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The goal of this paper is to study the convergence of the double trigonometric series

$$S(x) := 3D \sum_{m=3D1}^{\infty} \sum_{n=3D1}^{\infty} \frac{\sin mnx}{m^2 + n^2}\,.$$

We prove that this series converges for all real $x$, and that $S(x)$ is bounded as a function of $x$. The proof will use some elementary arithmetical considerations, namely, the approximation of $x/(2\pi)$ by the rational numbers in the Farey tree.

The interest to this series was motivated by the recent results of M.Z. Garaev [3], the author's [5], as well as an earlier result of G.I. Arkhipov and the author [1]. In [3], the sequence of partial sums

$$h_N(x) := 3D \sum_{m=3D1}^{N} \sum_{n=3D1}^{N} \frac{\sin mnx}{mn}, \quad N = 3D1, 2, \ldots$$

was considered. Garaev proved that there exist real numbers $x$ for which the sequence $h_N(x)$ *diverges* as $N \to \infty$.

Garaev's investigation was motivated by the *convergence* result [1] for sequences of discrete Hilbert transforms with the polynomial phase $p(\cdot)$

$$H_N(p) := 3D \sum_{1 \le |n| \le N} \frac{e^{ip(n)}}{n} = 3D \sum_{n=3D1}^{N} \frac{e^{ip(n)} - e^{ip(-n)}}{n}\,.$$

If $p = 3Dp(\cdot)$ is an algebraic polynomial with the real coefficients then the sequence $H_N(p)$ converges as $N \to \infty$. Moreover, for every fixed $r \in \mathbb{N}$

$$\sup_{p \in \mathcal{P}^r} \sup_{N \in \mathbb{N}} |H_N(p)| < \infty \tag{1}$$

where $\mathcal{P}^r$ denotes the set of all uni-variate algebraic polynomials of degree $r$ with the real coefficients. The proof of this statement in [1] was based on the Arkhipov's version [2] circular method of Hardy – Littlewood – Vinogradov [11]. Independently and somewhat later than in [1], (1) was established by E.M. Stein and S. Wainger [8], see also [9], Ch. 8, Section 5.

The latter result has found applications in several fields, such as the spectral problems of the classical theory of trigonometric series, the study of solutions of time-dependent Schrödinger type equations with the periodic initial data, and the variational properties of the incomplete Gauss' sums. A survey can be found in [6], see also [7].

The recent paper [5] considers an extension of [1] for the multiple sums with the additive polynomial multi-phase. For $r \in \mathbb{N}$ and $d = 3D2, 3, \ldots$, denote $\mathcal{P}^{r,d}$ the set of $d$-element collections $\vec{p} = 3D(p_1, \ldots, p_d)$ of algebraic polynomials, where $p_k \in \mathcal{P}^r$ ; $\mathcal{P}^{\infty,d} := 3D\bigcup_r \mathcal{P}^{r,d}$;
$\mathbf{e}_k := 3D(0, \ldots, 0, \overset{k}{1}, 0, \ldots, 0),\ k = 3D1, \ldots, d$ – the standard basis in $\mathbb{R}^d$;
for $= \mathbf{n} = 3D(n_1, \ldots, n_k, \ldots, n_d) \in \mathbb{N}^d,\ \mathbf{m} = 3D(m_1, \ldots, m_k, \ldots, m_d)\imath = n\mathbb{N}^d$, denote $\mathbf{n}_k := 3D\mathbf{n} - n_k\mathbf{e}_k$, and $\square_{\mathbf{m}}$ – the parallelepiped $\{\mathbf{n} :\ \mathbf{n} \in \mathbb{N}^d,\ n_k \le m_k,\ k = 3D1, \ldots, d\}$;
for a $d$-indexed sequence $f(\cdot) :\ \mathbb{N}^d \mapsto \mathcal{C}$, and $\mathbf{m} \in \mathbb{N}^d,\ \vec{p} \in \mathcal{P}^{\infty,d}$, let

$$H_{\mathbf{m}}(f; \vec{p}) := 3D \sum_{\mathbf{n} \in \square_{\mathbf{m}}} f(\mathbf{n}) \frac{e^{ip_1(n_1)} - e^{ip_1(=-n_1)}}{n_1} \cdots \frac{e^{ip_d(n_d)} - e^{ip_1(-n_d)}}{n_d}.$$

Further, a sequence $f(\cdot) :\ \mathbb{N}^d \mapsto \mathcal{C}$ is called coordinate-wise slow (notation: $f \in \mathcal{S}^d$) if $f$ is bounded and satisfies the Littlewood – Paley condition (see [12], Ch. 15) uniformly on all lines, parallel to coordinate axes:

$$\|f\|_{\mathcal{S}^d} := 3D \sup_{\mathbf{n} \in \mathbb{N}^d} \max_{1 \le k \le d} \left( |f(\mathbf{n})| + \sup_{n \in \mathbb{N}^1} \sum_{m \in [n, 2n]} |f(\mathbf{n}_k + m\mathbf{e}_k) - f(\mathbf{n}_k + (m+1)\mathbf{e}_k)| \right) < \infty.$$

The following are three typical examples of coordinate-wise slow sequences.
1) $f(\cdot)$ – the characteristic function of a coordinate-wise convex domain $D \subset \mathbb{R}^d_+$. The latter means (see [10]) that the intersection of $D$ with any line parallel to one of the coordinate axes consists of a single interval, possibly, empty.
2) for $d = 3D2,\ f(m, n) := 3Dmn(m^2 + n^2)^{-1}$.
3) $f(\cdot)$ – the Riemann's $\zeta$-multiplier, i. e. $f(n_1, \ldots, n_d) := 3Dn_1^{it_1} \cdots n_d^{it_d} = 3De^{i(t_1 \ln n_1 + \cdots + t_d \ln n_d)}$ where $t_1, \ldots, t_d$ are fixed real numbers (parameters).

The main result of [5] is the global boundedness and the convergence of the sequence $H_{\mathbf{m}}(f; \vec{p})$. If $f :\ \mathbb{N}^d \mapsto C$ is a coordinate-wise slow sequence, then for every fixed $r$

$$\sup_{\mathbf{m} \in \mathbb{N}^d} \sup_{\vec{p} \in \mathcal{P}^{r,d}} |H_{\mathbf{m}}(f; \vec{p})| < \infty,$$

and the limit

$$H(f, \vec{p}) := 3D \lim_{\min m_k \to \infty} H_{\mathbf{m}}(f; \vec{p})$$

exists for every fixed collection of polynomials $\vec{p} \in \mathcal{P}^{\infty,d}$.

Here we prove the following theorem.

**Theorem 1** *Assume that a bivariate sequence $f: \mathbb{N}^2 \mapsto \mathcal{C}$ is coordinate-wise slow. Then the sequence*

$$S_{M,N}(f;x) := 3D \sum_{m=3D1}^{M} \sum_{n=3D1}^{N} f(m,n) \frac{\sin mnx}{m^2+n^2}$$

*is uniformly bounded:*

$$\sup_{x\in\mathbb{R}} \sup_{(M,N)\in\mathbb{N}^2} |S_{M,N}(f;x)| < \infty, \tag{2}$$

*and there exists the limit $S(x) := 3D \lim_{\min\{M,N\}\to\infty} S_{M,N}(f;x)$.*

**Proof.** Without loss of generality, we will assume that $\|f\|_{\mathcal{S}^2} \leq 1$. For $n$, $m \in \mathbb{N}$, $n \geq m$ and $y \in \mathbb{R}$, let

$$R_{n,m}(y) := 3D \sum_{\nu=3Dn}^{\infty} \frac{\sin 2\pi\nu y}{m^2+\nu^2}, \quad \tau_m(y) := 3D \sup_{N\geq m} \left| \sum_{n=3Dm}^{N} \frac{\sin 2\pi n y}{m^2+n^2} \right|,$$

$$\sigma_m(y) = 3D\sigma_m(f;y) := 3D \sup_{N\geq m} \left| \sum_{n=3Dm}^{N} f(m,n) \frac{\sin 2\pi n y}{m^2+n^2} \right|.$$

Clearly, it is enough to prove that

$$\sup_x \sum_{m=3D1}^{\infty} \sigma_m(mx) < \infty. \tag{3}$$

For a fixed $y \in \mathbb{R}$, the following estimates are true

$$\sigma_m(y) \ll \frac{\alpha(m\langle y\rangle)}{m}, \quad \alpha(\eta) := 3D \begin{cases} \eta \log \frac{e}{\eta} & \text{if} \quad \eta \leq 1, \\ \frac{1}{\eta} & \text{if} \quad \eta > 1, \end{cases} \tag{4}$$

where $\langle y\rangle$ denotes the distance from $y$ to the nearest integer.

Indeed, we have

$$|R_{n,m}(y)| \ll \frac{1}{(m^2+n^2)\langle y\rangle}, \quad \tau_m(y) \ll \frac{1}{m^2\langle y\rangle}, \quad \sigma_m(y) \ll \frac{1}{m^2\langle y\rangle}. \tag{5}$$

The estimates of $R$ and $\tau$ follow by application of the partial summation (Abel's transform) and the well-known estimate of the Dirichlet kernel

$$\sup_{M,N} \left| \sum_{n=3DM}^{N} e^{2\pi i n y} \right| \ll \frac{1}{\langle y\rangle}.$$

To prove the estimate for $\sigma$, let us consider a slow sequence $f(n),\ n \in \mathbb{N}$ and also another numerical sequence $R(n),\ n \in \mathbb{N}$ such that

$$\sum_{n\in\mathbb{N}} |R(n) - R(n+1)| < \infty.$$

Then, applying Abel's transformation, and the dyadic blocks summation we see that

$$\sum_{n\geq m} f(n)(R(n) - R(n+1)) = 3Df(m)R(m) + \sum_{k=3D0}^{\infty} \sum_{n\in(2^k m, 2^{k+1}m]} (f(n) - f(n-1))R(n).$$

From here and the definition of a slow sequence, it follows that

$$\sup_N \left| \sum_{n=3Dm}^{N} f(n)(R(n) - R(n+1)) \right| \leq 2\|f\|_{\mathcal{S}^1} \sum_{k=3D0}^{\infty} \max_{n\in(2^k m, 2^{k+1}m]} |R(n)|.$$

The estimate for $\sigma$ is a corollary of this relation and the estimate of $\tau$ (also, recall the assumption $\|f\|_{\mathcal{S}^2} \leq 1$):

$$\left| \sum_{n=3Dm}^{N} f(m,n) \frac{\sin 2\pi ny}{m^2+n^2} \right| = 3D \left| \sum_{n=3Dm}^{N} f(m,n)(R_{n,m}(y) - R_{n+1,m}(y)) \right|$$
$$\ll \sum_{k=3D0}^{\infty} \tau_{2^k m}(y) \ll \sum_{k=3D0}^{\infty} \frac{1}{(2^k m)^2 \langle y \rangle} \ll \frac{1}{m^2 \langle y \rangle}\,.$$

This completes the proof of (4) in the case, when $m\langle y\rangle \geq 1$. On the other hand, if $m\langle y\rangle \leq 1$, we have

$$\sup_{N\in[m,1/\langle y\rangle]} \left| \sum_{n\in[m,N]} f(m,n) \frac{\sin 2\pi ny}{m^2+n^2} \right| \ll \sum_{n\in[m,1/\langle y\rangle]} \frac{n\langle y\rangle}{n^2+m^2} \ll \langle y\rangle \log \frac{e}{m\langle y\rangle},$$
$$\sup_{N\geq 1/\langle y\rangle} \left| \sum_{n\in[1/\langle y\rangle,N]} f(m,n) \frac{\sin 2\pi ny}{m^2+n^2} \right| \leq \sigma_{(1/\langle y\rangle)}(y) \ll \frac{1}{(1/\langle y\rangle)^2 \langle y\rangle} = 3D\langle y\rangle,$$

and (4) follows.

Let us prove that for every fixed $x \in [0,1]$

$$\sum_{m=3D1}^{\infty} \frac{\alpha(m\langle mx\rangle)}{m} \ll 1. \tag{6}$$

Clearly, it is sufficient to establish this estimate for the irrational $x$. Let us consider the Farey sequence, see [4], Section 6.10, and a pair of neighboring fractions $\left(\frac{a}{q}, \frac{a'}{Q}\right)$ in $\mathcal{F}$, that is adjacent to $x$, i. e. $Q > q$ and

$$(a, q) = 3D(a', Q) = 3D1, \quad \left|\frac{a}{q} - \frac{a'}{Q}\right| = 3D\frac{1}{qQ}, \quad x = 3D\frac{a}{q} + \delta, \quad \delta = 3D\frac{\theta}{qQ}, \quad = |\theta| \leq 1. \tag{7}$$

Let

$$A := 3D \sum_{m \in [q,Q)} \frac{\alpha(m\langle mx\rangle)}{m},$$

and denote $B$ and, respectively, $C$ the parts of the sum $A$ that corresponds to $m \in [q, Q)$ with the "large" and "small" values of $\langle mx\rangle$, namely,

$$B := 3D \sum_{m\in[q,Q),\ \langle mx\rangle \geq 1/q} \frac{\alpha(m\langle mx\rangle)}{m}, \quad C := 3D \sum_{m\in[q,Q),\ \langle mx\rangle < 1/q} \frac{\alpha(m\langle mx\rangle)}{m}.$$

Since $(a, q) = 3D1$, for each fixed $k \in \mathbb{N}$ the set of numbers $ma = 3D(kq + l)a$, $l = 3D0, 1, \ldots, q - 1$ represents all residues mod$q$. Therefore,

$$\sum_{kq<m<(k+1)q} \left\langle \frac{ma}{q} \right\rangle^{-1} = 3D \sum_{0<l<q} \left\langle \frac{l}{q} \right\rangle^{-1} \ll q \log(eq).$$

Moreover, if $q \leq m < Q$, then by (7) we also have

$$\left| mx - \frac{ma}{q} \right| \leq \frac{m}{qQ} \leq \frac{1}{q},$$

and hence

$$B \ll \sum_{k=3D1}^{\infty} \sum_{m\in(kq,(k+1)q),\ \langle mx\rangle \geq 1/q} (m^2\langle mx\rangle)^{-1} \ll \sum_{k=3D1}^{\infty} (kq)^{-2} q \log(eq) \ll \frac{\log(eq)}{q}. \tag{8}$$

Now we prove that

$$C \ll \frac{1}{q}. \tag{9}$$

Without loss of generality, we may assume that $\delta > 0$ in (7). Then the set of natural numbers

$$\mathcal{C} := 3D \left\{ m \in [q, Q), \quad \langle mx\rangle < \frac{1}{q} \right\}$$

consists of two finite progressions

$$\mathcal{C} = 3D\mathcal{C}_1 \cup \mathcal{C}_2, \quad \mathcal{C}_1 := 3D\left\{m = 3Dkq,\ 1 \le k < \frac{Q}{q}\right\}, \quad \mathcal{C}_2 := 3D\left\{m = 3Dkq + l^*,\ 1 \le k < \frac{Q}{q}\right\}$$

where $l^*$ is the residue of the number $-1 \mod q$, i. e. (cf. (7)) $l^* = 3Dq\left\{\frac{aQ}{q}\right\}$, and $\{\cdot\}$ denotes the fractional part function.

Let us first consider the progression $\mathcal{C}_1$. If $m \in \mathcal{C}_1$, we have $m\langle mx\rangle = 3Dm^2\delta$, and the condition $m\langle mx\rangle < 1$ is equivalent to $m^2\delta < 1$, or $m < 1/\sqrt{\delta}$. Consequently, $k = 3Dm/q < 1/\Delta$ where $\Delta := 3D1/(q\sqrt{\delta})$, and it follows from the definition of the function $\alpha$ in (4) that

$$\sum_{m\in\mathcal{C}_1,\ m\langle mx\rangle<1} \frac{\alpha(m\langle mx\rangle)}{m} \le \sum_{1\le k<1/\Delta} kq\delta \log\frac{e}{(kq)^2\delta} = 3D\frac{1}{q}\sum_{1\le k<1/\Delta} \Delta\eta_k \log\frac{e}{\eta_k^2}$$

where $\eta_k := 3Dk\Delta$. Clearly, the latter sum is $\ll 1$, because it is the Riemannian sum for the integral $\int_0^1 \eta\log(e/\eta^2)\,d\eta$, so that

$$\sum_{m\in\mathcal{C}_1,\ m\langle mx\rangle<1} \frac{\alpha(m\langle mx\rangle)}{m} \ll \frac{1}{q}.$$

Further, if $m \in \mathcal{C}_1$ and $m\langle mx\rangle \ge 1$, we have $\alpha(m\langle mx\rangle) = 3D(m^2\delta)^{-1}$. Thus

$$\sum_{m\in\mathcal{C}_1,\ m\langle mx\rangle\ge1} \frac{\alpha(m\langle mx\rangle)}{m} \le \sum_{k\ge1/(q\sqrt{\delta})} \frac{1}{(kq)^3\delta} \ll \frac{1}{q}.$$

Finally, let us consider the remaining progression $\mathcal{C}_2$. If $m \in \mathcal{C}_2$, we have $\langle mx\rangle = 3D(1/q) - m\delta$, so that

$$\sum_{m\in\mathcal{C}_2} \frac{\alpha(m\langle mx\rangle)}{m} \le \sum_{m\in\mathcal{C}_2,\ 2qm\delta\le1} \frac{q}{m^2(1-qm\delta)} + \sum_{m\in\mathcal{C}_2,\ Q/2<m<Q} \frac{1}{m}$$

$$= \quad \le \frac{1}{q}\left(2\sum_{k=3D1}^{\infty}\frac{1}{k^2} + \sum_{Q/(2q)<m<=}\right.$$

Q/q $1\frac{}{k\ll\frac{1}{q}.}$Summarizing, we see that the sum $A$ satisfies the estimate

$$A \ll \frac{\log(eq)}{q}.$$

Now let us consider the whole sequence of convergents $\left\{\frac{a_j}{q_j}\right\}_1^\infty$ of $x$ in the Farey tree $\mathcal{F}$. Then by the well-known property of $\mathcal{F}$ we have $q_{j+1} = 3Dq_j + q_{j-1}$, and consequently

$$\sum_{m=3D1}^{\infty} \frac{\alpha(m\langle mx\rangle)}{m} = 3D\sum_j \sum_{m\in[q_j,q_{j+1})} \frac{\alpha(m\langle mx\rangle)}{m} = 3D\sum_j A_j \ll \sum_j \frac{\log(eq_j)}{q_j} \ll 1.$$

From here (6) and (3) follow, and the proof is complete.

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