Farey tree and the convergence of a double trigonometric series

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The goal of this paper is to study the convergence of the double trigonometric series

\[ S(x) := 3D \sum_{m=3D1}^{\infty} \sum_{n=3D1}^{\infty} \frac{\sin mn x}{m^2 + n^2}. \]

We prove that this series converges for all real \( x \), and that \( S(x) \) is bounded as a function of \( x \). The proof will use some elementary arithmetical considerations, namely, the approximation of \( x/(2\pi) \) by the rational numbers in the Farey tree.

The interest to this series was motivated by the recent results of M.Z. Garaev [3], the author’s [5], as well as an earlier result of G.I. Arkhipov and the author [1]. In [3], the sequence of partial sums

\[ h_N(x) := 3D \sum_{m=3D1}^{N} \sum_{n=3D1}^{N} \frac{\sin mn x}{mn}, \quad N = 3D1, 2, \ldots \]

was considered. Garaev proved that there exist real numbers \( x \) for which the sequence \( h_N(x) \) diverges as \( N \to \infty \).

Garaev’s investigation was motivated by the convergence result [1] for sequences of discrete Hilbert transforms with the polynomial phase \( p(\cdot) \)

\[ H_N(p) := 3D \sum_{1 \leq |n| \leq N} \frac{e^{ip(n)}}{n} = 3D \sum_{n=3D1}^{N} \frac{e^{ip(n)} - e^{ip(-n)}}{n}. \]

If \( p = 3Dp(\cdot) \) is an algebraic polynomial with the real coefficients then the sequence \( H_N(p) \) converges as \( N \to \infty \). Moreover, for every fixed \( r \in \mathbb{N} \)

\[ \sup_{p \in \mathcal{P}^r} \sup_{N \in \mathbb{N}} |H_N(p)| < \infty \quad (1) \]

where \( \mathcal{P}^r \) denotes the set of all uni-variate algebraic polynomials of degree \( r \) with the real coefficients. The proof of this statement in [1] was based on the Arkhipov’s version [2] circular
method of Hardy – Littlewood – Vinogradov [11]. Independently and somewhat later than in
[1], (1) was established by E.M. Stein and S. Wainger [8], see also [9], Ch. 8, Section 5.

The latter result has found applications in several fields, such as the spectral problems of
the classical theory of trigonometric series, the study of solutions of time-dependent Schrödinger
type equations with the periodic initial data, and the variational properties of the incomplete
Gauss’ sums. A survey can be found in [6], see also [7].

The recent paper [5] considers an extension of [1] for the multiple sums with the additive
polynomial multi-phase. For \( r \in \mathbb{N} \) and \( d = 3D2, 3, \ldots \), denote \( \mathcal{P}_{r,d} \) the set of \( d \)-element collections \( \vec{p} = 3D(p_1, \ldots, p_d) \) of algebraic polynomials, where \( p_k \in \mathcal{P}_r ; \mathcal{P}^{\infty,d} := 3D \cup \mathcal{P}_{r,d} ; \)
\( \mathbf{e}_k := 3D(0, \ldots, 0, 1, 0, \ldots, 0), \ k = 3D1, \ldots, d \) – the standard basis in \( \mathbb{R}^d \),
for \( n = 3D(n_1, \ldots, n_k, \ldots, n_d) \in \mathbb{N}^d \), \( \mathbf{m} = 3D(m_1, \ldots, m_k, \ldots, m_d)1 = n\mathbb{N}^d \), denote \( \mathbf{n}_k := 3D(n_k) \), \( \square \mathbf{m} \) – the parallelepiped \{ \( \mathbf{n} : \mathbf{n} \in \mathbb{N}^d, n_k \leq m_k, k = 3D1, \ldots, d \) \};
for a \( d \)-indexed sequence \( f(\cdot) : \mathbb{N}^d \to C \), and \( \mathbf{m} \in \mathbb{N}^d, \vec{p} \in \mathcal{P}^{\infty,d} \), let
\[
H_m(f; \vec{p}) := 3D \sum_{\mathbf{n} \in \square \mathbf{m}} f(\mathbf{n}) \frac{e^{ip_1(n_1)} - e^{ip_1(-n_1)}}{n_1} \ldots \frac{e^{ip_d(n_d)} - e^{ip_1(-n_d)}}{n_d}.
\]
Further, a sequence \( f(\cdot) : \mathbb{N}^d \to C \) is called coordinate-wise slow (notation: \( f \in \mathcal{S}^d \)) if \( f \) is
bounded and satisfies the Littlewood – Paley condition (see [12], Ch. 15) uniformly on all lines,
parallel to coordinate axes:
\[
\| f \|_{\mathcal{S}^d} := 3D \sup_{\mathbf{n} \in \mathbb{N}^d} \max_{1 \leq k \leq d} \left( |f(\mathbf{n})| + \sup_{\mathbf{n} \in \mathbb{N}^d} \sum_{m \in [n, 2n]} \left| f(\mathbf{n}_k + m\mathbf{e}_k) - f(\mathbf{n}_k + (m + 1)\mathbf{e}_k) \right| \right) < \infty.
\]

The following are three typical examples of coordinate-wise slow sequences.
1) \( f(\cdot) \) – the characteristic function of a coordinate-wise convex domain \( D \subset \mathbb{R}^d_+ \). The latter
means (see [10]) that the intersection of \( D \) with any line parallel to one of the coordinate axes
consists of a single interval, possibly, empty.
2) for \( d = 3D2 \), \( f(m, n) := 3Dmn(m^2 + n^2)^{-1} \).
3) \( f(\cdot) \) – the Riemann’s \( \zeta \)-multiplier, i. e. \( f(n_1, \ldots, n_d) := 3Dn_1^{t_1} \ldots n_d^{t_d} = 3De^{i(t_1 \ln n_1 + \ldots + t_d \ln n_d)} \)
where \( t_1, \ldots, t_d \) are fixed real numbers (parameters).

The main result of [5] is the global boundedness and the convergence of the sequence
\( H_m(f; \vec{p}) \). If \( f : \mathbb{N}^d \to C \) is a coordinate-wise slow sequence, then for every fixed \( r \)
\[
\sup_{\mathbf{m} \in \mathbb{N}^d} \sup_{\vec{p} \in \mathcal{P}^{r,d}} |H_m(f; \vec{p})| < \infty,
\]
and the limit
\[
H(f, \vec{p}) := 3D \lim_{\min_m \to \infty} H_m(f; \vec{p})
\]
exists for every fixed collection of polynomials $\tilde{p} \in \mathcal{P}^{\infty,d}$.

Here we prove the following theorem.

**Theorem 1** Assume that a bivariate sequence $f : \mathbb{N}^2 \mapsto \mathcal{C}$ is coordinate-wise slow. Then the sequence

$$S_{M,N}(f; x) := 3D \sum_{m=3D1}^{M} \sum_{n=3D1}^{N} f(m, n) \frac{\sin mnx}{m^2 + n^2}$$

is uniformly bounded:

$$\sup_{x \in \mathbb{R}} \sup_{(M,N) \in \mathbb{N}^2} |S_{M,N}(f; x)| < \infty,$$

(2)

and there exists the limit $S(x) := 3D \lim_{\min\{M,N\} \to \infty} S_{M,N}(f; x)$.

**Proof.** Without loss of generality, we will assume that $\|f\|_{S^2} \leq 1$. For $n, m \in \mathbb{N}$, $n \geq m$ and $y \in \mathbb{R}$, let

$$R_{n,m}(y) := 3D \sum_{\nu=3Dn}^{\infty} \frac{\sin 2\pi ny}{m^2 + \nu^2}, \quad \tau_{m}(y) := 3D \sup_{N \geq m} \left| \sum_{n=3Dm}^{N} \frac{\sin 2\pi ny}{m^2 + n^2} \right|,$$

$$\sigma_{m}(y) = 3D \sigma_{m}(f; y) := 3D \sup_{N \geq m} \left| \sum_{n=3Dm}^{N} f(m, n) \frac{\sin 2\pi ny}{m^2 + n^2} \right|.$$

Clearly, it is enough to prove that

$$\sup_{m=3D1}^{\infty} \sum_{x} \sigma_{m}(mx) < \infty.$$  \hspace{1cm} (3)

For a fixed $y \in \mathbb{R}$, the following estimates are true

$$\sigma_{m}(y) \ll \frac{\alpha(m\langle y \rangle)}{m}, \quad \alpha(\eta) := 3D \left\{ \begin{array}{ll} \eta \log \frac{\eta}{\eta} & \text{if } \eta \leq 1, \\ \frac{1}{\eta} & \text{if } \eta > 1, \end{array} \right.$$  \hspace{1cm} (4)

where $\langle y \rangle$ denotes the distance from $y$ to the nearest integer.

Indeed, we have

$$|R_{n,m}(y)| \ll \frac{1}{(m^2 + n^2)\langle y \rangle}, \quad \tau_{m}(y) \ll \frac{1}{m^2\langle y \rangle}, \quad \sigma_{m}(y) \ll \frac{1}{m^2\langle y \rangle}.$$  \hspace{1cm} (5)

The estimates of $R$ and $\tau$ follow by application of the partial summation (Abel’s transform) and the well-known estimate of the Dirichlet kernel

$$\sup_{M,N} \left| \sum_{n=3DM}^{N} e^{2\pi i ny} \right| \ll \frac{1}{\langle y \rangle}.$$
To prove the estimate for $\sigma$, let us consider a slow sequence $f(n), \ n \in \mathbb{N}$ and also another numerical sequence $R(n), \ n \in \mathbb{N}$ such that
\[ \sum_{n \in \mathbb{N}} |R(n) - R(n + 1)| < \infty. \]

Then, applying Abel’s transformation, and the dyadic blocks summation we see that
\[ \sum_{n \geq m} f(n)(R(n) - R(n + 1)) = 3Df(m)R(m) + \sum_{k=3D0}^{\infty} \sum_{n \in (2^k m, 2^{k+1} m]} (f(n) - f(n - 1))R(n). \]

From here and the definition of a slow sequence, it follows that
\[ \sup_N \left| \sum_{n=3Dm}^{N} f(n)(R(n) - R(n + 1)) \right| \leq 2\|f\|_{S^1} \sum_{k=3D0}^{\infty} \max_{n \in (2^k m, 2^{k+1} m]} |R(n)|. \]

The estimate for $\sigma$ is a corollary of this relation and the estimate of $\tau$ (also, recall the assumption $\|f\|_{S^2} \leq 1$):
\[ \left| \sum_{n=3Dm}^{N} f(m, n) \frac{\sin 2\pi ny}{m^2 + n^2} \right| = 3D \left| \sum_{n=3Dm}^{N} f(m, n)(R_{n,m}(y) - R_{n+1,m}(y)) \right| \leq \sum_{k=3D0}^{\infty} \tau_{2^k m}(y) \leq \sum_{k=3D0}^{\infty} \frac{1}{(2^k m)^2 \langle y \rangle} \leq \frac{1}{m^2 \langle y \rangle}. \]

This completes the proof of (4) in the case, when $m\langle y \rangle \geq 1$. On the other hand, if $m\langle y \rangle \leq 1$, we have
\[ \sup_{N \in [m, 1/\langle y \rangle]} \left| \sum_{n=m, N} f(m, n) \frac{\sin 2\pi ny}{m^2 + n^2} \right| \leq \sum_{n=m, 1/\langle y \rangle} \frac{n \langle y \rangle}{n^2 + m^2} \ll \langle y \rangle \log \frac{e}{m \langle y \rangle}, \]
\[ \sup_{N \geq 1/\langle y \rangle} \left| \sum_{n=1/\langle y \rangle, N} f(m, n) \frac{\sin 2\pi ny}{m^2 + n^2} \right| \leq \sigma(1/\langle y \rangle)(y) \ll \frac{1}{(1/\langle y \rangle)^2 \langle y \rangle} = 3D\langle y \rangle, \]
and (4) follows.

Let us prove that for every fixed $x \in [0, 1]$
\[ \sum_{m=3D1}^{\infty} \frac{a(m \langle mx \rangle)}{m} \ll 1. \]
Clearly, it is sufficient to establish this estimate for the irrational $x$. Let us consider the Farey sequence, see [4], Section 6.10, and a pair of neighboring fractions $\left(\frac{a}{q}, \frac{a'}{q'}\right)$ in $\mathcal{F}$, that is adjacent to $x$, i.e. $Q > q$ and

$$(a, q) = 3D(a', Q) = 3D1, \quad \left|\frac{a}{q} - \frac{a'}{Q}\right| = 3D \frac{1}{qQ}, \quad x = 3D \frac{a}{q} + \delta, \quad \delta = 3D \frac{\theta}{qQ}, \quad = |\theta| \leq 1. \tag{7}$$

Let

$$A := 3D \sum_{m \in [q, Q]} \frac{\alpha(m\langle mx \rangle)}{m},$$

and denote $B$ and, respectively, $C$ the parts of the sum $A$ that corresponds to $m \in [q, Q)$ with the “large” and “small” values of $\langle mx \rangle$, namely,

$$B := 3D \sum_{m \in [q, Q), \langle mx \rangle \geq 1/q} \frac{\alpha(m\langle mx \rangle)}{m}, \quad C := 3D \sum_{m \in [q, Q), \langle mx \rangle < 1/q} \frac{\alpha(m\langle mx \rangle)}{m}.$$ 

Since $(a, q) = 3D1$, for each fixed $k \in \mathbb{N}$ the set of numbers $ma = 3D(kq + l)a, \ l = 3D0, 1, \ldots, q - 1$ represents all residues mod $q$. Therefore,

$$\sum_{kq < m < (k+1)q} \langle \frac{ma}{q} \rangle^{-1} = 3D \sum_{0 < l < q} \langle \frac{l}{q} \rangle^{-1} \ll q \log (eq).$$

Moreover, if $q \leq m < Q$, then by (7) we also have

$$\left|\frac{mx - ma}{q}\right| \leq \frac{m}{qQ} \leq \frac{1}{q},$$

and hence

$$B \ll \sum_{k=3D1}^{\infty} \sum_{m \in (kq, (k+1)q), \langle mx \rangle \geq 1/q} (m^2 \langle mx \rangle)^{-1} \ll \sum_{k=3D1}^{\infty} (kq)^{-2} q \log (eq) \ll \frac{\log (eq)}{q}. \tag{8}$$

Now we prove that

$$C \ll \frac{1}{q}. \tag{9}$$

Without loss of generality, we may assume that $\delta > 0$ in (7). Then the set of natural numbers

$$C := 3D \left\{ m \in [q, Q), \langle mx \rangle < \frac{1}{q} \right\}$$
consists of two finite progressions

\[ C = 3Dc_1 \cup c_2, \quad c_1 := 3D \left\{ m = 3Dkq, \ 1 \leq k < \frac{Q}{q} \right\}, \quad c_2 := 3D \left\{ m = 3Dkq + l^*, \ 1 \leq k < \frac{Q}{q} \right\} \]

where \( l^* \) is the residue of the number \(-1 \mod q\), i.e. (cf. (7)) \( l^* = 3Dq \left\{ \frac{mQ}{q} \right\} \), and \( \{ \cdot \} \) denotes the fractional part function.

Let us first consider the progression \( c_1 \). If \( m \in c_1 \), we have \( m\langle mx \rangle = 3Dm^2\delta \), and the condition \( m\langle mx \rangle < 1 \) is equivalent to \( m^2\delta < 1 \), or \( m < 1/\sqrt{\delta} \). Consequently, \( k = 3Dm/q < 1/\Delta \) where \( \Delta := 3D1/(q\sqrt{\delta}) \), and it follows from the definition of the function \( \alpha \) in (4) that

\[
\sum_{m \in c_1, m\langle mx \rangle < 1} \frac{\alpha(m\langle mx \rangle)}{m} \leq \sum_{1 \leq k < 1/\Delta} kq\delta \log \frac{e}{(kq)^2\delta} = 3D\frac{1}{q} \sum_{1 \leq k < 1/\Delta} \Delta \eta_k \log \frac{e}{\eta_k^2}
\]

where \( \eta_k := 3Dk\Delta \). Clearly, the latter sum is \( \ll 1 \), because it is the Riemannian sum for the integral \( \int_0^1 \eta \log(e/\eta^2) \, d\eta \), so that

\[
\sum_{m \in c_1, m\langle mx \rangle < 1} \frac{\alpha(m\langle mx \rangle)}{m} \ll \frac{1}{q}.
\]

Further, if \( m \in c_1 \) and \( m\langle mx \rangle \geq 1 \), we have \( \alpha(m\langle mx \rangle) = 3D(m^2\delta)^{-1} \). Thus

\[
\sum_{m \in c_1, m\langle mx \rangle \geq 1} \frac{\alpha(m\langle mx \rangle)}{m} \leq \sum_{k \geq 1/(q\sqrt{\delta})} \frac{1}{(kq)^3\delta} \ll \frac{1}{q}.
\]

Finally, let us consider the remaining progression \( c_2 \). If \( m \in c_2 \), we have \( \langle mx \rangle = 3D(1/q) - m\delta \), so that

\[
\sum_{m \in c_2} \frac{\alpha(m\langle mx \rangle)}{m} \leq \sum_{m \in c_2, 2qm\delta \leq 1} \frac{q}{m^2(1 - qm\delta)} + \sum_{m \in c_2, Q/2 < m < Q} \frac{1}{m} = \leq \frac{1}{q} \left( 2 \sum_{k = 3D1}^{\infty} \frac{1}{k^2} + \sum_{Q/2 < m < Q} \frac{1}{m} \right).
\]

\[
\frac{Q}{q} \ll \sum_{k \leq \sqrt{q}} \frac{1}{k}.
\]

Summarizing, we see that the sum \( A \) satisfies the estimate

\[
A \ll \frac{\log(eq)}{q}.
\]
Now let us consider the whole sequence of convergents \( \left\{ \frac{a_n}{q_n} \right\}_1^\infty \) of \( x \) in the Farey tree \( \mathcal{F} \). Then by the well-known property of \( \mathcal{F} \) we have \( q_{j+1} = 3Dq_j + q_{j-1} \), and consequently

\[
\sum_{m=3D1}^\infty \frac{\alpha(m\langle mx \rangle)}{m} = 3D \sum_j \sum_{m=q_j,q_{j+1}} \frac{\alpha(m\langle mx \rangle)}{m} = 3D \sum_j A_j \ll \sum_j \frac{\log(eq_j)}{q_j} \ll 1.
\]

From here (6) and (3) follow, and the proof is complete.

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References


