerning double trigonometric series with the polynomial phase. The reasons for the author's interest in the convergence problems for such series are explained in [8].

Our goal is to study the convergence and the divergence of the following three trigonometric series

$$S(x) := \sum_{n=1}^{\infty} \frac{\{nx\}}{n}; \quad T(x) := \sum_{n=1}^{\infty} d(n) \frac{\sin 2\pi nx}{\pi n}; \quad U(x) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin 2\pi mn x}{\pi mn}$$

Here $\{\cdot\} := \frac{1}{2} - \text{frac}(\cdot)$, and $\text{frac}(x)$ denotes the fractional part of the real number x ; $d(n) := \sum_{d|n} 1$ – the divisor function.

It will be proved that for every real x these series are equiconvergent (divergent), and the sums are the same, wherever they exist. In particular, we establish an exact description of the set, where the series diverge. Naturally, this exceptional set turns out to be rather “thin”. To describe it, one needs to address the arithmetical terms, namely, the continued fractions. We emphasize that the material of this paper is rather coherent with some papers by G.H. Hardy¹

¹*I have never done anything “useful”. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.* G.H. Hardy, *Apology*, see [3], p. 3.

and J.E. Littlewood on diophantine approximation, cf. [3]. In fact, $S(x)$ coincides with the Dirichlet's series

$$\mathcal{H}(s, x) := \sum_{n=1}^{\infty} \frac{\{nx\}}{n^s}, \quad s = \sigma + it,$$

at the point $s = 1$. This series was introduced by E. Hecke [6]. As a function of the complex variable s for fixed x , $\mathcal{H}(s, x)$ was studied by Hecke in [6], and by G.H. Hardy and J.E. Littlewood in [4], [5], see also [3], pp. 197 – 252.

That S and T coincide *formally*, follows by the application to the double series U of two different summation procedures:

$$S(x) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{n} \left(\lim_{M \rightarrow \infty} \sum_{m=1}^M \frac{\sin 2\pi mn x}{\pi m} \right), \quad T(x) = \lim_{N \rightarrow \infty} \sum_{(m,n): nm \leq N} \frac{\sin 2\pi mn x}{\pi mn}.$$

However, it is a priori not clear, whether these limits exist, and if they do exist, whether their values are the same. This question is non-trivial, as it was confirmed by a result of M. Garaev [2]. Garaev considered the sequence of the square partial sums of U

$$U_{\square_{N,N}}(x) := \sum_{m=1}^N \sum_{n=1}^N \frac{\sin 2\pi mn x}{\pi mn}, \quad N = 1, 2, \dots$$

He established, that there exist $x \in \mathbb{R}$ for which this sequence is divergent as $N \rightarrow \infty$. As an example of such x , Garaev pointed out

$$x = \sum_{j=1}^{\infty} \frac{1}{p_j}$$

where $p_1 := 2$, $p_{j+1} := p_j^{j p_j + 1}$, $j = 1, 2, \dots$

We apply the summation of the series U over expanding families of coordinate-wise convex domains:

$$U_{\Omega}(x) := \sum_{(m,n) \in \Omega} \frac{\sin 2\pi mn x}{\pi mn}.$$

A domain Ω in the first quadrant \mathbb{R}_+^2 on the real plane \mathbb{R}^2 is called coordinate-wise convex, iff for every line, parallel to one of the coordinate axes, the intersection of Ω with this line is an interval (possibly, empty or infinite). Let us denote \mathcal{U} the class of all coordinate-wise convex domains on \mathbb{R}_+^2 . Obviously, domains that are convex in the usual sense (such as squares, rectangles, discs) are also coordinate-wise convex. Along with this, non-convex domains such as *hyperbolic crosses* $\mathfrak{X}_N := \{(m, n) : 1 \leq mn \leq N\}$, are nevertheless coordinate-wise convex.

This type of summation of the multiple Fourier series was introduced by S.A. Telyakovskii [10] in relation with the study of the multiplicative sin-polynomials

$$\sum_{(m,n) \in \Omega} \frac{(\sin 2\pi mx)(\sin 2\pi nx)}{mn}.$$

To approximate the full sum of a double series, a domain Ω has to include a “large” square of the type $[1, M] \times [1, M]$. Therefore, the following characteristic of Ω is significant in the problem of convergence:

$$M(\Omega) := \max\{M : [1, M] \times [1, M] \subset \Omega\}.$$

Let us call a sequence of domains $\{\Omega_r\}_{r=1}^{\infty} \subset \mathfrak{U}$ *expanding*, if $M(\Omega_r) \rightarrow \infty$, $r \rightarrow \infty$. If, in addition to this property, the equation $M(\Omega_r) = M$ has a solution $r = r(M)$ for every sufficiently large $M \in \mathbb{N}$, we will say that the sequence of domains is *expanding and full*.

For $x \in (0, 1)$, consider its continued fraction and the sequence of the convergents, cf. [7], Chapter 10, or [11], Chapter 1:

$$x = \frac{1}{k_1 + \frac{1}{k_2 + \dots}} = [k_1, k_2, \dots], \quad \frac{a_j}{q_j} := [k_1, k_2, \dots, k_j], \quad j = 1, 2, \dots \quad (1)$$

The natural numbers k_1, k_2, \dots in this representation are known as partial quotients of x .

Denote $q_j(x)$ the denominators of the convergents of x , and consider the following series

$$\Xi(x) := \sum_{j=1}^{\infty} \frac{(-1)^j \ln q_{j+1}(x)}{q_j(x)}, \quad \Upsilon(x) := \sum_{j=0}^{\infty} \frac{\ln q_{j+1}(x)}{q_j(x)}$$

(if x is a rational number, $x = \frac{a}{q}$ where $(a, q) = 1$, then these series are finite sums, and $\max q_{j+1} = q$).

The following statement provides a complete description of the divergence set of the series U in the light of the given definitions.

Theorem 1 A. *If for a given $x \in \mathbb{R}$ the series $\Xi(x)$ converges, then the limit*

$$U(x) = \lim_{r \rightarrow \infty} U_{\Omega_r}(x)$$

exists and its value is unique for all expanding sequences $\{\Omega_r\}_{r=1}^{\infty}$ of coordinate-wise convex domains.

B. *If $\Xi(x)$ diverges, and the sequence of domains $\{\Omega_r\}_{r=1}^{\infty} \subset \mathfrak{U}$ is expanding and full, then the corresponding sequence of the partial sums $\{U_{\Omega_r}(x)\}_{r=1}^{\infty}$ is also divergent.*

In particular, the sequences of the partial sums

$$U_{\square_{M,N}}(x) := \sum_{m=1}^M \sum_{n=1}^N \frac{\sin 2\pi mn x}{\pi mn}; \quad U_{\square_{N,N}}(x) = \sum_{m=1}^N \sum_{n=1}^N \frac{\sin 2\pi mn x}{\pi mn};$$

$$S_N(x) := \sum_{n=1}^N \frac{\{nx\}}{n}; \quad T_N(x) := \sum_{n=1}^N d(n) \frac{\sin 2\pi n x}{\pi n}$$

are equiconvergent as $M, N \rightarrow \infty$ for all real x . The limits of these sequences are the same, wherever they exist, i. e. for $x \in \mathbb{R} \setminus \mathcal{T}$, where \mathcal{T} is the divergence set of the series Ξ .

Remark 1. Recently, the author [8] studied a modification of the series U with “a somewhat bigger denominator”

$$V(x) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin 2\pi mn x}{m^2 + n^2}.$$

In contrast to U , the series V converges for all real x , and its sum is everywhere bounded.

Denote \mathcal{T}' the divergence set of the series Υ . Obviously, $\mathcal{T} \subset \mathcal{T}'$; as we already mentioned, the sets $\mathcal{T}, \mathcal{T}'$ are rather “thin”. To quantify this property, let us apply two following characteristics:

1) integral estimates of the remainder terms of the series Υ :

$$\Upsilon_j := \sum_{\nu=j}^{\infty} \varphi_{\nu}, \quad \varphi_{\nu}(x) := \frac{\ln q_{\nu+1}(x)}{q_{\nu}(x)}, \quad \nu, j = 0, 1, \dots;$$

2) estimates of the “logarithmic dimension” of the set \mathcal{T}' .

In the next theorem F_j denotes the j -th Fibonacci number, i. e. $F_0 = F_1 := 1$, $F_{j+1} = F_j + F_{j-1}$, $j = 1, 2, \dots$

Theorem 2 A) *The function Υ is exponentially integrable:*

$$\int_0^1 e^{\lambda \Upsilon(x)} dx < \infty, \quad 0 < \lambda < 1.$$

Moreover, if a sequence of positive numbers λ_j satisfies the condition $\lambda_j = o(F_j/j)$, $j \rightarrow \infty$, then

$$\int_0^1 (e^{\lambda_j \Upsilon_j(x)} - 1) dx \rightarrow 0.$$

B) Let $\alpha > 0$ and ε - an arbitrarily small positive number. Then there exist a family of intervals $\mathcal{I} = \mathcal{I}(\alpha, \varepsilon) = \{I\}$ such that²

$$\mathcal{T}' \subset \bigcup_{I \in \mathcal{I}(\alpha, \varepsilon)} I, \quad \sum_{I \in \mathcal{I}(\alpha, \varepsilon)} \left(\ln \frac{1}{|I|} \right)^{-\alpha} \leq \varepsilon^\alpha.$$

Corollary. *The Hausdorff dimension (cf. [1]) of the exceptional sets $\mathcal{T}, \mathcal{T}'$ equals 0.*

Remark 2. It can be proved, that in contrast to the sin-series U , the sum of its trigonometric conjugate, i. e. the cos-series

$$\tilde{U}(x) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos 2\pi mn x}{\pi mn} = \sum_{n=1}^{\infty} d(n) \frac{\cos 2\pi n x}{\pi n}$$

is not exponentially integrable for any positive λ in the exponent. Therefore, the function $U(x)$ does not belong to the class BMO, i. e. does not possess the bounded mean oscillation, see e. g. [9], Ch. 4.

Proof of the theorems. For a fixed bounded domain $\Omega \in \mathfrak{U}$, denote $M := M(\Omega)$. Split the corresponding sum U_Ω along the diagonal $m = n$:

$$U_\Omega(x) = \sum_{m=1}^M \frac{1}{m} \sum_{n=m}^{n_m} \frac{\sin 2\pi mn x}{\pi n} + \sum_{n=1}^M \frac{1}{n} \sum_{m=n}^{m_n} \frac{\sin 2\pi mn x}{\pi m} - \sum_{m=1}^M \frac{\sin 2\pi m^2 x}{\pi m^2}.$$

For $m \in \mathbb{N}$, the following well-known estimate is true

$$R_m(y) := \frac{1}{m} \sup_{N \geq m} \left| \sum_{n=m}^N \frac{\sin 2\pi n y}{\pi n} \right| = O \left(\frac{1}{m} \min \left(1, \frac{1}{m \|y\|} \right) \right), \quad (2)$$

where $\|y\|$ denotes the distance from $y \in \mathbb{R}$ to the nearest integer. Moreover, the following “asymptotic formula” holds

$$\sigma_m(y) := \frac{1}{m} \sum_{n=m}^{n_m} \frac{\sin 2\pi n y}{\pi n} = \frac{\text{sign } y}{2m} + O(|y|) + O \left(\frac{1}{m} \min \left(1, \frac{1}{M|y|} \right) \right), \quad |y| \leq \frac{1}{2}. \quad (3)$$

In the estimate of the second remainder term we utilized the inequality $n_m \geq M$ which follows from the definition of $M(\Omega)$.

²Here and in the sequel we use the notation $|I|$ for the length of the interval $I \subset \mathbb{R}$. Also, everywhere below the constants in the symbol O are absolute.

Obviously, we have

$$\sum_{m=1}^M \frac{1}{m} \sum_{n=m}^{n_m} \frac{\sin 2\pi mnx}{\pi n} = \sum_{m=1}^M \sigma_m(mx).$$

Let $q_0 := 1$, and subdivide the summation into the intervals of the form $[q_j, q_{j+1})$, where q_j are the denominators of the continued fraction (1):

$$\sum_{m=1}^M \sigma_m(mx) = \sum_{j=0}^{J-1} \sum_{m \in [q_j, q_{j+1})} \sigma_m(mx) + \sum_{m \in [q_J, M]} \sigma_m(mx), \quad J := \max\{j : q_j \leq M\}.$$

The following properties of the partial quotients and the convergents are well known, cf.[7], Chapter 10, or [11], Chapter 1:

$$\begin{aligned} x &= \frac{a_j}{q_j} + \delta_j, \quad \frac{1}{2q_j q_{j+1}} < |\delta_j| \leq \frac{1}{q_j q_{j+1}}, \quad \text{sign } \delta_j = (-1)^j; \\ q_{j+1} &= k_{j+1} q_j + q_{j-1}; \quad (a_j, q_j) = 1; \quad \frac{a_{j+1}}{q_{j+1}} - \frac{a_j}{q_j} = \frac{(-1)^j}{q_j q_{j+1}}, \quad j = 0, 1, \dots \end{aligned} \quad (4)$$

where $a_0 := 0$.

Fix an integer $j \geq 0$, and for brevity denote $a_j := a$, $q_j := q$, $k_{j+1} := K$, $q_{j+1} := Q$, $\delta_j := \delta$. Now we estimate the sum

$$A := \sum_{m \in [q, Q]} \sigma_m(mx).$$

We split this sum as follows:

$$A = B + C, \quad B := \sum_{m \in \mathcal{B}} \sigma_m(mx), \quad C := \sum_{m \in \mathcal{C}} \sigma_m(mx)$$

where

$$\mathcal{B} := \left\{ m \in [q, Q), \quad \|mx\| \geq \frac{1}{q} \right\}, \quad \mathcal{C} := \left\{ m \in [q, Q), \quad \|mx\| < \frac{1}{q} \right\}.$$

Since $(a, q) = 1$, for each fixed $k \in \mathbb{N}$ the set of numbers $ma = (kq + l)a$, $l = 0, 1, \dots, q - 1$ represents all residues mod q . Therefore,

$$\sum_{kq < m < (k+1)q} \left\| \frac{ma}{q} \right\|^{-1} = \sum_{0 < l < q} \left\| \frac{l}{q} \right\|^{-1} = O(q \ln(eq)).$$

Moreover, if $q \leq m < Q$, then by (4) we also have

$$\left| mx - \frac{ma}{q} \right| \leq \frac{m}{qQ} \leq \frac{1}{q},$$

and hence, using the estimate $|\sigma_m| \leq R_m$, we obtain from (2):

$$\begin{aligned} |B| &\leq \sum_{m \in \mathcal{B}} |R_m(mx)| \ll \sum_{k=1}^{\infty} \sum_{m \in (kq, (k+1)q), \|mx\| \geq 1/q} \frac{1}{m^2 \|mx\|} \\ &\ll \sum_{k=1}^{\infty} \frac{q \ln(eq)}{k^2 q^2} \ll \frac{\ln(eq)}{q}. \end{aligned} \quad (5)$$

Turn to the estimate of the sum C . One can assume that $\delta_j = \delta > 0$ in (4). Then \mathcal{C} is contained in the union of the two following (finite) progressions

$$\mathcal{C} \subset \mathcal{C}_1 \cup \mathcal{C}_2, \quad \mathcal{C}_1 := \{m = kq, 1 \leq k \leq K\}, \quad \mathcal{C}_2 := \{m = kq + l^*, 1 \leq k \leq K\}$$

where l^* is the residue of the number $-1 \pmod{q}$, i. e. (cf. (4)) $l^* = q \operatorname{frac}\left(\frac{aQ}{q}\right)$. We have

$$\|mx\| = m\delta, \quad m \in \mathcal{C}_1; \quad \|mx\| = \frac{1}{q} - m\delta, \quad m \in \mathcal{C}_2.$$

For the sum of $\sigma_m(mx)$ over the part of \mathcal{C}_1 where $\|mx\| \geq 1/m$, and also the whole progression \mathcal{C}_2 , we again apply the estimate $|\sigma_m| \leq R_m$ and (2):

$$\begin{aligned} &\sum_{m \in \mathcal{C}_2} |\sigma_m(mx)| + \sum_{m \in \mathcal{C}_1, \|mx\| \geq 1/m} |\sigma_m(mx)| \\ &\ll \sum_{m \in \mathcal{C}_1, \|mx\| \geq 1/m} \frac{1}{m^2 \|mx\|} + \sum_{m \in \mathcal{C}_2, m \leq Q/2} \frac{1}{m^2 \|mx\|} + \sum_{m \in \mathcal{C}_2, Q/2 \leq m < Q} \frac{1}{m} \\ &\ll \sum_{k \geq 1/(q\sqrt{\delta})} \frac{1}{k^3 q^3 \delta} + \sum_{k \geq 1} \frac{q}{k^2 q^2} + \sum_{K/2 \leq k \leq K} \frac{1}{kq} \ll \frac{1}{q}. \end{aligned} \quad (6)$$

Now we consider the sum of $\sigma_m(mx)$ over the part of the progression \mathcal{C}_1 where $\|mx\| < 1/m$, i. e. $m = kq$ and $k < 1/(q\sqrt{\delta})$. On this part of \mathcal{C}_1 , we use the ‘‘asymptotic formula’’ (3) for σ_m . The summation of the first remainder term in this formula, with $m = kq$, $y = \|mx\| = kq\delta$, $1 \leq k < 1/(q\sqrt{\delta})$, results in the quantity

$$\sum_{m \in \mathcal{C}_1, \|mx\| < 1/m} \|mx\| = \sum_{1 \leq k < 1/(q\sqrt{\delta})} kq\delta \ll \frac{1}{q}. \quad (7)$$

In dealing with the sum of the second remainder terms in (3), we can ignore the condition $\|mx\| < 1/m$, but instead need to utilize the lower estimate $\delta > 1/(2qQ)$, cf. (4). Thereby, we

obtain for this sum the following estimate:

$$\begin{aligned} \sum_{m \in \mathcal{C}_1} \frac{1}{m} \min \left(1, \frac{1}{M \|mx\|} \right) &\leq \sum_{k \geq 1} \frac{1}{kq} \min \left(1, \frac{1}{Mkq\delta} \right) \\ &= \sum_{1 \leq k \leq 1/(Mq\delta)} \frac{1}{kq} + \sum_{k \geq 1/Mq\delta} \frac{1}{Mk^2q^2\delta} \ll \frac{1}{q} \ln \left(1 + \frac{Q}{M} \right) e. \end{aligned} \quad (8)$$

For the sum of the main terms in (3), making use of the two-sided estimate $1/(qQ) \geq \delta > 1/(2qQ)$, cf. (4), we have

$$\sum_{m \in \mathcal{C}_1, \|mx\| \leq 1/m} \frac{1}{2m} = \sum_{k \leq 1/(q\sqrt{\delta})} \frac{1}{2kq} = \frac{1}{2q} \ln \left(\frac{e}{q\sqrt{\delta}} \right) + O \left(\frac{1}{q} \right) = \frac{\ln eQ}{4q} + O \left(\frac{\ln eQ}{q} \right). \quad (9)$$

Here we used the assumption that $\delta = \delta_j > 0$. In a general case, according to (3) and (4), the result has to be multiplied by $\text{sign } \delta_j = (-1)^j$. Thus, summarizing the estimates (5) – (9), we see that if $M = M(\Omega) \in [q_J, q_{J+1})$ then

$$U_\Omega(x) = \frac{1}{2} \sum_{j=0}^J \left[\frac{(-1)^j \ln q_{j+1}}{q_j} + \varepsilon_j^M \right] + O \left(\frac{1}{q_J} \ln \frac{q_{J+1}}{M} \right), \quad \varepsilon_j^M = O \left(\frac{\ln q_j}{q_j} \right). \quad (10)$$

with an absolute constant in the symbol O . In addition, for each fixed j the limit ε_j^∞ of the quantities ε_j^M as $M = M(\Omega) \rightarrow \infty^3$ exists, and (cf. (3))

$$\begin{aligned} \varepsilon_j^\infty &= \lim_{M \rightarrow \infty} \varepsilon_j^M = 2 \sum_{m \in [q_j, q_{j+1})} \frac{1}{m} \sum_{n=m}^{\infty} \frac{\sin 2\pi mn x}{\pi n} - \sum_{m \in [q_j, q_{j+1})} \frac{\sin 2\pi m^2 x}{\pi m^2} - \frac{(-1)^j \ln q_{j+1}}{2q_j}, \\ \sum_j |\varepsilon_j^\infty| &\leq \sum_j \sup_M |\varepsilon_j^M| \ll \sum_j \frac{\ln q_j}{q_j} \ll 1. \end{aligned} \quad (11)$$

Now we finish the proof of theorem 1. If the series $\Xi(x)$ converges, i. e. $x \in (0, 1) \setminus \mathcal{T}$, then in particular, $(\ln q_{J+1})/q_J \rightarrow 0$ as $J \rightarrow \infty$. Since $M \geq q_J$ we conclude with the help of (10) and (11) that the sequence $\{U_{\Omega_r}(x)\}$ has a limit whenever the sequence of domains $\{\Omega_r\} \in \mathcal{U}$ is expanding. The relation (11) also implies the independence of this limit from the latter sequence, and the claim A follows.

Let us assume that a sequence of domains $\{\Omega_r\} \in \mathcal{U}$ is expanding and full. The latter means, that for every sufficiently large J there exists r_J such that $M(\Omega_{r_J}) = q_{J+1} - 1$. Then the corresponding “remainder term” in (10) tends to 0 for $J \rightarrow \infty$:

$$O \left(\frac{1}{q_J} \ln \frac{q_{J+1}}{M(\Omega_{r_J})} \right) = O \left(\frac{1}{q_J} \right).$$

³This means, that the domain $\Omega \in \mathcal{U}$ is expanding

Therefore, also keeping in mind (11), we infer that the relations (10) imply the equiconvergence of the sequence of the partial sums of the series $\Xi(x)$ and a subsequence of the sequence of the sums $\{U_{\Omega_r}(x)\}$:

$$U_{\Omega_{r_J}}(x) - \frac{1}{2} \sum_{j=0}^J \frac{(-1)^j \ln q_{j+1}}{q_j} - \sum_{j=0}^J \varepsilon_j^\infty \rightarrow 0, \quad J \rightarrow \infty.$$

Consequently, if $x \in \mathcal{T}$, i. e. the series $\Xi(x)$ diverges, so does the sequence $\{U_{\Omega_r}(x)\}$, which completes the proof of the claim B.

Let us prove theorem 2. For a collection of natural numbers $\mathbf{k} = (k_1, \dots, k_{j-1}, k_j) \in \mathbb{N}^j$, let (see (1))

$$[k_1, \dots, k_{j-1}] = \frac{a_{j-1}}{q_{j-1}}$$

and consider an interval $\omega = \omega(\mathbf{k})$ with the end-points

$$[k_1, \dots, k_{j-1}, k_j] = \frac{a_j}{q_j}, \quad [k_1, \dots, k_{j-1}, k_j + 1] = \frac{a'_j}{q'_j}. \quad (12)$$

By the basic property of the continued fraction, see (4), we have

$$\left| \frac{a_{j-1}}{q_{j-1}} - \frac{a_j}{q_j} \right| = \frac{1}{q_j q_{j-1}},$$

and further,

$$(i) \ q_j(x) = q_j, \ x \in \omega; \quad (ii) \ |\omega| = \left| \frac{a_j}{q_j} - \frac{a'_j}{q'_j} \right| = \frac{1}{q_j q'_j}, \quad (iii)^4 \ \bigcup_{\mathbf{k} \in \mathbb{N}^j} \omega(\mathbf{k}) = (0, 1). \quad (13)$$

Consider the partitioning of ω by the points

$$\frac{a_{j+1,k}}{q_{j+1,k}} = [k_1, \dots, k_{j-1}, k_j, k] = \frac{a_j k + a_{j-1}}{q_j k + q_{j-1}}, \quad k = 1, 2, \dots$$

Denote ω_k the sub-interval of ω whose endpoints are two consecutive fractions $\frac{a_{j+1,k}}{q_{j+1,k}}$ and $\frac{a_{j+1,k+1}}{q_{j+1,k+1}}$. Then

$$(i) \ \varphi_j(x) = \frac{\ln q_{j+1,k}}{q_j} = \frac{\ln(q_j k + q_{j-1})}{q_j}, \quad x \in \omega_k;$$

$$(ii) \ |\omega_k| = \left| \frac{a_{j+1,k}}{q_{j+1,k}} - \frac{a_{j+1,k+1}}{q_{j+1,k+1}} \right| = \frac{1}{q_{j+1,k} q_{j+1,k+1}}, \quad (iii) \ \bigcup_{k=1}^{\infty} \omega_k = \omega. \quad (14)$$

⁴Here and in the similar partitions below, we disregard all rational points

Let

$$\mathcal{E}_\omega(z) := \{x : x \in \omega, \varphi_j(x) > z\}, \quad \mu(\omega, z) := \text{meas } \mathcal{E}_\omega(z). \quad (15)$$

Then using (14) and (13, ii) we conclude that

$$\begin{aligned} \mathcal{E}_\omega(z) &\subset \bigcup_{k: q_{j+1,k} > e^{q_j z}} \omega_k, \quad \mu(\omega, z) = \sum_{k: q_{j+1,k} > e^{q_j z}} |\omega_k| = \sum_{k: q_{j+1,k} > e^{q_j z}} \left| \frac{a_{j+1,k}}{q_{j+1,k}} - \frac{a_{j+1,k+1}}{q_{j+1,k+1}} \right| \\ &\leq \frac{1}{e^{q_j z} q_j} = q_j' e^{-q_j z} |\omega| \leq 2q_j e^{-q_j z} |\omega|; \quad \mu(\omega, z) \leq \min(1, 2q_j e^{-q_j z}) |\omega|. \end{aligned} \quad (16)$$

Since $q_j \geq F_j$ and

$$\{x : x \in (0, 1), \varphi_j > z\} = \bigcup_{\mathbf{k} \in \mathbb{N}^j} \mathcal{E}_{\omega(\mathbf{k})}(z)$$

it follows that

$$\begin{aligned} \mu(\omega, z) &\leq \min(1, 2F_j e^{-F_j z}) |\omega|, \\ \mu_j(z) &\leq \min(1, 2F_j e^{-F_j z}) \sum_{\mathbf{k} \in \mathbb{N}^j} |\omega(\mathbf{k})| = \min(1, 2F_j e^{-F_j z}). \end{aligned} \quad (17)$$

In particular, φ_j is exponentially integrable, and for $\lambda < F_j$

$$\begin{aligned} \int_0^1 e^{\lambda \varphi_j(x)} dx &= - \int_0^\infty e^{\lambda z} d\mu_j(z) = 1 + \lambda \int_0^\infty e^{\lambda z} \mu_j(z) dz \\ &\leq 1 + \lambda \int_0^\infty e^{\lambda z} \min(1, 2F_j e^{-F_j z}) dz = e^{\lambda \xi_j} \left(1 + \frac{1}{F_j - \lambda} \right), \quad \xi_j := \frac{\ln 2F_j}{F_j}. \end{aligned} \quad (18)$$

For a fixed j , let us consider the sequence of functions

$$\Upsilon_{j,l} := \sum_{\nu=j}^l \varphi_\nu, \quad l = j, j+1, \dots$$

On each interval $\omega(\mathbf{k})$, $\mathbf{k} \in \mathbb{N}^l$ all functions φ_ν with $\nu < l$ are constant, so that the sum $\Upsilon_{j,l-1}$ is also constant. Thus, applying (18) and (16) and induction in l with $l \geq j$, for $\lambda < F_j$ we obtain the estimate

$$\int_0^1 e^{\lambda \Upsilon_{j,l}(x)} dx \leq \prod_{\nu=j}^l e^{\lambda \xi_\nu} \left(1 + \frac{1}{F_\nu - \lambda} \right).$$

Letting $l \rightarrow \infty$, from here we derive the claims A) of theorem 2.

To prove the claim B), for a given positive number ε and $j = 0, 1, \dots$ let us consider the sets

$$\mathcal{F}_j(\varepsilon) := \left\{ x : x \in (0, 1), \varphi_j(x) > z_j, \quad z_j := \frac{1}{(j+1)^2\varepsilon} \right\}, \quad \mathcal{F}(\varepsilon) := \bigcup_{j=0}^{\infty} \mathcal{F}_j(\varepsilon).$$

According to (15), (16), for each interval $\omega = \omega(\mathbf{k})$, $\mathbf{k} \in \mathbb{N}^j$, the part $\mathcal{E}_\omega(z_j)$ of $\mathcal{F}_j(\varepsilon)$ is an interval $I = I_{j,\varepsilon}(\mathbf{k})$ of the length

$$|I| \leq \frac{1}{q_j} \exp\left(-\frac{q_j}{(j+1)^2\varepsilon}\right),$$

and keeping in mind that $\alpha > 2$, $|\omega| \geq q_j^{-2}$, we have

$$\mathcal{E}_\omega(z_j) = I, \quad \left(\ln \frac{1}{|I|}\right)^{-\alpha} \leq \left(\ln q_j + \frac{q_j}{(j+1)^2\varepsilon}\right)^{-\alpha} \leq \frac{\varepsilon^\alpha(j+1)^{2\alpha}}{q_j^\alpha} \leq \frac{\varepsilon^\alpha(j+1)^{2\alpha}}{F_j^{\alpha-2}} |\omega|.$$

Therefore,

$$(i) \mathcal{F}_j(\varepsilon) = \bigcup_{\mathbf{k} \in \mathbb{N}^j} I_{j,\varepsilon}(\mathbf{k}); \quad (ii) \sum_{\mathbf{k} \in \mathbb{N}^j} \left(\ln \frac{1}{|I_{j,\varepsilon}(\mathbf{k})|}\right)^{-\alpha} \leq \frac{\varepsilon^\alpha(j+1)^{2\alpha}}{F_j^{\alpha-2}} \sum_{\mathbf{k} \in \mathbb{N}^j} |\omega(\mathbf{k})| = \frac{\varepsilon^\alpha(j+1)^{2\alpha}}{F_j^{\alpha-2}}.$$

Consequently, the set $\mathcal{F}(\varepsilon)$ is covered by the collection of intervals

$$\mathcal{I}(\alpha, \varepsilon) := \left\{ \{I_{j,\varepsilon}(\mathbf{k})\}_{\mathbf{k} \in \mathbb{N}^j} \right\}_{j=0}^{\infty},$$

and

$$\sum_{I \in \mathcal{I}(\alpha, \varepsilon)} \left(\ln \frac{1}{|I|}\right)^{-\alpha} \leq \varepsilon^\alpha \sum_{j=0}^{\infty} \frac{(j+1)^{2\alpha}}{F_j^{\alpha-2}} \ll \varepsilon^\alpha, \quad \alpha > 2.$$

In the complement of the set $\mathcal{F}(\varepsilon)$ the series Υ converges:

$$\Upsilon(x) = \sum_{j=0}^{\infty} \varphi_j(x) \leq \sum_{j=0}^{\infty} \frac{1}{(j+1)^2\varepsilon} < \infty,$$

which completes the proof of the claim B) of theorem 2.

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