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Algorithms for nonlinear
piecewise polynomial
approximation: Theoretical
aspects

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1 Introduction

Highly detailed Digital Terrain Elevation Data (DTED) and associated imagery are now becoming widely available for most of the earth's surface. However, algorithms for effective approximation of data of this type are not yet available. A primary motivation for this work is the development of effective algorithms for nonlinear piecewise polynomial approximation of DTED maps from a redundant hierarchical system over (possibly) irregular triangulations which are constructive in nature. Application of the ideas and theory from [3] to the resulting framework will permit optimal entropy tree encoding of the elevation data, enable progressive view-dependent refinements which may be focused to user localized regions, and permit the registration of similarly encoded image textures to the surface (see [10, 3] for more details).

Our philosophy is that dependable practical approximation procedures can be built only upon a solid theoretical basis. Accordingly, we have two primary goals in this paper. The first is to better understand nonlinear piecewise polynomial approximation, in particular, to understand the nature of the global smoothness conditions (spaces) which govern the rate of approximation. The second goal is to develop or refine existing constructive approximation methods for nonlinear approximation which capture the rate of the best approximation and can be implemented effectively in practice.

This paper addresses nonlinear n -term approximation from Courant elements generated by multilevel nested triangulations. More precisely, for a given bounded polygonal domain $E \subset \mathbb{R}^2$, let $(\mathcal{T}_m)_{m \geq 0}$ be a sequence of triangulations such that each level \mathcal{T}_m is a triangulation of E consisting of closed triangles with disjoint interiors and a refinement of the previous level \mathcal{T}_{m-1} . We impose some mild natural conditions on the triangulations in order to prevent them from possible deterioration, but our results are valid for fairly general triangulations with sharp angles. We define $\mathcal{T} := \bigcup_{m \geq 0} \mathcal{T}_m$. Each such multilevel triangulation \mathcal{T} generates a ladder of spaces $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots$ consisting of piecewise linear functions, where \mathcal{S}_m ($m \geq 0$) is spanned by all Courant elements φ_θ supported on cells θ at the m -th level \mathcal{T}_m .

Utilizing these primal elements, we consider nonlinear approximation from n -term piecewise linear functions of the form $S = \sum_{j=1}^n a_{\theta_j} \varphi_{\theta_j}$, where θ_j may come from different levels and locations. Our first goal is to characterize the approximation spaces consisting of all functions with a given rate of approximation. For approximation in L_p , $p < \infty$, this is done in [11], where a collection of smoothness spaces (called B -spaces) have been introduced and utilized. In this paper, we develop this theory in the more complicated case of approximation in the uniform norm ($p = \infty$). Our program consists of the following steps. First, in order to quantify the approximation process, we develop a collection of smoothness spaces $B_r^\alpha(\mathcal{T})$ which depend on \mathcal{T} and will govern the best approximation. Second, we prove companion Jackson and Bernstein estimates and, third, we characterize the approximation spaces by interpolation space methods.

Our second and primary goal is, by using the B -spaces and the related techniques, to develop (or refine) algorithms for nonlinear n -term Courant element approximation so that the new algorithms are capable of achieving the rate of the best approximation. In the present paper, we develop three such algorithms for n -term Courant element approximation in L_p , which we call "Threshold" ($p < \infty$), "Trim and Cut" ($0 < p \leq \infty$), and "Push the Error" ($p = \infty$) algorithms.

The first step of each of these algorithms is a decomposition step. We denote by Θ the

set of all cells (supports of Courant elements) generated by \mathcal{T} . The set $(\varphi_\theta)_{\theta \in \Theta}$ is obviously redundant and, therefore, every function f has infinitely many representations of the form

$$f = \sum_{\theta \in \Theta} b_\theta(f) \varphi_\theta. \quad (1.1)$$

It is crucial to have a sufficiently efficient (sparse) initial representation of the function f which is being approximated. In our case, this means that the representation (1.1) of f should allow a realization of the corresponding B-norm $\|f\|_{B_\varepsilon^q(\mathcal{T})}$. Thus the problem for obtaining an efficient initial representation of the functions is tightly related to the development of the B -spaces. We achieve such efficiency by using good projectors into the spaces \mathcal{S}_m , $m = 0, 1, \dots$

For completeness and comparison, we first consider the natural “Threshold” algorithm for n -term Courant element approximation which is valid only in L_p , $0 < p < \infty$. This algorithm simply takes the largest (in L_p) n -terms from (1.1). Using the results from [11], it is easy to show that the “Threshold” algorithm captures the rate of the best n -term Courant element approximation in L_p ($p < \infty$).

The second algorithm, which we call “Trim & Cut”, originates from the proof of the Jackson estimate in [7] and uses the following idea. First, we partition Θ through a coloring into a family of disjoint trees Θ^ν (with respect to the inclusion relation): $\Theta := \bigcup_{\nu=1}^K \Theta^\nu$. Second, we “trim” each tree by removing cells $\theta \in \Theta^\nu$ corresponding to insignificant small terms $a_\theta \varphi_\theta$ from (1.1), located near the tips of the branches. Third, we divide (“cut”) the remaining parts of each tree Θ^ν into sections of small “energy”. Finally, we rewrite the significant part of each section as a linear combination of small number of Courant elements. The resulting terms determine the final approximant. We shall show that “Trim & Cut” is capable of achieving the rate of the best approximation in L_p ($0 < p \leq \infty$).

Pivotal in our development is the “Push the Error” algorithm, the name of which was coined by Nira Dyn. The idea for this algorithm appears in [5] and may be roughly described in L_∞ as follows. For a fixed $\varepsilon > 0$, we “Push the Error” with ε , starting from the coarsest level Θ_0 and proceeding to finer levels. Namely, we denote by Λ_0 the set of all $\theta \in \Theta_0$ such that $|a_\theta| > \varepsilon$ ($\|\varphi_\theta\|_\infty = 1$) and define $\mathcal{A}_0 := \sum_{\theta \in \Lambda_0} a_\theta \varphi_\theta$. Then we rewrite all remaining terms $a_\theta \varphi_\theta$ at the next level and add the resulting terms to the existing terms $a_\theta \varphi_\theta$, $\theta \in \Theta_1$. We denote the new terms by $d_\theta \varphi_\theta$, $\theta \in \Theta_1$, and select in Λ_1 all $\theta \in \Theta_1$ such that $|d_\theta| > \varepsilon$. We continue pushing the error in this way to the finer levels in the representation of f . Finally, we define our approximant by $\mathcal{A} := \sum_{m \geq 0} \mathcal{A}_m$. Thus terms $d_\theta \varphi_\theta$ with $|d_\theta| \leq \varepsilon$ are discarded only at a very fine level and hence the error (in L_∞) is $\leq \varepsilon$.

Of course, this *naive* “Push the Error” algorithm cannot achieve the rate of the best approximation. However, as we shall show in §3.3 and §5, after some substantial improvements, the algorithm is capable of achieving the rate of convergence of the best n -term Courant element approximation in the uniform norm.

A focal point of our developments is the characterization of the approximation spaces generated by the best n -term Courant element approximation in L_∞ and the characterization of certain approximation spaces associated with the three algorithms developed, which show that they capture the rate of convergence of the best approximation.

The outline of the paper is as follows. In §2, we collect all facts needed regarding multilevel triangulations, local approximation, quasi-interpolants, and B -spaces. In §3, we de-

velop and explore the three algorithms for nonlinear n -term Courant element approximation: “Threshold” algorithm (in §3.1), “Trim & Cut” algorithm (in §3.2), and “Push the Error” algorithm (in §3.3). Section 4 is devoted to establishing Jackson and Bernstein inequalities in order to study best n -term Courant element approximation. In §5, we show that the three algorithms capture the rate of the best n -term Courant element approximation and identify the associated approximation spaces as B -spaces. In §6, we discuss some of the main issues of nonlinear Courant element approximation. We postpone until the Appendix the proof of an important coloring lemma used in § 3.2 for tree approximation in the “Trim and Cut” algorithm.

For convenience, we use the convention that positive constants are denoted by c, c_1, \dots throughout and they may vary at every occurrence. The notation $A \approx B$ means that $c_1 A \leq B \leq c_2 A$.

2 Preliminaries

In this section we collect all the facts needed regarding multilevel triangulations, local approximation, quasi-interpolants, and other results which were developed in [11] and earlier papers. The essentials are presented for clarity but without proofs.

2.1 Triangulations

By definition $E \subset \mathbb{R}^2$ is a bounded polygonal domain if E can be represented as the union of a finite set \mathcal{T}_0 of closed triangles with disjoint interiors: $E = \bigcup_{\Delta \in \mathcal{T}_0} \Delta$. We shall always assume that there exists an initial triangulation \mathcal{T}_0 of E of this form. We call

$$\mathcal{T} = \bigcup_{m=0}^{\infty} \mathcal{T}_m$$

a *multilevel triangulation* of E with levels (\mathcal{T}_m) if the following conditions are fulfilled:

- (a) Every level \mathcal{T}_m is a partition (or triangulation) of E , that is, $E = \bigcup_{\Delta \in \mathcal{T}_m} \Delta$ and \mathcal{T}_m consists of closed triangles with disjoint interiors.
- (b) The levels (\mathcal{T}_m) of \mathcal{T} are nested, i.e., \mathcal{T}_{m+1} is a refinement of \mathcal{T}_m .
- (c) Each triangle $\Delta \in \mathcal{T}_m$ has at least one and at most M_0 children (subtriangles) in \mathcal{T}_{m+1} , where $M_0 \geq 4$ is a constant.
- (d) The valence N_v of each vertex v of any triangle $\Delta \in \mathcal{T}_m$ (the number of the triangles from \mathcal{T}_m which share v as a vertex) is at most N_0 , where $N_0 \geq 3$ is a constant.
- (e) *No hanging vertices condition*: No vertex of any triangle $\Delta \in \mathcal{T}_m$ which belongs to the interior of E lies in the interior of an edge of another triangle from \mathcal{T}_m .

We denote by \mathcal{V}_m the set of all vertices of triangles from \mathcal{T}_m , where if $v \in \mathcal{V}_m$ is on the boundary of E , we include in \mathcal{V}_m as many copies of v as is its multiplicity. With this understanding, we set $\mathcal{V} = \bigcup_{m \geq 0} \mathcal{V}_m$.

We now introduce three types of multilevel nested triangulations which will play an essential role in our developments:

• **Locally regular triangulations.** We call a multilevel triangulation $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_m$ of E , a compact polygonal domain in \mathbb{R}^2 , a locally regular triangulation, or briefly an LR-triangulation, if \mathcal{T} satisfies the following additional conditions:

- (i) There exist constants $0 < r < \rho < 1$ ($r \leq \frac{1}{4}$), such that for each $\Delta \in \mathcal{T}$ and any child Δ' of Δ which belongs to \mathcal{T}

$$r|\Delta| \leq |\Delta'| \leq \rho|\Delta|. \quad (2.2)$$

- (ii) There exists a constant $0 < \delta \leq 1$ such that for each $\Delta', \Delta'' \in \mathcal{T}_m$ ($m \geq 0$) with a common vertex

$$\delta \leq \frac{|\Delta'|}{|\Delta|} \leq \frac{1}{\delta}. \quad (2.3)$$

• **Strong locally regular triangulations.** We call a multilevel triangulation $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_m$ of E , a compact polygonal domain in \mathbb{R}^2 , a strong locally regular triangulation, or simply an SLR-triangulation, if \mathcal{T} satisfies condition (2.2) and also the following condition (which replaces (2.3)):

- (iii) *Affine transform angle condition:* There exists a constant $\beta = \beta(\mathcal{T}) > 0$, ($0 < \beta < \frac{\pi}{3}$) such that if $\Delta_0 \in \mathcal{T}_m$ ($m \geq 0$) and $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an affine transform mapping Δ_0 one-to-one, onto an equilateral reference triangle, then for every triangle $\Delta \in \mathcal{T}_m$ with a common vertex with Δ_0 , we have

$$\min \text{angle}(A(\Delta)) \geq \beta \quad (2.4)$$

where $A(\Delta)$ is the image of Δ under A and is therefore also a triangle.

• **Regular triangulations.** By definition a multilevel triangulation \mathcal{T} of $E \subset \mathbb{R}^2$ is called a regular triangulation if \mathcal{T} satisfies the following condition:

- (iv) There exists a constant $\beta = \beta(\mathcal{T}) > 0$ such that the minimal angle of each $\Delta \in \mathcal{T}$ is greater or equal to β .

The remainder of this subsection makes several observations to better understand the nature of multilevel triangulations. First, it is clear that the classes of LR- and SLR-triangulations are each invariant under affine transforms. We next observe that each SLR-triangulation is an LR-triangulation, but that the converse statement does not hold. Moreover, each regular triangulation is an SLR-triangulation, but again the converse is in general false. Counterexamples are given in [11].

Each type of triangulation depends on several parameters which are not completely independent. For instance, the parameters of LR-triangulations are $M_0, N_0, r, \rho, \delta$, and $\#\mathcal{T}_0$ (the

cardinality of \mathcal{T}_0). We could set $M_0 = \frac{1}{r}$, $\rho = 1 - r$ and eliminate these as parameters, but this would tend to obscure the actual dependence of the estimates upon given triangulations.

We next briefly describe a simple standard procedure for constructing multilevel triangulations. We start from an initial triangulation \mathcal{T}_0 of the given compact polygonal domain $E \subset \mathbb{R}^2$. We then select a point on each edge of every triangle $\Delta \in \mathcal{T}_0$ and join them within Δ by edges to subdivide Δ into four children. The collection of all such children becomes the first generation of triangles, which we denote by \mathcal{T}_1 . We recursively refine in this way to produce succeeding generations $\mathcal{T}_2, \mathcal{T}_3, \dots$. The resulting collection $\mathcal{T} := \bigcup_{m \geq 0} \mathcal{T}_m$ is a multilevel triangulation of E .

It is important to know how the quantities $|\Delta|$, $\min \text{angle}(\Delta)$, and $\max \text{edge}(\Delta)$ of a triangle $\Delta \in \mathcal{T}$ may change as Δ moves away from a fixed triangle Δ° within the same level or through the nested refinements. Consider the case when \mathcal{T} is an LR-triangulation. Then conditions (i)-(ii) suggest a geometric rate of change of $|\Delta|$ (at the same level). In fact, the rate is polynomial [11]. Furthermore, if $\Delta', \Delta'' \in \mathcal{T}_m$, ($m \geq 1$) have a common vertex and are also children of some $\Delta \in \mathcal{T}_{m-1}$, then, as shown in [11], it is possible that Δ' is equilateral (or close to such), but Δ'' has an uncontrollably sharp angle (see Figure 1).

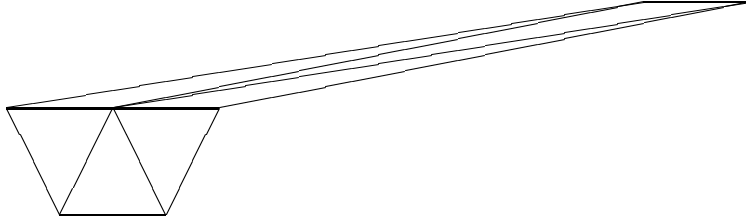


Figure 1: A skewed cell

If \mathcal{T} is an SLR-triangulation, the above configuration is impossible, but the triangles from \mathcal{T} still may have uncontrollably sharp angles. In this case, $\min \text{angle}(\Delta)$ changes gradually from one triangle to the adjacent ones.

For any vertex $v \in \mathcal{V}_m$ ($m \geq 0$), we denote by θ_v the cell at level m associated with v , i.e., θ_v is the union of all triangles from \mathcal{T}_m which have v as a common vertex. We denote by Θ_m the set of all such cells θ_v with $v \in \mathcal{V}_m$ and set $\Theta = \bigcup_{m \geq 0} \Theta_m$.

2.2 Local piecewise linear approximation and quasi-interpolants

We denote by Π_k the set of all algebraic polynomials of total degree less than k . We shall often refer to the following lemma (see [11]), which establishes the equivalence of different norms of polynomials over different sets.

Lemma 2.1. *Let $P \in \Pi_k$, $k \geq 1$, and $0 < p, q \leq \infty$.*

(a) *For any triangle $\Delta \subset \mathbb{R}^2$,*

$$\|P\|_{L_p(\Delta)} \approx |\Delta|^{\frac{1}{p} - \frac{1}{q}} \|P\|_{L_q(\Delta)}.$$

(b) If Δ and Δ' are two triangles such that $\Delta' \subset \Delta$ and $|\Delta| \leq c_1 |\Delta'|$, then

$$\|P\|_{L_p(\Delta)} \leq c \|P\|_{L_p(\Delta')}.$$

(c) If $\Delta' \subset \Delta$ and $|\Delta'| \leq c_1 |\Delta|$ with $0 < c_1 < 1$, then

$$\|P\|_{L_p(\Delta)} \leq c \|P\|_{L_p(\Delta \setminus \Delta')} \approx |\Delta|^{\frac{1}{p} - \frac{1}{q}} \|P\|_{L_q(\Delta \setminus \Delta')}.$$

In the above expressions, the constants depend at most on the corresponding parameters and the constant c_1 .

The nohanging vertices condition (e) of triangulations guarantees the existence of Courant elements. Namely, for any vertex $v \in \mathcal{V}_m$ ($m \geq 0$) there exists a unique Courant element φ_{θ_v} supported on $\theta_v \in \Theta_m$ which is the unique continuous piecewise linear function on E which is supported on θ_v and satisfies $\varphi_{\theta_v}(v) = 1$. We denote $\Phi := \Phi(\mathcal{T}) := (\varphi_\theta)_{\theta \in \Theta}$. We also denote by \mathcal{S}_m the space of all continuous piecewise linear functions over \mathcal{T}_m . Clearly, $S \in \mathcal{S}_m$ if and only if $S = \sum_{v \in \mathcal{V}_m} S(v) \varphi_{\theta_v}$. Throughout the remainder of this section, we assume that \mathcal{T} is an LR-triangulation of E . We shall often use the following stability estimates for $(\varphi_\theta)_{\theta \in \Theta_m}$.

Lemma 2.2. *Let $0 < q \leq \infty$ and $S = \sum_{\theta \in \Theta_m} a_\theta \varphi_\theta$, $m \geq 0$, with coefficients $a_\theta \in \mathbb{R}$. Then for every $\Delta \in \mathcal{T}_m$, we have*

$$\|S\|_{L_q(\Delta)} \approx \left(\sum_{\theta \in \Theta_m, \theta \supset \Delta} \|a_\theta \varphi_\theta\|_q^q \right)^{\frac{1}{q}}$$

and hence

$$\|S\|_{L_q(E)} \approx \left(\sum_{\theta \in \Theta_m} \|a_\theta \varphi_\theta\|_q^q \right)^{\frac{1}{q}}$$

with constants of equivalence depending only on the parameters of \mathcal{T} . In these estimates the ℓ_q -norm is replaced by the sup-norm if $q = \infty$.

The proof of this lemma is fairly simple and can be found in [11].

• **Local piecewise linear approximation.** The local approximation by continuous piecewise linear functions will be an important tool in our further development. For $f \in L_\eta(E)$, $\eta > 0$, and any $\Delta \in \mathcal{T}_m$ ($m \geq 0$), we denote the error of $L_\eta(\Omega_\Delta)$ -approximation to f from \mathcal{S}_m by

$$\mathbb{S}_\Delta(f)_\eta := \mathbb{S}_\Delta(f, \mathcal{T})_\eta := \inf_{S \in \mathcal{S}_m} \|f - S\|_{L_\eta(\Omega_\Delta)} \quad (2.5)$$

where Ω_Δ is the union of all triangles from \mathcal{T}_m which have a vertex in common with Δ .

• **Quasi-interpolants.** The set $\Phi(\mathcal{T})$ of all Courant elements is obviously redundant. To obtain a good (i.e., sparse) representation of a given function f , we shall use the following well known quasi-interpolant:

$$Q_m(f) := Q_m(f, \mathcal{T}) := \sum_{\theta \in \Theta_m} \langle f, \tilde{\varphi}_\theta \rangle \varphi_\theta \quad (2.6)$$

where $\langle f, g \rangle := \int_E fg$ and $(\tilde{\varphi}_\theta)$ are the duals of (φ_θ) defined by

$$\tilde{\varphi}_\theta := \sum_{\Delta \in \mathcal{T}_m, \Delta \subset \theta} \mathbb{1}_\Delta \tilde{\lambda}_{\Delta, \theta} \quad (2.7)$$

with $\tilde{\lambda}_{\Delta, \theta}$ the linear polynomial which is equal to $\frac{9}{N_v |\Delta|}$ at v_θ , the ‘‘central vertex’’ of θ , and $-\frac{3}{N_v |\Delta|}$ at the other two vertices of Δ (recall that N_v is the valence of v). It is easily seen that

$$\langle \varphi_\theta, \tilde{\varphi}_{\theta'} \rangle = \delta_{\theta\theta'}, \quad \text{for } \theta, \theta' \in \Theta_m.$$

Evidently, Q_m is a linear projector, i.e., $Q_m(S) = S$ for $S \in \mathcal{S}_m$. It is crucial that $\tilde{\varphi}_\theta \in L_\infty$ and $\tilde{\varphi}_\theta$ is locally supported. Consequently, Q_m is locally bounded and provides good local approximation.

Lemma 2.3. (a) *If $f \in L_\eta(E)$, $1 \leq \eta \leq \infty$, and $\Delta \in \mathcal{T}_m$, $m \geq 0$, then*

$$\|Q_m(f)\|_{L_\eta(\Delta)} \leq c \|f\|_{L_\eta(\Omega_\Delta)}.$$

(b) *If $0 < \eta \leq \infty$ and $g = \sum_{\Delta \in \mathcal{T}_m} \mathbb{1}_\Delta \cdot P_\Delta$ with $P_\Delta \in \Pi_2$ and $m \geq 0$, then*

$$\|Q_m(g)\|_{L_\eta(\Delta)} \leq c \|g\|_{L_\eta(\Omega_\Delta)}, \quad \text{for } \Delta \in \mathcal{T}_m.$$

The constants above depend only on η and the parameters of \mathcal{T} .

For a proof of this lemma, see [11].

From the above lemma, we see that $Q_m : L_\eta(E) \rightarrow \mathcal{S}_m$ ($1 \leq \eta \leq \infty$) is a locally bounded linear projector. There is a well-known scheme for extending Q_m to a nonlinear projector $Q_m : L_\eta(E) \rightarrow \mathcal{S}_m$ for $0 < \eta < 1$. This is needed for nonlinear approximation in L_p ($0 < p \leq 1$). To describe this extension, let $P_{\Delta, \eta} : L_\eta(\Delta) \rightarrow \Pi_2$ ($0 < \eta \leq \infty$) be a projector (linear if $\eta \geq 1$ and nonlinear if $0 < \eta < 1$) such that

$$\|f - P_{\Delta, \eta}(f)\|_{L_\eta(\Delta)} \leq c E_2(f, \Delta) \quad \text{for } f \in L_\eta(\Delta),$$

where $E_2(f, \Delta)$ is the error of the best $L_\eta(\Delta)$ -approximation to f from Π_2 (the linear polynomials). We define

$$p_{m, \eta}(f) := \sum_{\Delta \in \mathcal{T}_m} \mathbb{1}_\Delta \cdot P_{\Delta, \eta}(f)$$

and set

$$T_{m, \eta}(f) := Q_m(p_{m, \eta}(f)), \quad \text{for } f \in L_\eta(\Delta). \quad (2.8)$$

Clearly, $T_{m, \eta} : L_\eta(E) \rightarrow \mathcal{S}_m$ is a projector (linear if $\eta \geq 1$ and nonlinear if $0 < \eta < 1$).

The next lemma, established in [11], shows that Q_m and $T_{m, \eta}$ provide good local approximation from \mathcal{S}_m .

Lemma 2.4. (a) If $f \in L_\eta(E)$, $1 \leq \eta \leq \infty$, and $\Delta \in \mathcal{T}_m$, $m \geq 0$, then

$$\|f - Q_m(f)\|_{L_\eta(\Delta)} \leq c\mathbb{S}_\Delta(f)_\eta.$$

(b) If $f \in L_\eta(E)$, $0 < \eta \leq \infty$, and $\Delta \in \mathcal{T}_m$, $m \geq 0$, then

$$\|f - T_m(f)\|_{L_\eta(\Delta)} \leq c\mathbb{S}_\Delta(f)_\eta.$$

The constants above depend only on η and the parameters of \mathcal{T} .

The needed convergence of $Q_m(f)$ and $T_m(f)$ to f is provided by the following result (see Lemma 2.15 from [11]).

Lemma 2.5. If $f \in L_\eta(E)$, then

$$\begin{aligned} \|f - Q_m(f)\|_{L_\eta(E)} &\rightarrow 0 \text{ as } m \rightarrow \infty, & \text{if } 1 \leq \eta \leq \infty, \text{ and} \\ \|f - T_{m,\eta}(f)\|_{L_\eta(E)} &\rightarrow 0 \text{ as } m \rightarrow \infty, & \text{if } 0 < \eta \leq \infty. \end{aligned}$$

Now, we apply a well known scheme for obtaining sparse Courant element representation of functions. We define

$$q_m := Q_m - Q_{m-1} \text{ and } t_{m,\eta} := T_{m,\eta} - T_{m-1,\eta}, \text{ for } m \geq 0, \quad (2.9)$$

where $Q_{-1} := 0$, $T_{-1,\eta} := 0$. Clearly, $q_m(f), t_{m,\eta}(f) \in \mathcal{S}_m$.

For a given function $f \in L_\eta(E)$, $1 \leq \eta \leq \infty$, we define the sequence $\mathbf{b}(f) := (b_\theta(f))_{\theta \in \Theta_m}$ from the expression

$$q_m(f) =: \sum_{\theta \in \Theta_m} b_\theta(f)\varphi_\theta, \quad m \geq 0. \quad (2.10)$$

Using Lemma 2.5, we have

$$f = \sum_{m \geq 0} q_m(f) = \sum_{m \geq 0} \sum_{\theta \in \Theta_m} b_\theta(f)\varphi_\theta \text{ in } L_\eta. \quad (2.11)$$

If $f \in L_\eta(E)$, $0 < \eta < 1$, we define the sequence $\mathbf{b}_\eta(f) := (b_{\theta,\eta}(f))_{\theta \in \Theta_m}$ by

$$t_{m,\eta}(f) =: \sum_{\theta \in \Theta_m} b_{\theta,\eta}(f)\varphi_\theta, \quad m \geq 0, \quad (2.12)$$

and again by Lemma 2.5, we have

$$f = \sum_{m \geq 0} t_{m,\eta}(f) = \sum_{m \geq 0} \sum_{\theta \in \Theta_m} b_{\theta,\eta}(f)\varphi_\theta \text{ in } L_\eta. \quad (2.13)$$

Clearly, $\mathbf{b}(\cdot)$ is a linear operator while $\mathbf{b}_\eta(\cdot)$ ($0 < \eta < 1$) is nonlinear.

2.3 B-spaces

In this section, we include the necessary tools for the B -spaces which we need for nonlinear n -term Courant element approximation. The B -spaces over multilevel nested triangulations of \mathbb{R}^2 are introduced in [11] and used there for nonlinear n -term Courant element approximation in $L_p(\mathbb{R}^2)$ ($0 < p < \infty$). In the present paper, we shall use the B -spaces for n -term Courant element approximation in $L_p(E)$ ($0 < p \leq \infty$), where E is a compact polygonal domain in \mathbb{R}^2 . We shall put the emphasis on approximation in the uniform norm ($p = \infty$). There are three types of B -spaces (skinny, slim, and fat B -spaces) which were introduced in [11] to serve different purposes. For Courant element approximation, we need the slim B -spaces which we shall simply call B -spaces.

Throughout this paper, we assume that \mathcal{T} is an LR -triangulation of a compact polygonal domain E in \mathbb{R}^2 . Moreover, the B -spaces $B_\tau^\alpha(\mathcal{T})$, with parameters set $1/\tau := \alpha + 1/p$ according to two specific choices: (a) $p = \infty$ and $\alpha \geq 1$; or (b) $0 < p < \infty$ and $\alpha > 0$, will arise naturally in our algorithms and error estimates. These spaces have several equivalent definitions, which we briefly describe.

• **Definition of $B_\tau^\alpha(\mathcal{T})$ via local approximation.** We define $B_\tau^\alpha(\mathcal{T})$ as the set of all functions $f \in L_\tau(E)$ such that

$$|f|_{B_\tau^\alpha(\mathcal{T})} := \left(\sum_{\Delta \in \mathcal{T}} (|\Delta|^{-\alpha} \mathbb{S}_\Delta(f)_\tau)^\tau \right)^{1/\tau} < \infty, \quad (2.14)$$

where $\mathbb{S}_\Delta(f)_\tau$ is the error of $L_\tau(\Omega_\Delta)$ -approximation (local) to f from \mathcal{S}_m for $\Delta \in \mathcal{T}_m$ (see (2.5)). It is readily seen that $|f + g|_{B_\tau^\alpha} \leq |f|_{B_\tau^\alpha} + |g|_{B_\tau^\alpha}$ with $\tau^* := \min\{\tau, 1\}$, and $|f + s|_{B_\tau^\alpha} = |f|_{B_\tau^\alpha}$ for $s \in \mathcal{S}_0$. Hence $|\cdot|_{B_\tau^\alpha}$ is a semi-norm if $\tau \geq 1$ and a semi-quasi-norm if $\tau < 1$.

By Theorem 2.7 and Theorem 2.9 below, it follows that if $f \in B_\tau^\alpha(\mathcal{T})$, then $f \in L_p(E)$. Therefore, it is natural to define a (quasi-)norm in $B_\tau^\alpha(\mathcal{T})$ by

$$\|f\|_{B_\tau^\alpha(\mathcal{T})} := \|f\|_p + |f|_{B_\tau^\alpha(\mathcal{T})}. \quad (2.15)$$

More generally, for $0 < \eta < p$, we define

$$N_{\mathbb{S}, \eta}(f) := \|f\|_p + \left(\sum_{\Delta \in \mathcal{T}} (|\Delta|^{1/p-1/\eta} \mathbb{S}_\Delta(f)_\eta)^\tau \right)^{1/\tau}. \quad (2.16)$$

Evidently, $N_{\mathbb{S}, \tau}(f) = \|f\|_{B_\tau^\alpha(\mathcal{T})}$ and is dependent upon \mathcal{T} , $N_{\mathbb{S}, \tau}(f) = N_{\mathbb{S}, \tau}(f, \mathcal{T})$.

• **Definition of norm in $B_\tau^\alpha(\mathcal{T})$ via atomic decomposition.** For $f \in L_\tau(E)$, we define

$$N_\Phi(f) := \inf_{f = \sum_{\theta \in \Theta} c_\theta \varphi_\theta} \left(\sum_{\theta \in \Theta} (|\theta|^{-\alpha} \|c_\theta \varphi_\theta\|_\tau)^\tau \right)^{1/\tau}, \quad (2.17)$$

where the infimum is taken over all representations $f = \sum_{\theta \in \Theta} c_\theta \varphi_\theta$ in $L_\tau(E)$. Note that the existence of such representations of f follows by (2.11) and (2.13). By Theorem 2.7 below,

$$\sum_{\theta \in \Theta} (|\theta|^{-\alpha} \|c_\theta \varphi_\theta\|_\tau)^\tau < \infty \quad \text{implies} \quad \left\| \sum_{\theta \in \Theta} c_\theta \varphi_\theta(\cdot) \right\|_p < \infty$$

and hence $f \in L_p(E)$ and the series $\sum_{\theta \in \Theta} |c_\theta \varphi_\theta(\cdot)|$ converges a.e. and in $L_p(E)$. Therefore, the way in which the terms of the series are ordered is not essential and the convergence in $L_\tau(E)$ implies a stronger (absolute) convergence in $L_p(E)$ ($\tau < p$). By Lemma 2.1, it follows that

$$\begin{aligned} N_\Phi(f) &\approx \inf_{f = \sum_{\theta \in \Theta} c_\theta \varphi_\theta} \left(\sum_{\theta \in \Theta} (|\theta|^{1/p} |c_\theta|)^\tau \right)^{1/\tau} \\ &\approx \inf_{f = \sum_{\theta \in \Theta} c_\theta \varphi_\theta} \left(\sum_{\theta \in \Theta} \|c_\theta \varphi_\theta\|_p^\tau \right)^{1/\tau}. \end{aligned} \quad (2.18)$$

If $p = \infty$, then

$$N_\Phi(f) \approx \inf_{f = \sum_{\theta \in \Theta} c_\theta \varphi_\theta} \left(\sum_{\theta \in \Theta} |c_\theta|^\tau \right)^{1/\tau}.$$

• **Definition of norms in $B_\tau^\alpha(\mathcal{T})$ via projectors.** For $f \in L_\eta(E)$, we let

$$f = \sum_{\theta \in \Theta} b_{\theta,\eta}(f) \varphi_\theta \quad (2.19)$$

be the representation of f from (2.11) if $\eta \geq 1$ and from (2.13) if $0 < \eta < 1$. We define

$$N_{Q,\tau}(f) := \left(\sum_{\theta \in \Theta} (|\theta|^{-\alpha} \|b_{\theta,\tau}(f) \varphi_\theta\|_\tau)^\tau \right)^{1/\tau} \quad (2.20)$$

and, more generally (in accordance with (2.16)),

$$N_{Q,\eta}(f) := \left(\sum_{\theta \in \Theta} (|\theta|^{1/p-1/\eta} \|b_{\theta,\eta}(f) \varphi_\theta\|_\eta)^\tau \right)^{1/\tau}. \quad (2.21)$$

By Lemma 2.1 and Lemma 2.2, we have

$$N_{Q,\eta}(f) \approx \left(\sum_{\Delta \in \mathcal{T}} (|\Delta|^{1/p-1/\eta} \|q_m(f)\|_{L_\eta(\Delta)})^\tau \right)^{1/\tau}, \quad \text{if } \eta \geq 1, \quad (2.22)$$

$$N_{Q,\eta}(f) \approx \left(\sum_{\Delta \in \mathcal{T}} (|\Delta|^{1/p-1/\eta} \|t_{m,\eta}(f)\|_{L_\eta(\Delta)})^\tau \right)^{1/\tau}, \quad \text{if } 0 < \eta < 1, \quad (2.23)$$

and

$$N_{Q,\eta}(f) \approx \left(\sum_{\theta \in \Theta} (|\theta|^{1/p} |b_{\theta,\eta}(f)|)^\tau \right)^{1/\tau} \approx \left(\sum_{\theta \in \Theta} \|b_{\theta,\eta}(f) \varphi_\theta\|_p^\tau \right)^{1/\tau}. \quad (2.24)$$

In the most interesting case of $p = \infty$,

$$N_{Q,\eta}(f) \approx \left(\sum_{\theta \in \Theta} |b_{\theta,\eta}(f)|^\tau \right)^{1/\tau}. \quad (2.25)$$

• **General B-spaces.** A more general B-space $B_{pq}^\alpha(\mathcal{T})$, $\alpha > 0$, $0 < p, q \leq \infty$, is defined as the set of all $f \in L_p(E)$ such that

$$\|f\|_{B_{pq}^\alpha(\mathcal{T})} := \inf_{f = \sum_{\theta \in \Theta} c_\theta \varphi_\theta} \left(\sum_{m \in \mathbb{Z}} \left[2^{m\alpha} \left(\sum_{\theta \in \Theta, 2^{-m} \leq |\theta| < 2^{-m+1}} \|c_\theta \varphi_\theta\|_p^p \right)^{1/p} \right]^q \right)^{1/q} < \infty,$$

where the ℓ_q -norm is replaced by the sup-norm if $q = \infty$. In this paper, we do not need the B-spaces in such generality.

• **Embedding theorems and equivalence of norms.** We recall our assumptions. We have $0 < p \leq \infty$, and $\alpha \geq 1$ if $p = \infty$ and $\alpha > 0$ if $p < \infty$. In both cases, $1/\tau := \alpha + 1/p$ ($1/\tau := \alpha$ if $p = \infty$). We record estimates and embeddings from [11], along with the necessary modifications, which are necessary for the development of the main results of this paper. The first embedding result appears as Theorem 2.16 in [11].

Theorem 2.6. *For $0 < \tau < p < \infty$ or $p = \infty, \tau \leq 1$, then for any sequence of real numbers $(c_\theta)_{\theta \in \Theta}$, we have*

$$\left\| \sum_{\theta \in \Theta} |c_\theta| \varphi_\theta \right\|_p \leq c \left(\sum_{\theta \in \Theta} \|c_\theta \varphi_\theta\|_p^\tau \right)^{1/\tau}, \quad (2.26)$$

where c depends only on τ, p , and the parameters of \mathcal{T} .

Theorem 2.7. *If $f \in L_\eta(E)$ with $0 < \eta < p$, and $N_{Q,\eta}(f) < \infty$, then $f \in L_p(E)$ ($f \in C(E)$ if $p = \infty$) and f has the representation $f = \sum_{\theta \in \Theta} b_{\theta,\eta}(f) \varphi_\theta$ with the series converging absolutely a.e. in E and in L_p (respectively, in $C(E)$) and*

$$\|f\|_p \leq \left\| \sum_{\theta \in \Theta} |b_{\theta,\eta}(f)| \varphi_\theta \right\|_p \leq c N_{Q,\eta}(f), \quad (2.27)$$

where c is independent of f .

Proof. For $0 < p < \infty$, the result follows from (2.11), (2.13), and Theorem 2.9 below. If $p = \infty$, the theorem follows by (2.11), (2.13), (2.25), and the following estimates

$$\left\| \sum_{\theta \in \Theta} b_{\theta,\eta}(f) \varphi_\theta \right\|_\infty \leq \left(\sum_{\theta \in \Theta} |b_{\theta,\eta}(f)|^\tau \right)^{1/\tau} \leq c N_{Q,\eta}(f), \quad (\tau = 1/\alpha \leq 1). \quad \square$$

Remark 2.8. *It is easily seen that Theorem 2.7 is not true when $p = \infty$ and $\alpha < 1$. For this reason we impose the restriction $\alpha \geq 1$ when $p = \infty$ throughout.*

Theorem 2.9. *The norms $\|\cdot\|_{B_{pq}^\alpha(\mathcal{T})}$, $N_{\mathbb{S},\eta}(\cdot)$ ($0 < \eta < p$), $N_\Phi(\cdot)$, and $N_{Q,\eta}(\cdot)$ ($0 < \eta < p$), defined in (2.15), (2.16), (2.17), and (2.21) are equivalent with constants of equivalence depending only on p, τ, η , and the parameters of \mathcal{T} .*

Proof. One proceeds exactly as in [11] (see the proof of Theorem 2.17 of that reference) and proves that

$$\begin{aligned} |f|_{B_{pq}^\alpha(\mathcal{T})} &\approx \left(\sum_{\Delta \in \mathcal{T}} (|\Delta|^{1/p-1/\eta} \mathbb{S}_\Delta(f)_\eta)^\tau \right)^{1/\tau} \approx \inf_{f = \sum_{\theta \in \Theta} c_\theta \varphi_\theta} \left(\sum_{\theta \in \Theta \setminus \Theta_0} \|c_\theta \varphi_\theta\|_p^\tau \right)^{1/\tau} \\ &\approx \left(\sum_{\theta \in \Theta \setminus \Theta_0} \|b_{\theta,\eta}(f) \varphi_\theta\|_p^\tau \right)^{1/\tau}, \end{aligned} \quad (2.28)$$

provided $0 < \eta < p$. To obtain the norm estimate from these semi-norm equivalences, we use Theorem 2.6 to give $\|f\|_p \leq cN_\Phi(f)$. Using this, (2.28), and the remark after the definition of $N_\Phi(f)$ in (2.17), we obtain

$$\|f\|_{B_\tau^\alpha(\mathcal{T})} \approx N_{S,\eta}(f) \leq cN_\Phi(f) \leq cN_{Q,\eta}(f), \quad 0 < \eta < p.$$

For the reverse inequality, we use Lemma 2.2, Theorem 2.6, and (2.28) to obtain

$$\begin{aligned} \left(\sum_{\theta \in \Theta_0} \|b_{\theta,\eta}(f)\varphi_\theta\|_p^\tau \right)^{1/\tau} &\leq c(\#\Theta_0, \tau, p) \left(\sum_{\theta \in \Theta_0} \|b_{\theta,\eta}(f)\varphi_\theta\|_p^p \right)^{1/p} \\ &\leq c \left\| \sum_{\theta \in \Theta_0} b_{\theta,\eta}(f)\varphi_\theta \right\|_p \leq c \left(\|f\|_p + \left\| \sum_{\theta \in \Theta \setminus \Theta_0} b_{\theta,\eta}(f)\varphi_\theta \right\|_p \right) \\ &\leq c\|f\|_p + c \left(\sum_{\theta \in \Theta \setminus \Theta_0} \|b_{\theta,\eta}(f)\varphi_\theta\|_p^\tau \right)^{1/\tau} \\ &\leq c\|f\|_{B_\tau^\alpha(\mathcal{T})}. \end{aligned}$$

This and (2.28) imply $N_{Q,\eta}(f) \leq c\|f\|_{B_\tau^\alpha(\mathcal{T})}$. \square

The next embedding theorem of Sobolev type follows immediately from (2.18) or (2.24).

Theorem 2.10. *For $0 < \alpha_0 < \alpha_1$ and $\tau_j := (\alpha_j + 1/p)^{-1}$, $j = 0, 1$, we have the following continuous embedding*

$$B_{\tau_1}^{\alpha_1}(\mathcal{T}) \subset B_{\tau_0}^{\alpha_0}(\mathcal{T}), \quad (2.29)$$

i.e., if $f \in B_{\tau_1}^{\alpha_1}(\mathcal{T})$, then $f \in B_{\tau_0}^{\alpha_0}(\mathcal{T})$ and $\|f\|_{B_{\tau_0}^{\alpha_0}(\mathcal{T})} \leq c\|f\|_{B_{\tau_1}^{\alpha_1}(\mathcal{T})}$.

• **Interpolation.** We first recall some basic definitions from the real interpolation method. We refer the reader to [1] and [2] as general references for interpolation theory. For a pair of quasi-normed spaces X_0, X_1 , embedded in a Hausdorff space, the space $X_0 + X_1$ is defined as the collection of all functions f that can be represented as $f_0 + f_1$ with $f_0 \in X_0$ and $f_1 \in X_1$. The quasi-norm in $X_0 + X_1$ is defined by

$$\|f\|_{X_0+X_1} := \inf_{f=f_0+f_1} \|f_0\|_{X_0} + \|f_1\|_{X_1}.$$

The K-functional is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t) := K(f, t; X_0, X_1) := \inf_{f=f_0+f_1} \|f_0\|_{X_0} + t\|f_1\|_{X_1}. \quad (2.30)$$

The real interpolation space $(X_0, X_1)_{\lambda,q}$ with $0 < \lambda < 1$ and $0 < q \leq \infty$ is defined as the set of all $f \in X_0 + X_1$ such that

$$\|f\|_{(X_0, X_1)_{\lambda,q}} := \left(\int_0^\infty (t^{-\lambda} K(f, t))^q \frac{dt}{t} \right)^{1/q} < \infty,$$

where the L_q -norm is replaced by the sup-norm if $q = \infty$.

It is easily seen that if $X_1 \subset X_0$ (X_1 continuously embedded in X_0), then $K(f, t) \approx \|f\|_{X_0}$ for $f \in X_0$ and $t \geq 1$ and, consequently,

$$\|f\|_{(X_0, X_1)_{\lambda, q}} \approx \|f\|_{X_0} + \left(\sum_{\nu=0}^{\infty} [2^{\nu\lambda} K(f, 2^{-\nu})]^q \right)^{1/q}. \quad (2.31)$$

Theorem 2.11. *Suppose $0 < p \leq \infty$ and further assume that both $\alpha_0, \alpha_1 \geq 1$ in the case $p = \infty$, and $\alpha_0, \alpha_1 > 0$ otherwise. Furthermore, let $\tau_j := (\alpha_j + 1/p)^{-1}$, $j = 0, 1$. Then*

$$(B_{\tau_0}^{\alpha_0}(\mathcal{T}), B_{\tau_1}^{\alpha_1}(\mathcal{T}))_{\lambda, \tau} = B_{\tau}^{\alpha}(\mathcal{T}) \quad (2.32)$$

with equivalent norms, provided $\alpha = (1 - \lambda)\alpha_0 + \lambda\alpha_1$ with $0 < \lambda < 1$ and $\tau := (\alpha + 1/p)^{-1}$.

Proof. We shall prove (2.32) only in the case $p > 1$. For a proof of (2.32) when $p \leq 1$, see [4].

We shall use the abbreviated notation $B^{\alpha} := B_{\tau}^{\alpha}(\mathcal{T})$ and $B^{\alpha_j} := B_{\tau_j}^{\alpha_j}(\mathcal{T})$, $j = 0, 1$. Also, we denote by ℓ_q the space of all sequences $\mathbf{a} = (a_{\theta})_{\theta \in \Theta}$ of real numbers such that $\|\mathbf{a}\|_{\ell_q} := (\sum_{\theta \in \Theta} |a_{\theta}|^q)^{1/q} < \infty$.

We set $\eta := 1$ and normalize the Courant elements in L_p , that is, $\|\varphi_{\theta}\|_p = 1$. We also renormalize the duals $\tilde{\varphi}_{\theta}$ from (2.7) accordingly. We denote again by $\mathbf{b}(f) = (b_{\theta})_{\theta \in \Theta}$ the sequence from (2.10) with respect to the normalized Courant elements. By (2.24), Theorem 2.7, and Theorem 2.9, if $f \in B^{\alpha_j}$, $j = 0, 1$, then

$$f = \sum_{\theta \in \Theta} b_{\theta}(f) \varphi_{\theta} \quad \text{and} \quad \|f\|_{B^{\alpha_j}} \approx \|\mathbf{b}(f)\|_{\ell_{\tau_j}}, \quad (2.33)$$

recalling that the elements φ_{θ} are normalized in L_p . The corresponding statement holds for functions $f \in B^{\alpha}$ as well.

We shall next employ the following interpolation theorem (see, e.g., §5.1 of [2] or [1]) which follows directly from the definition of the K -functional and the norms of the interpolation spaces. Suppose T is a linear operator which maps boundedly X_0 into Y_0 and X_1 into Y_1 , where (X_0, X_1) and (Y_0, Y_1) are couples of quasi-normed spaces as above. Then for $0 < \lambda < 1$ and $0 < q \leq \infty$, T maps boundedly $(X_0, X_1)_{\lambda, q}$ into $(Y_0, Y_1)_{\lambda, q}$.

We introduce linear operators \mathcal{I} and \mathcal{P} as follows: \mathcal{I} is defined by $\mathcal{I}(f)_{\theta} := b_{\theta}(f)$, $\theta \in \Theta$, and \mathcal{P} is given by $\mathcal{P}(\mathbf{a}) := \sum_{\theta \in \Theta} a_{\theta} \varphi_{\theta}$, $\mathbf{a} = (a_{\theta})_{\theta \in \Theta}$. By (2.33), $\|\mathbf{b}(f)\|_{\ell_{\tau_j}} \leq c \|f\|_{B^{\alpha_j}}$ for $f \in B^{\alpha_j}$, $j = 0, 1$, and hence $\mathcal{I} : B^{\alpha_j} \rightarrow \ell_{\tau_j}$ (boundedly). By the above mentioned interpolation theorem,

$$\mathcal{I} : (B^{\alpha_0}, B^{\alpha_1})_{\lambda, \tau} \rightarrow (\ell_{\tau_0}, \ell_{\tau_1})_{\lambda, \tau} \quad (\text{boundedly}). \quad (2.34)$$

Similarly, if $\mathbf{a} \in \ell_{\tau_j}$, then by Theorem 2.7 and Theorem 2.9, we may conclude that $\mathcal{P}(\mathbf{a}) \stackrel{L_p}{=} \sum_{\theta \in \Theta} a_{\theta} \varphi_{\theta}$ is well defined so if we set $f = \mathcal{P}(\mathbf{a})$, then

$$\|\mathcal{P}(\mathbf{a})\|_{B^{\alpha_j}} \leq c \inf_{f = \sum_{\theta \in \Theta} c_{\theta} \varphi_{\theta}} \|(c_{\theta})_{\theta \in \Theta}\|_{\ell_{\tau_j}} \leq c \|\mathbf{a}\|_{\ell_{\tau_j}}, \quad j = 0, 1.$$

Thus $\mathcal{P} : \ell_{\tau_j} \rightarrow B^{\alpha_j}$ (boundedly) and by interpolation

$$\mathcal{P} : (\ell_{\tau_0}, \ell_{\tau_1})_{\lambda, \tau} \rightarrow (B^{\alpha_0}, B^{\alpha_1})_{\lambda, \tau} \quad (\text{boundedly}). \quad (2.35)$$

Finally, we recall the well known interpolation result (see, e.g. [1, 2]):

$$(\ell_{\tau_0}, \ell_{\tau_1})_{\lambda, \tau} = \ell_{\tau}, \quad \text{where } \frac{1}{\tau} = \frac{1-\lambda}{\tau_0} + \frac{\lambda}{\tau_1} \text{ with } 0 < \lambda < 1. \quad (2.36)$$

Clearly, (2.32) follows by (2.33)-(2.36). \square

• **Skinny B-spaces.** The skinny B -spaces were introduced in [11] and used for characterization of nonlinear (discontinuous) piecewise polynomial approximation on \mathbb{R}^2 . We next adapt that definition to the case of approximation on a compact polygonal domain $E \subset \mathbb{R}^2$. Suppose \mathcal{T} is a multilevel nested triangulation of E which additionally satisfies condition (2.2) (see §2.1 and [11]). The skinny B -space $\mathcal{B}_\tau^{\alpha k}(\mathcal{T})$, where $k \geq 1$ and α and τ are as above, is defined as the set of all $f \in L_\tau(E)$ such that

$$|f|_{\mathcal{B}_\tau^{\alpha k}(\mathcal{T})} := \left(\sum_{\Delta \in \mathcal{T}} (|\Delta|^{-\alpha} \omega_k(f, \Delta)_\tau)^\tau \right)^{1/\tau} < \infty, \quad (2.37)$$

where $\omega_k(f, \Delta)_\tau$ is a k -th modulus of smoothness of f in $L_\tau(\Delta)$ defined by $\omega_k(f, \Delta)_\tau := \sup_{h \in \mathbb{R}^2} \|\Delta_h^k(f, \cdot)\|_{L_\tau(\Delta)}$ and $\Delta_h^k(f, \cdot)$ is the k -th difference of f . The norm in $\mathcal{B}_\tau^{\alpha k}(\mathcal{T})$ is defined by $\|\cdot\|_{\mathcal{B}_\tau^{\alpha k}(\mathcal{T})} := \|\cdot\|_p + |\cdot|_{\mathcal{B}_\tau^{\alpha k}(\mathcal{T})}$.

• **Fat B-spaces: The link to Besov spaces.** Suppose \mathcal{T} is an SLR -triangulation of a compact polygonal domain $E \subset \mathbb{R}^2$. Similarly as in [11], we define the fat B -space $\mathbb{B}_\tau^{\alpha k}(\mathcal{T})$, where $k \geq 1$ and α and τ are as above, as the set of all functions $f \in L_\tau(E)$ such that

$$|f|_{\mathbb{B}_\tau^{\alpha k}(\mathcal{T})} := \left(\sum_{\Delta \in \mathcal{T}} (|\Delta|^{-\alpha} \omega_k(f, \Omega_\Delta)_\tau)^\tau \right)^{1/\tau} < \infty. \quad (2.38)$$

We endow $\mathbb{B}_\tau^{\alpha k}(\mathcal{T})$ with the norm $\|\cdot\|_{\mathbb{B}_\tau^{\alpha k}(\mathcal{T})} := \|\cdot\|_p + |\cdot|_{\mathbb{B}_\tau^{\alpha k}(\mathcal{T})}$. Using Whitney's theorem, it readily follows that $c_1 \omega_2(f, \Delta)_\tau \leq \mathbb{S}_\Delta(f)_\tau \leq c_2 \omega_2(f, \Omega_\Delta)_\tau$ and hence $|f|_{\mathbb{B}_\tau^{\alpha 2}(\mathcal{T})} \leq c |f|_{B_\tau^\alpha(\mathcal{T})} \leq c |f|_{\mathbb{B}_\tau^{\alpha 2}(\mathcal{T})}$. The space $\mathbb{B}_\tau^{\alpha 2}(\mathcal{T})$ is a natural candidate to replace $B_\tau^\alpha(\mathcal{T})$ in nonlinear n -term Courant element approximation. This is, however, only possible for sufficiently small α ($0 < \alpha < \alpha_0$). Otherwise $\mathbb{B}_\tau^{\alpha 2}(\mathcal{T})$ is too “fat” and is not suitable for these purposes.

• **Comparison between B-spaces and Besov spaces.** As shown in [11] (Theorem 2.28), if \mathcal{T} is a regular triangulation and $0 < \alpha < 1 + 1/p$, then

$$B_\tau^\alpha(\mathcal{T}) = B_\tau^{2\alpha}(L_\tau) \quad (= \mathbb{B}_\tau^{\alpha 2}(\mathcal{T})) \quad (2.39)$$

with equivalent norms, where $B_\tau^{2\alpha}(L_\tau)$ is the Besov space of all Lebesgue measurable functions f on \mathbb{R}^2 such that

$$|f|_{B_\tau^{2\alpha}(L_\tau)} := \left(\int_0^\infty (t^{-2\alpha} \omega_2(f, t)_\tau)^\tau \frac{dt}{t} \right)^{1/\tau} < \infty \quad (2.40)$$

with $\omega_2(f, t)_\tau$ the second modulus of smoothness of f in $L_\tau(\mathbb{R}^2)$. (Notice that the smoothness parameters of B-spaces and Besov spaces are normalized differently, α corresponds to 2α .)

Moreover, the identification (2.39) is no longer valid if $\alpha \geq 1 + 1/p$. More precisely (see [11]), for each $\theta \in \Theta$, $|\varphi_\theta|_{B_\tau^{2\alpha}(L_\tau)} = \infty$, if $\alpha \geq 1 + 1/p$, while $|\varphi_\theta|_{B_\tau^\alpha(\mathcal{T})} \approx \|\varphi_\theta\|_p$.

A peculiar situation occurs when $p = \infty$ since, in that case, the constraints on α provide no range of α for which (2.39) holds.

The univariate case is another interesting case to be mentioned. Jackson and Bernstein estimates and the identification of approximation spaces determined by nonlinear approximation rates was established in [13] for uniform grids. B-spaces similar to these from the present article can be defined in dimension $d = 1$ but all of them will be equivalent to the corresponding Besov spaces and hence are not needed. For more complete discussion of this and other related issues, see [11]. Our constructive algorithms are never-the-less valid and should prove very useful in this simplified setting as well.

3 Algorithms for n-term Courant approximation

• **Decomposition step for all approximation algorithms.** The first step of each of the three approximation algorithms that we consider in this section is a decomposition step. This step is not trivial since the set $\Phi(\mathcal{T}) := (\varphi_\theta)_{\theta \in \Theta}$ of all Courant elements is redundant and, therefore, each function has infinitely many representations using Courant elements. For each algorithm, it is crucial to have a sufficiently efficient initial representation of the function f which is being approximated. This means that the representation of f should allow a realization of the corresponding B-norm.

To construct the initial representation, we consider two cases of metric approximation. If the approximation takes place in L_p , $1 < p \leq \infty$, we utilize the decomposition of f via quasi-interpolation from (2.11) with $1 \leq \eta < p$, while if $0 < p \leq 1$, we use (2.13) with $0 < \eta < p$. In both cases, we have an initial desirable sparse representation of f of the form

$$f = \sum_{\theta \in \Theta} b_\theta \varphi_\theta, \quad b_\theta = b_\theta(f), \quad (3.1)$$

which allows a realization of the B-norm (see (2.24)-(2.25) and Theorem 2.9). For the remainder of this subsection, in order to more easily track the dependency of the constants appearing in inequalities, we redefine $\|f\|_{B_r^\alpha(\mathcal{T})}$ by

$$\|f\|_{B_r^\alpha(\mathcal{T})} := \left(\sum_{\theta \in \Theta} (|b_\theta| |\theta|^{1/p})^\tau \right)^{1/\tau} \approx \left(\sum_{\theta \in \Theta} \|b_\theta \varphi_\theta\|_p^\tau \right)^{1/\tau}. \quad (3.2)$$

which is an equivalent norm in $B_r^\alpha(\mathcal{T})$ (see Theorem 2.9). Without loss of generality we may assume (when needed) that there is a final level Θ_L ($L < \infty$), in (3.1).

3.1 “Threshold” algorithm ($p < \infty$ only)

In this algorithm we utilize the usual thresholding strategy used for n -term approximation from a basis in L_p ($1 < p < \infty$). The resulting procedure performs extremely well due to the sparse representation realized by the first step. We note, however, that the derived error estimates involve constants that depend on p and become unbounded as $p \rightarrow \infty$. The “Push the Error” and “Trim & Cut” algorithms described later in this section will be shown to achieve the corresponding estimates for the uniform norm ($p = \infty$). For this subsection we

therefore assume that $f \in L_p$, $0 < p < \infty$.

• **Description of the “Threshold” algorithm.**

Step 1. (*Decompose*) We use the decomposition of $f \in L_p(E)$ from (3.1).

Step 2. (*Select the n largest terms*) We order the terms $(b_\theta \varphi_\theta)_{\theta \in \Theta}$ in a sequence $(b_{\theta_j} \varphi_{\theta_j})_{j=1}^\infty$ so that

$$\|b_{\theta_1} \varphi_{\theta_1}\|_p \geq \|b_{\theta_2} \varphi_{\theta_2}\|_p \geq \dots \quad (3.3)$$

Then we define the approximant $A_n^T(f)_p$ by $A_n^T(f)_p := \sum_{j=1}^n b_{\theta_j} \varphi_{\theta_j}$.

• **Error estimation for the “Threshold” algorithm.**

We denote the corresponding error of approximation of this threshold algorithm by $\mathbb{A}_n^T(f)_p := \|f - A_n^T(f)_p\|_p$. The argument used in establishing the Jackson error estimate in [11] may be modified in obvious ways to prove the following error estimate.

Theorem 3.1. *If $f \in B_\tau^\alpha(\mathcal{T})$, $\alpha > 0$, $1/\tau := \alpha + 1/p$ ($0 < p < \infty$), then*

$$\mathbb{A}_n^T(f)_p \leq cn^{-\alpha} \|f\|_{B_\tau^\alpha(\mathcal{T})}, \quad (3.4)$$

where c depends on α , p , and the parameters of \mathcal{T} .

In §5, we shall need the following result:

Lemma 3.2. *If $f = \sum_{\theta \in \Theta} b_\theta \varphi_\theta$ is the decomposition of f from (3.1), then*

$$\mathbb{A}_{2n}^T(f)_p \leq cn^{-\alpha} \left(\sum_{j=n+1}^\infty \|b_{\theta_j} \varphi_{\theta_j}\|_p^\tau \right)^{1/\tau},$$

where $(b_{\theta_j} \varphi_{\theta_j})_{j=1}^\infty$ is as in **Step 2** and c depends on α , p , and the parameters of \mathcal{T} .

Proof. Applying Theorem 3.4 from [11] to $(b_{\theta_j} \varphi_{\theta_j})_{j=n+1}^\infty$, immediately provides the desired result. \square

Remark 3.3. *As we have mentioned, the main drawback of the “Threshold” algorithm is that it is not applicable to approximation in the uniform norm since the constant $c = c(\alpha, p)$ in (3.4) tends to infinity as $p \rightarrow \infty$ and the performance of the algorithm deteriorates as p gets large. The obvious reason for this behavior is that f can be built out of many terms $(b_\theta \varphi_\theta)$ which have small coefficients and are supported at the same location. These terms can pile up to an essential contribution but the algorithm will fail to anticipate their future significance.*

3.2 “Trim & Cut (the tree)” algorithm

The idea of this algorithm has its origins in the proof of the Jackson estimate in [7] (see §5, pages 272-276). The approximation considered there is by wavelets or splines over a uniform partition in the uniform norm. We shall refine this idea to develop an algorithm for n -term Courant element approximation in $L_p(E)$, $0 < p \leq \infty$, over LR-triangulations. We begin with a brief description of the algorithm and then elaborate on the details of each of the main steps.

- **Description of the “Trim & Cut” algorithm.**

Step 1. (*Decompose*) We use the common decomposition of $f \in L_p(E)$ given in (3.1).

Step 2. (*Organize the cells of Θ into manageable trees Θ^ν*) We develop an algorithm (procedure) for coloring the cells of Θ in such a way that the cells of the same color form a tree structure as described in Lemma 3.4 below. This organization greatly simplifies the management of the estimates, both the approximation construction and the enumeration of “active” Courant elements in our approximant.

Step 3. (*Trim each tree*) Since all the elements may initially effect the B -space norm of a function, we need to preprocess each tree by pruning all branches which may have many leaves, but do not have significant contribution to the norm of the function f . We do this by running a stopping time argument from the finest level to coarser, until a significant cumulative contribution is met. We prune the branch just below that element.

Step 4. (*Partition the remaining trees into “segments”*) We continue to partition the remainders of each of the K trees by cutting them at each of the joins of branches to form chains from the tree. We will easily be able to track the number of chains produced by this procedure. A second stopping time argument is then applied to cut the chains into “segments” in order to control the number of significant elements added to the approximant (at most $N_0 + 1$ from each segment) and to guarantee that the cumulative effect of the left-over elements (i.e., error) can be controlled by the final Step 5.

Step 5. (*Rewrite the “segments” to control error*) Here each segment is rewritten at its finest level and its terminal element (with the new coefficients) and some of its neighboring elements are added to the approximant. This allows for a void to be created so that the residual of the segment will have disjoint support with all remaining segments as well as the residuals of those previously processed. This insures that the cumulative pointwise error remains under control.

We now describe these rather vague steps in more detail. Step 1 is clear from our earlier discussion.

Step 2. In the following lemma, we construct a procedure for coloring the elements of Θ with K colors ν , so that no two Courant elements of the same color from the same level have supports that intersect, in fact corresponding cells of the same color will have a tree structure with set inclusion as the order relation. This allows us to partition Θ into a disjoint

union of sets Θ^ν ($1 \leq \nu \leq K$), and correspondingly organize f as the sum $f = \sum_{\nu=1}^K f_\nu$, where $f_\nu := \sum_{\theta \in \Theta^\nu} b_\theta \varphi_\theta$. We can then proceed to process each of the f_ν without worry about its terms from the same level overlapping and at worst a factor of K will come into the constants for the estimates that we derive. For its proof, see the Appendix.

Lemma 3.4. [Coloring lemma] *For any multilevel-triangulation \mathcal{T} of E , the set $\Theta := \Theta(\mathcal{T})$ of all cells generated by \mathcal{T} can be represented as a finite disjoint union of its subsets $(\Theta^\nu)_{\nu=1}^K$ with $K = K(N_0, M_0)$ (N_0 is the maximal valence and M_0 is the maximal number of children of a triangle in \mathcal{T}), such that each Θ^ν has a tree structure with respect to set inclusion, i.e., if $\theta', \theta'' \in \Theta^\nu$ with $(\theta')^\circ \cap (\theta'')^\circ \neq \emptyset$, then either $\theta' \subset \theta''$ or $\theta'' \subset \theta'$.*

In order to complete the remaining **Steps 3-5** we must consider two variations in the details of the algorithm depending whether $p = \infty$ or $0 < p < \infty$. The case of the uniform metric is presented in Subsection 3.2.1, while the case of L_p ($0 < p < \infty$) is given in Subsection 3.2.2.

3.2.1 The case $p = \infty$.

Fix $\varepsilon > 0$ and let $\varepsilon^* := \frac{\varepsilon}{2K}$ where we recall that K is the number of colors representing the tree structures.

Step 3. *Trimming of Θ^ν ($1 \leq \nu \leq K$) with ε^* .*

We trim each Θ^ν , starting from the finest level Θ_f^ν and proceeding to the coarsest level. We remove from Θ^ν every cell θ^\diamond such that

$$\sum_{\theta \subset \theta^\diamond} |b_\theta| \leq \varepsilon^*. \quad (3.5)$$

We denote by Γ^ν the set of all $\theta \in \Theta^\nu$ which have been retained after completing this procedure and by Γ_f^ν the set of all final cells in Γ^ν , i.e., $\theta^\diamond \in \Gamma_f^\nu$ iff there is no $\theta \in \Gamma^\nu$ such that $\theta \subsetneq \theta^\diamond$. Clearly, for each $\theta^\diamond \in \Gamma_f^\nu$,

$$\sum_{\theta \subset \theta'} |b_\theta| \leq \varepsilon^* \text{ for each } \theta' \subsetneq \theta^\diamond, \text{ but } \sum_{\theta \subset \theta^\diamond} |b_\theta| > \varepsilon^*. \quad (3.6)$$

We denote $f_{\Gamma^\nu} := \sum_{\theta \in \Gamma^\nu} b_\theta \varphi_\theta$. Therefore,

$$\|f_\nu - f_{\Gamma^\nu}\|_\infty \leq \max_{\theta^\diamond \notin \Gamma^\nu} \left\| \sum_{\theta \subset \theta^\diamond} b_\theta \varphi_\theta \right\|_\infty \leq \max_{\theta^\diamond \notin \Gamma^\nu} \sum_{\theta \subset \theta^\diamond} |b_\theta| \leq \varepsilon^* \quad (3.7)$$

and hence if we set $f_\Gamma := \sum_{\nu=1}^K f_{\Gamma^\nu}$, then

$$\|f - f_\Gamma\|_\infty \leq K\varepsilon^* = \varepsilon/2. \quad (3.8)$$

Step 4. *Partitioning the branches of each tree Γ^ν into chains and the chains into “segments”.* For each of the tree structures Γ^ν ($1 \leq \nu \leq K$), we denote by Γ_b^ν the set of all *branching*

cells in Γ^ν (cells with more than one child in Γ^ν) and by Γ_{ch}^ν the set of all *chain* cells in Γ^ν (cells with exactly one child in Γ^ν). It is easy to see that

$$\#\Gamma_b^\nu \leq \#\Gamma_f^\nu. \quad (3.9)$$

In fact, one proceeds by induction from the finest to coarser levels, associating each branch cell from Γ_b^ν by a cell from Γ_f^ν . For each branch cell, there is always at least one member of Γ_f^ν still available from each descendant edge. Only one is used to associate with the current branch cell thereby leaving at least one available for its next ancestor branch cell in that line.

On the other hand, $\#\Gamma_{ch}^\nu$ may be much larger than $\#\Gamma_f^\nu$ and so we will need to process these elements. A collection of cells $\theta_1 \supset \theta_2 \supset \cdots \supset \theta_l$ is called a *chain* if for $j = 1, \dots, l-1$ θ_{j+1} is a child of θ_j and $\theta_j \in \Gamma_{ch}^\nu$, and the terminal cell $\theta_l \in \Gamma_f^\nu \cup \Gamma_b^\nu$. We partition the tree Γ^ν into chains. Namely, we start at the coarsest level and construct (maximal) chains which will terminate with either a final cell (in Γ_f^ν) or a branching cell (in Γ_b^ν). We continue this procedure to the finest level.

We next “section” each chain into *segments* using ε^* as a threshold. Namely, if λ is a chain and $\lambda = (\theta_j)_{j=1}^l$ with $\theta_1 \supset \theta_2 \supset \cdots \supset \theta_l$, then we start from the coarsest element θ_1 and sum the coefficients of each cell moving to the next child of the chain until the sum exceeds the threshold. At this point we cut the chain to form the first (significant) segment and start this procedure again with the next child in line until this is not possible (i.e., ending without the threshold being crossed). We call this type of segment a “*remnant segment*”. Therefore, this procedure cuts λ into disjoint segments σ of the form $(\theta_j)_{j=i}^{i+\mu}$, $\mu \geq 0$, so that each segment satisfies exactly one of the following conditions:

(a) σ consists of a single “significant cell”:

$$|b_{\theta_i}| > \varepsilon^* \text{ (case of } \mu = 0), \quad (3.10)$$

(b) σ is a “significant segment”:

$$\sum_{j=i}^{i+\mu-1} |b_{\theta_j}| \leq \varepsilon^*, \text{ but } \sum_{j=i}^{i+\mu} |b_{\theta_j}| > \varepsilon^*, \text{ (case of } \mu > 0), \quad (3.11)$$

(c) σ is an “remnant segment”:

$$\sum_{j=i}^l |b_{\theta_j}| \leq \varepsilon^*. \quad (3.12)$$

We denote by Σ^ν the set of all such segments $\sigma = (\theta_j)_{j=i}^{i+\mu}$ resulting from this procedure.

Step 5. *Rewriting elements from certain segments of Σ^ν .*

Let $\sigma = (\theta_j)_{j=1}^\mu$ be any segment from Σ^ν and suppose that the finest cell θ_μ of σ belongs to Θ_m . We rewrite the Courant elements $(\sum_{j=1}^\mu b_{\theta_j} \varphi_{\theta_j})$ of the segment at its finest (m -th) level, finding coefficients (c_θ) such that

$$\sum_{\theta \in \Theta_m, \theta^\circ \cap \theta_\mu \neq \emptyset} c_\theta \varphi_\theta = \sum_{j=1}^\mu b_{\theta_j} \varphi_{\theta_j} \text{ on } \theta_\mu.$$

We denote $\mathcal{X}_\sigma := \{\theta \in \Theta_m : \theta^\circ \cap \theta_\mu \neq \emptyset \text{ and } \theta \subset \theta_1\}$. Obviously, if $\mu = 1$ (i.e., the segment consists of a single cell), then the coefficient remains unchanged and $\mathcal{X}_\sigma = \sigma = \{\theta_1\}$. Observe in any case that $\#\mathcal{X}_\sigma \leq N_0 + 1$ and $\cup_{\theta \in \mathcal{X}_\sigma} \theta \subset \theta_1$. Finally, set $\Sigma := \cup_{\nu=1}^K \Sigma^\nu$, and correspondingly define

$$A_\varepsilon^{TC}(f) := \sum_{\sigma \in \Sigma} \sum_{\theta \in \mathcal{X}_\sigma} c_\theta \varphi_\theta. \quad (3.13)$$

as our approximant produced by the ‘‘Trim & Cut’’ algorithm.

• **Error estimation for the ‘‘Trim & Cut’’ algorithm (Case $p = \infty$).**

Suppose that the ‘‘Trim & Cut’’ procedure has been applied to a function f with $\varepsilon > 0$, and $A_\varepsilon^{TC}(f) = \sum_{\theta \in \Lambda_\varepsilon} c_\theta \varphi_\theta$ is the resulting approximant from (3.13), where $\Lambda_\varepsilon = \cup_{\sigma \in \Sigma} \mathcal{X}_\sigma$. We denote

$$n(\varepsilon) := n_f(\varepsilon) := \#\Lambda_\varepsilon, \quad \mathbb{A}_{n(\varepsilon)}^{TC}(f)_\infty := \|f - A_\varepsilon^{TC}(f)\|_\infty,$$

and

$$\mathbb{A}_n^{TC}(f)_\infty := \inf \{ \mathbb{A}_{n(\varepsilon)}^{TC}(f)_\infty : n(\varepsilon) \leq n \}.$$

Note that each of these quantities depend implicitly on \mathcal{T} . To complete our results for the ‘‘Trim & Cut’’ algorithm, we first show in Lemma 3.5 that this is a good approximation to f , and that the number of elements which are used in the approximant satisfies the correct estimates (see Theorem 3.7 below).

Lemma 3.5. *Suppose that $A_\varepsilon^{TC}(f)$ is the approximant for f given in equation (3.13) which has been constructed using the ‘‘Trim & Cut’’ algorithm, then there holds the error estimate*

$$\|f - A_\varepsilon^{TC}(f)\|_\infty \leq \varepsilon. \quad (3.14)$$

Proof. Following the definition (3.13) of $A_\varepsilon^{TC}(f)$, we define

$$A^\nu := \sum_{\sigma' \in \Sigma^\nu} \sum_{\theta \in \mathcal{X}_{\sigma'}} c_\theta \varphi_\theta,$$

then obviously $A_\varepsilon^{TC}(f) = \sum_{\nu=1}^K A^\nu$. Since $\varepsilon^* = \frac{\varepsilon}{2K}$ it suffices to show that $\|f_{\Gamma^\nu} - A^\nu\|_\infty \leq \varepsilon^*$.

In Step 5 we extracted the heart of each segment $\sigma = (\theta_j)_{j=1}^\mu$, added its contribution to the approximant (3.13), and cleared room for descendant cells. To estimate the associated error, we introduce the *ring* for σ as $R_\sigma := \theta_1 \setminus \theta_\mu$, then $R_\sigma = \emptyset$ when σ consists of a significant cell (i.e., condition (3.10) holds). For any nonempty ring R_σ ($\sigma \in \Sigma^\nu$), set $\sigma' := (\theta_j)_{j=1}^{\mu-1}$ and observe that at worst

$$\begin{aligned} \|f_{\Gamma^\nu} - A^\nu\|_{L_\infty(R_\sigma)} &= \left\| \sum_{\theta \in \sigma} b_\theta \varphi_\theta - \sum_{\theta \in \mathcal{X}_\sigma} c_\theta \varphi_\theta \right\|_{L_\infty(R_\sigma)} \\ &\leq \left\| \sum_{\theta \in \sigma'} b_\theta \varphi_\theta \right\|_{L_\infty(\theta_1)} \leq \sum_{\theta \in \sigma'} |b_\theta| \leq \varepsilon^*. \end{aligned} \quad (3.15)$$

It is easy to see that all rings R_σ ($\sigma \in \Sigma^\nu$) are disjoint and the set where A^ν may differ from f_{Γ^ν} is contained in $\cup_{\sigma \in \Sigma^\nu} R_\sigma$. Hence, by summing over all segments σ and then over all colors

ν , it follows that

$$\|f_{\Gamma} - A_{\varepsilon}^{TC}(f)\|_{\infty} \leq \sum_{\nu=1}^K \sum_{\sigma \in \Sigma^{\nu}} \|f_{\Gamma^{\nu}} - A^{\nu}\|_{L_{\infty}(R_{\sigma})} \leq K\varepsilon^* = \frac{\varepsilon}{2}.$$

This together with estimate (3.8) implies the desired error estimate (3.14). \square

Remark 3.6. *Conditions (3.5), (3.11), and (3.12) can be relaxed by replacing every sum $\sum |b_{\theta}|$ by $\|\sum b_{\theta}\varphi_{\theta}\|_{\infty}$. This would not change the rate of approximation, but may improve the constants in a practical implementation.*

Theorem 3.7. *If $f \in B_{\tau}^{\alpha}(\mathcal{T})$, $\alpha \geq 1$, $\tau := 1/\alpha$, then for each $\varepsilon > 0$*

$$\mathbb{A}_{n(\varepsilon)}^{TC}(f)_{\infty} \leq \varepsilon \quad \text{and} \quad n(\varepsilon) \leq c\varepsilon^{-\tau} \|f\|_{B_{\tau}^{\alpha}(\mathcal{T})}^{\tau}, \quad (3.16)$$

where $c = c(N_0, M_0, \alpha)$. Therefore,

$$\mathbb{A}_n^{TC}(f)_{\infty} \leq c n^{-\alpha} \|f\|_{B_{\tau}^{\alpha}(\mathcal{T})}. \quad (3.17)$$

Proof. We have already shown in Lemma 3.5 that $\mathbb{A}_{n(\varepsilon)}^{TC}(f)_{\infty} \leq \varepsilon$, so we only need to establish $n(\varepsilon) \leq c\varepsilon^{-\tau} \|f\|_{B_{\tau}^{\alpha}(\mathcal{T})}^{\tau}$. We first observe that it is enough to estimate $\#\Sigma^{\nu}$, since contributions to the approximant occur only as each segment from Σ^{ν} is processed. Note that at most one element is contributed for segments consisting of a single significant cell (3.10) and at most $N_0 + 1$ contributions for the segments satisfying instead either (3.11) or (3.12).

In order to estimate $\#\Sigma^{\nu}$ we first estimate $\#\Gamma_f^{\nu}$ since it will estimate certain terms. The stopping criterium (3.6) in Step 3

$$\varepsilon^* < \sum_{\theta \subseteq \theta^{\circ}} |b_{\theta}| \quad (3.18)$$

must hold for each $\theta^{\circ} \in \Gamma_f^{\nu}$, so if we apply the τ -th power to both sides, use the embedding of the sequence spaces ($\tau \leq 1$), sum over all $\theta \in \Gamma_f^{\nu}$, and observe that the supports of the cells in Γ_f^{ν} have disjoint interiors, then we obtain

$$\#\Gamma_f^{\nu} (\varepsilon^*)^{\tau} < \sum_{\theta^{\circ} \in \Gamma_f^{\nu}} \sum_{\theta \subseteq \theta^{\circ}} |b_{\theta}|^{\tau} \leq \|f\|_{B_{\tau}^{\alpha}(\mathcal{T})}^{\tau}. \quad (3.19)$$

The right most inequality follows immediately by our definition of the norm of $B_{\tau}^{\alpha}(\mathcal{T})$ (see (3.2)).

To complete the proof of the theorem, we only need to establish a similar estimate for the number of elements of Σ^{ν} . Recall, however, that the segments σ are formed as disjoint segments of cells from the tree structure and come as one of two types, Σ_{sig} , those exceeding the threshold (see conditions (3.10) or (3.11)) and, Σ_{rem} , those that do not (see condition (3.12)). From the construction it follows that remnant segments terminate with either a unique final cell or a unique branching cell, and so by (3.9)

$$\#\Sigma_{\text{rem}} \leq \#\Gamma_b^{\nu} + \#\Gamma_f^{\nu} \leq 2 \#\Gamma_f^{\nu} \quad (3.20)$$

which has just been shown in (3.19) to satisfy the desired bound.

Therefore we are reduced to estimating $\#\Sigma_{\text{sig}}$. But the same idea used in estimating $\#\Gamma_f^\gamma$ (see estimates (3.18)-(3.19)) may be employed once again. Indeed we just replace the condition (3.18) with

$$\varepsilon^* < \sum_{\theta \in \sigma} |b_\theta|, \quad (3.21)$$

and use the fact that the segments are disjoint (considered as part of the tree structure) in order to obtain

$$\#\Sigma_{\text{sig}} (\varepsilon^*)^\tau < \sum_{\sigma \in \Sigma_{\text{sig}}} \sum_{\theta \in \sigma} |b_\theta|^\tau \leq \|f\|_{B_\tau^q(\mathcal{T})}^\tau. \quad \square \quad (3.22)$$

Although not required here, the following lemma will be needed in §5 and can now be established using the techniques of this section.

Lemma 3.8. *Let $f = f^0 + f^1$, where $f = \sum_{\theta \in \Theta} b_\theta \varphi_\theta$, $f^j = \sum_{\theta \in \Theta} b_\theta^j \varphi_\theta$ ($j = 0, 1$) with $b_\theta = b_\theta^0 + b_\theta^1$ (all $\theta \in \Theta$) and let*

$$\mathcal{N}_j := \left(\sum_{\theta \in \Theta} |b_\theta^j|^{\tau_j} \right)^{1/\tau_j} < \infty, \quad (j = 0, 1)$$

with $\alpha_j \geq 1$ and $\tau_j = 1/\alpha_j$. If the ‘‘Trim & Cut’’ algorithm with $\varepsilon = \varepsilon_0 + \varepsilon_1$ ($\varepsilon_j > 0$) has been applied to f , represented as above in place of Step 1, then

$$\mathbb{A}_{n(\varepsilon_0 + \varepsilon_1)}^{TC}(f)_\infty \leq \varepsilon_0 + \varepsilon_1, \quad (3.23)$$

$$n(\varepsilon_0 + \varepsilon_1) \leq c\varepsilon^{-\tau_0} \mathcal{N}_0^{\tau_0} + c\varepsilon^{-\tau_1} \mathcal{N}_1^{\tau_1}, \quad (3.24)$$

and consequently

$$\mathbb{A}_n^{TC}(f)_\infty \leq cn^{-\alpha_0} \mathcal{N}_0 + cn^{-\alpha_1} \mathcal{N}_1, \quad n = 1, 2, \dots \quad (3.25)$$

with c depending only on α_0 , α_1 , and the parameters of \mathcal{T} .

Proof. All the elements for the proof already appear in this subsection, especially in the proofs of Theorem 3.7 and Lemma 3.5 and we shall assume complete familiarity with the notation, terminology, and estimates given there. Denote the number of cells used in the ‘‘Trim & Cut’’ algorithm for (b_θ) , with approximation error ε , by $n(\varepsilon)$. Similarly, let $n_j(\varepsilon_j)$ be the corresponding number of cells used for f^j ($j = 0, 1$), again represented as $f^j = \sum_{\theta \in \Theta} b_\theta^j \varphi_j$, in place of Step 1. The theorem will be proved once we establish the estimate

$$n(\varepsilon_0 + \varepsilon_1) \leq 2(n_0(\varepsilon_0) + n_1(\varepsilon_1)) \quad (3.26)$$

for any $\varepsilon_0, \varepsilon_1 > 0$. Indeed, by combining this inequality with the results of Theorem 3.7 (in particular inequalities (3.16)-(3.17)), we can see that

$$n(\varepsilon_0 + \varepsilon_1) \leq 2c\varepsilon_0^{-\tau_0} \mathcal{N}_0^{\tau_0} + 2c\varepsilon_1^{-\tau_1} \mathcal{N}_1^{\tau_1} = n. \quad (3.27)$$

is true if we set $\varepsilon_j := (4c)^{1/\tau_j} n^{-1/\tau_j} \mathcal{N}_j$, $j = 0, 1$, where c is the constant appearing there. But the fact that $n \geq n(\varepsilon_0 + \varepsilon_1)$ and the definition of $n(\cdot)$ imply

$$\mathbb{A}_n^{TC}(f)_\infty \leq \mathbb{A}_{n(\varepsilon_0 + \varepsilon_1)}^{TC}(f)_\infty \leq \varepsilon_0 + \varepsilon_1.$$

Hence, by the definition of the ε_j , the right most terms of this last inequality are bounded by the desired terms on the right hand side of inequality (3.25).

In order to prove estimate (3.26), we only need to estimate the number of segments Σ for f . First observe in Step 3 of the algorithm that for the thresholding condition (3.6) to hold for f , with $\varepsilon := \varepsilon_0 + \varepsilon_1$, the condition must also be satisfied for that same cell θ^\diamond for at least one of the f^j with corresponding threshold ε_j ($j = 0, 1$). This shows that the tree $\Gamma^\nu = \Gamma^\nu(f, \varepsilon)$ determined by threshold ε is contained in the union of the corresponding trees $\Gamma^\nu(f^j, \varepsilon_j)$ ($j = 0, 1$). By the construction of segments σ from maximal chains of $\Gamma^\nu(f)$ in Step 4, the segments for f are disjoint and one of the conditions (3.10)-(3.12) must hold. If either of conditions (3.10) or (3.11) hold for a segment σ of f , then $\sum_{\theta \in \sigma} |b_\theta^0 + b_\theta^1| > \varepsilon_0 + \varepsilon_1$ implies the corresponding condition for at least one of f^0 (and ε_0) or f^1 (and ε_1). That is, for one of either $j = 0, 1$ there must hold $\sum_{\theta \in \sigma} |b_\theta^j| > \varepsilon_j$ and so for at least half of the segments of f this condition must persist for a fixed index j ($j = 0, 1$). The number of remnant segments (see (3.12)) on the other hand may be estimated by the sum of the number of remnant segments of f^0 and f^1 , plus the number of *new* branching cells which may arise within the union of the trees of f^0 and f^1 . These new cells are introduced in $\Gamma^\nu(f, \varepsilon)$ when two chains, exclusive to each of the $\Gamma^\nu(f^j, \varepsilon_j)$ meet, thereby dividing the existing chains for each of the trees and creating an additional segment. It is easy to see that the number of such new branching cells does not exceed $\min\{\Gamma_f^\nu(f^0, \varepsilon_0), \Gamma_f^\nu(f^1, \varepsilon_1)\}$.

This accounting of the three qualifying conditions (3.10)-(3.12) for segments gives

$$\begin{aligned} \left\lceil \frac{\#\Sigma}{2} \right\rceil &\leq \max\{\#\Sigma(f^0, \varepsilon_0), \#\Sigma(f^1, \varepsilon_1)\} + \min\{\Gamma_f^\nu(f^0, \varepsilon_0), \Gamma_f^\nu(f^1, \varepsilon_1)\} \\ &\leq \#\Sigma(f^0, \varepsilon_0) + \#\Sigma(f^1, \varepsilon_1) \end{aligned}$$

which implies the desired estimate (3.26) and completes the proof. \square

3.2.2 The case $0 < p < \infty$.

We now return to completing **Steps 3-5** in the case that $p < \infty$. The arguments are quite similar to the case $p = \infty$ in the previous subsection and we shall use the notation there and indicate only the differences. Introduce a new parameter ϱ , where $0 < \varrho < p$, and fix $\varepsilon > 0$.

Step 3. *Trimming of Θ^ν ($1 \leq \nu \leq K$) with ε .*

This step is the same as in Case 1 ($p = \infty$) with (3.5) replaced by

$$\left(\sum_{\theta \subset \theta^\diamond} (|b_\theta| |\theta|^{1/p})^\varrho \right)^{1/\varrho} \leq \varepsilon. \quad (3.28)$$

In contrast to the case $p = \infty$, the error $\|f_\nu - f_{\Gamma^\nu}\|_p$ is no longer controlled solely by ε . It will depend on the smoothness of the function f which is being approximated (see Theorem 3.9 below).

Step 4. *Partitioning the branches of each tree Γ^ν into chains and the chains into “segments”.*
 We proceed exactly as in the case $p = \infty$, replacing conditions (3.10)-(3.12) by the following

$$|b_{\theta_i}| |\theta_i|^{1/p} > \varepsilon \quad (\text{case of } \mu = 0), \quad (3.29)$$

$$\left(\sum_{j=i}^{i+\mu-1} (|b_{\theta_j}| |\theta_j|^{1/p})^\varrho \right)^{1/\varrho} \leq \varepsilon, \quad \text{but} \quad \left(\sum_{j=i}^{i+\mu} (|b_{\theta_j}| |\theta_j|^{1/p})^\varrho \right)^{1/\varrho} > \varepsilon, \quad (\text{case of } \mu > 0), \quad (3.30)$$

$$\left(\sum_{j=i}^l (|b_{\theta_j}| |\theta_j|^{1/p})^\varrho \right)^{1/\varrho} \leq \varepsilon. \quad (3.31)$$

Step 5. *Rewriting elements from certain segments of Σ^ν .*

This step is exactly the same as for the case $p = \infty$.

• **Error estimation for the “Trim & Cut” algorithm (Case $0 < p < \infty$).**

Suppose that the “Trim & Cut” algorithm has been applied to a function f with $0 < \varrho < p$ and $\varepsilon > 0$, as described above. Let $A_\varepsilon^{TC}(f)_p = \sum_{\theta \in \Lambda_\varepsilon} c_\theta \varphi_\theta$, $\Lambda_\varepsilon \subset \Theta$, be the approximant produced by the algorithm. We denote

$$n(\varepsilon) := \#\Lambda_\varepsilon \quad \mathbb{A}_{n(\varepsilon)}^{TC}(f)_p := \|f - A_\varepsilon^{TC}(f)_p\|_p,$$

and

$$\mathbb{A}_n^{TC}(f)_p := \inf\{\mathbb{A}_{n(\varepsilon)}^{TC}(f)_p : n(\varepsilon) \leq n\}.$$

Theorem 3.9. *If $f \in B_\tau^\alpha(\mathcal{T})$, where $\alpha \geq 1/\varrho - 1/p$ and $\tau = (\alpha + \frac{1}{p})^{-1}$, then for each $\varepsilon > 0$*

$$\mathbb{A}_{n(\varepsilon)}^{TC}(f)_p \leq c \varepsilon^{\alpha\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^{\tau/p} \quad \text{and} \quad n(\varepsilon) \leq c \varepsilon^{-\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau. \quad (3.32)$$

and hence

$$\mathbb{A}_n^{TC}(f)_p \leq c n^{-\alpha} \|f\|_{B_\tau^\alpha(\mathcal{T})}, \quad n = 1, 2, \dots, \quad (3.33)$$

where c depends on p, ϱ, α , and the parameters of \mathcal{T} .

Proof. We first estimate $n(\varepsilon)$. From the stopping time criterium (the converse inequality of (3.28)) in Step 3, it follows that

$$\varepsilon < \left(\sum_{\theta \subset \theta^\circ} (|b_\theta| |\theta|^{1/p})^\varrho \right)^{1/\varrho} \leq \left(\sum_{\theta \subset \theta^\circ} (|b_\theta| |\theta|^{1/p})^\tau \right)^{1/\tau} \quad (\text{since } \tau \leq \varrho) \quad (3.34)$$

for each $\theta^\circ \in \Gamma_f^\nu$, which enables us to repeat the arguments from the proof of Theorem 3.7 and obtain the estimate $\#\Gamma_f^\nu \leq c \varepsilon^{-\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau$. In going further, we use (3.30) in a similar fashion and the above to infer as in the proof of Theorem 3.7 that

$$\#\Sigma^\nu \leq c \varepsilon^{-\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau. \quad (3.35)$$

This implies the desired estimate for $n(\varepsilon)$.

It remains to estimate the error $\|f - A_{n(\varepsilon)}^{TC}(f)_p\|_p$. We first estimate $\|f_\nu - f_{\Gamma^\nu}\|_p$. To this end, we group the removed cells into collections of comparable B_τ^α -norms. We denote by

$$\Xi^\nu := \{\theta \in \Theta^\nu \setminus \Gamma^\nu : \theta \not\subseteq \theta' \text{ for any } \theta' \in \Theta^\nu \setminus \Gamma^\nu, \theta' \neq \theta\}$$

the set of all cells at which a trimmed branch starts. Note that for each $\theta^\circ \in \Xi^\nu$ the inequality (3.28) holds. Therefore, we can partition Ξ^ν into disjoint collections Ξ_j^ν , $j = 1, 2, \dots, L^\nu$, such that $\Xi^\nu = \cup_{j=1}^{L^\nu} \Xi_j^\nu$ and

$$\varepsilon^\rho < \sum_{\theta^\circ \in \Xi_j^\nu} \sum_{\theta \subset \theta^\circ} (|b_\theta| |\theta|^{1/p})^\rho \leq 2\varepsilon^\rho \quad (3.36)$$

for all $j = 1, 2, \dots, L^\nu$ except possibly for $j = L^\nu$ when the left most inequality may fail. Hence, since the cells from Ξ^ν have disjoint interiors and recalling that $|b_\theta| |\theta|^{1/p} \approx \|b_\theta \varphi_\theta\|_p$, we obtain

$$\begin{aligned} \|f_\nu - f_{\Gamma^\nu}\|_p &= \left\| \sum_{j=1}^{L^\nu} \sum_{\theta^\circ \in \Xi_j^\nu} \sum_{\theta \subset \theta^\circ} b_\theta \varphi_\theta \right\|_p \leq \left(\sum_{j=1}^{L^\nu} \left\| \sum_{\theta^\circ \in \Xi_j^\nu} \sum_{\theta \subset \theta^\circ} b_\theta \varphi_\theta \right\|_p^p \right)^{1/p} \\ &\leq c \left(\sum_{j=1}^{L^\nu} \left[\sum_{\theta^\circ \in \Xi_j^\nu} \sum_{\theta \subset \theta^\circ} (|b_\theta| |\theta|^{1/p})^\rho \right]^{p/\rho} \right)^{1/p} \\ &\leq c \left(\sum_{j=1}^{L^\nu} 2^{p/\rho} \varepsilon^p \right)^{1/p} = c (L^\nu)^{1/p} \varepsilon, \end{aligned} \quad (3.37)$$

where we used the embedding inequality (2.26). To estimate L^ν we once again exploit the idea used in estimating $\#\Gamma_j^\nu$ (see estimates (3.18)-(3.19)). Since $0 < \tau \leq \rho$, we have by estimate (3.36) that

$$\varepsilon < \left(\sum_{\theta^\circ \in \Xi_j^\nu} \sum_{\theta \subset \theta^\circ} (|b_\theta| |\theta|^{1/p})^\rho \right)^{1/\rho} \leq \left(\sum_{\theta^\circ \in \Xi_j^\nu} \sum_{\theta \subset \theta^\circ} (|b_\theta| |\theta|^{1/p})^\tau \right)^{1/\tau}.$$

We use this and the fact that the collections Ξ_j^ν are disjoint to obtain

$$L^\nu \cdot \varepsilon^\tau \leq c \sum_{j=1}^{L^\nu} \left(\sum_{\theta^\circ \in \Xi_j^\nu} \sum_{\theta \subset \theta^\circ} \|b_\theta \varphi_\theta\|_p^\tau \right) \leq c \|f_\nu\|_{B_\tau^\alpha(\mathcal{T})}^\tau. \quad (3.38)$$

Combining (3.37) and (3.38), we obtain

$$\|f_\nu - f_{\Gamma^\nu}\|_p \leq c (\varepsilon^{-\tau} \|f_\nu\|_{B_\tau^\alpha(\mathcal{T})}^\tau)^{1/p} \varepsilon = c \varepsilon^{\alpha\tau} \|f_\nu\|_{B_\tau^\alpha(\mathcal{T})}^{\tau/p},$$

and hence by standard subadditivity estimates for L_p ($0 < p < \infty$) we may estimate the sum

$$\begin{aligned} \|f - f_\Gamma\|_p &\leq \left(\sum_{\nu=1}^K \|f_\nu - f_{\Gamma^\nu}\|_p^{p^*} \right)^{1/p^*} \\ &\leq c \varepsilon^{\alpha\tau} \left(\sum_{\nu=1}^K (\|f_\nu\|_{B_\tau^\alpha(\mathcal{T})}^{\tau/p})^{p^*/p} \right)^{1/p^*} \leq c \varepsilon^{\alpha\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^{\tau/p}, \end{aligned} \quad (3.39)$$

where $p^* := \min\{1, p\}$.

To complete the proof of the theorem, we must estimate $\|f_{\Gamma^\nu} - A^\nu\|_p$. This differs from our earlier arguments in the case $p = \infty$ which involved the error estimate (3.15) over a ring of a segment. For any such ring R_σ ($\sigma \in \Sigma^\nu$) we use instead the estimate

$$\begin{aligned} \|f_{\Gamma^\nu} - A^\nu\|_{L_p(R_\sigma)} &= \left\| \sum_{\theta \in \sigma} b_\theta \varphi_\theta - \sum_{\theta \in \mathcal{X}_\sigma} c_\theta \varphi_\theta \right\|_{L_p(R_\sigma)} \leq \left\| \sum_{\theta \in \sigma'} b_\theta \varphi_\theta \right\|_{L_p(\theta_1)} \\ &\leq c \left(\sum_{\theta \in \sigma'} \|b_\theta \varphi_\theta\|_p^\tau \right)^{1/\tau} \leq c \left(\sum_{\theta \in \sigma'} (|b_\theta| |\theta|^{1/p})^\varrho \right)^{1/\varrho} \leq c \varepsilon, \end{aligned}$$

where we used the embedding inequality (2.26). From above, using that all rings $\{R_\sigma\}_{\sigma \in \Sigma^\nu}$ have disjoint interiors, we obtain

$$\|f_{\Gamma^\nu} - A^\nu\|_p \leq \left(\sum_{\sigma \in \Sigma^\nu} \|f_{\Gamma^\nu} - A^\nu\|_{L_p(R_\sigma)}^p \right)^{1/p} \leq c (\#\Sigma^\nu)^{1/p} \varepsilon. \quad (3.40)$$

Combining (3.40) and (3.35) yields

$$\|f_{\Gamma^\nu} - A^\nu\|_p \leq c \varepsilon^{\alpha\tau} \|f_\nu\|_{B_\tau^\varrho(\mathcal{T})}^{\tau/p},$$

and hence

$$\begin{aligned} \|f_\Gamma - A_\varepsilon^{TC}(f)_p\|_p &\leq \left(\sum_{\nu=1}^K \|f_{\Gamma^\nu} - A^\nu\|_p^{p^*} \right)^{1/p^*} \\ &\leq c \varepsilon^{\alpha\tau} \left(\sum_{\nu=1}^K (\|f_\nu\|_{B_\tau^\varrho(\mathcal{T})}^{\tau/p})^{p^*/p} \right)^{1/p^*} \leq c \varepsilon^{\alpha\tau} \|f\|_{B_\tau^\varrho(\mathcal{T})}^{\tau/p}, \end{aligned}$$

where $p^* := \min\{1, p\}$. From this and (3.39), we obtain the appropriate estimate which corresponds to (3.14) of the case for $p = \infty$,

$$\|f - A_\varepsilon^{TC}(f)_p\|_p \leq c \varepsilon^{\alpha\tau} \|f\|_{B_\tau^\varrho(\mathcal{T})}^{\tau/p}. \quad (3.41)$$

□

Lemma 3.10. *Let $f = f^0 + f^1$, where $f = \sum_{\theta \in \Theta} b_\theta \varphi_\theta$, $f^j = \sum_{\theta \in \Theta} b_\theta^j \varphi_\theta$ ($j = 0, 1$) with $b_\theta = b_\theta^1 + b_\theta^2$ (all θ) and let*

$$\mathcal{N}_j := \left(\sum_{\theta \in \Theta} (|b_\theta^j| |\theta|^{1/p})^{\tau_j} \right)^{1/\tau_j} < \infty, \quad (j = 0, 1)$$

with $\alpha_j \geq \frac{1}{\varrho} - \frac{1}{p}$ ($0 < \varrho < p$) and $\tau_j := 1/(\alpha_j + \frac{1}{p})^{-1}$, $j = 0, 1$. Furthermore, suppose the ‘‘Trim & Cut’’ algorithm has been applied to f , using the above representation of f in place of Step 1, with $0 < \varrho < p$ as above and $\varepsilon = \varepsilon_0 + \varepsilon_1$ for some $\varepsilon_0, \varepsilon_1 > 0$. Then we have

$$\mathbb{A}_{n(\varepsilon_0 + \varepsilon_1)}^{TC}(f)_p \leq c (\varepsilon_0 + \varepsilon_1) \left(\varepsilon_0^{-\tau_0} \mathcal{N}_0^{\tau_0} + \varepsilon_1^{-\tau_1} \mathcal{N}_1^{\tau_1} \right)^{1/p}, \quad (3.42)$$

$$n(\varepsilon_0 + \varepsilon_1) \leq c \left(\varepsilon_0^{-\tau_0} \mathcal{N}_0^{\tau_0} + \varepsilon_1^{-\tau_1} \mathcal{N}_1^{\tau_1} \right), \quad (3.43)$$

and, therefore,

$$\mathbb{A}_n^{TC}(f)_p \leq c \left(n^{-\alpha_0} \mathcal{N}_0 + n^{-\alpha_1} \mathcal{N}_1 \right), \quad n = 1, 2, \dots, \quad (3.44)$$

where c depends only on $p, \varrho, \alpha_0, \alpha_1$, and the parameters of \mathcal{T} .

Proof. The proof is very similar to the proof of Theorem 3.9 and we shall only indicate the differences using the notation and ideas from there. Those differences are in estimating $\#\Gamma_f^\nu$, $\#\Sigma^\nu$ and L^ν (see (3.35) and (3.38)). From the stopping criteria (converse inequality to (3.28)) in Step 3, it follows that, for $\theta^\diamond \in \Gamma_f^\nu$,

$$\begin{aligned} \varepsilon_0 + \varepsilon_1 &< \left(\sum_{\theta \subset \theta^\diamond} (|b_\theta| |\theta|^{1/p})^\varrho \right)^{1/\varrho} \\ &\leq c_\varrho \left(\sum_{\theta \subset \theta^\diamond} (|b_\theta^0| |\theta|^{1/p})^\varrho \right)^{1/\varrho} + c_\varrho \left(\sum_{\theta \subset \theta^\diamond} (|b_\theta^1| |\theta|^{1/p})^\varrho \right)^{1/\varrho} \\ &\leq c_\varrho \left(\sum_{\theta \subset \theta^\diamond} (|b_\theta^0| |\theta|^{1/p})^{\tau_0} \right)^{1/\tau_0} + c_\varrho \left(\sum_{\theta \subset \theta^\diamond} (|b_\theta^1| |\theta|^{1/p})^{\tau_1} \right)^{1/\tau_1} \end{aligned}$$

where $c_\varrho := \max\{1, 2^{1/\varrho-1}\}$ and we used the fact that $\tau_0, \tau_1 \leq \varrho$. Therefore, for each $\theta^\diamond \in \Gamma_f^\nu$, at least one of

$$\varepsilon_0 < c_\varrho \left(\sum_{\theta \subset \theta^\diamond} (|b_\theta^0| |\theta|^{1/p})^{\tau_0} \right)^{1/\tau_0} \quad \text{or} \quad \varepsilon_1 < c_\varrho \left(\sum_{\theta \subset \theta^\diamond} (|b_\theta^1| |\theta|^{1/p})^{\tau_1} \right)^{1/\tau_1}$$

must hold. Denoting by $\Gamma_{f_0}^\nu$ and $\Gamma_{f_1}^\nu$ the sets of all $\theta^\diamond \in \Gamma_f^\nu$ for which the first or second inequality, respectively, holds, we obtain

$$\mathcal{N}_j^{\tau_j} \geq c \sum_{\theta^\diamond \in \Gamma_{f_j}^\nu} \sum_{\theta \subset \theta^\diamond} (|b_\theta^j| |\theta|^{1/p})^{\tau_j} \geq c \#\Gamma_{f_j}^\nu \varepsilon_j^{\tau_j} \quad (j = 0, 1)$$

and hence

$$\#\Gamma_f^\nu \leq \#\Gamma_{f_0}^\nu + \#\Gamma_{f_1}^\nu \leq c \left(\varepsilon_0^{-\tau_0} \mathcal{N}_0^{\tau_0} + \varepsilon_1^{-\tau_1} \mathcal{N}_1^{\tau_1} \right). \quad (3.45)$$

We obtain similar estimates (with the same right-hand-side quantity) for $\#\Sigma^\nu$ and L^ν by using the same argument. The estimate for $\#\Sigma^\nu$ gives the desired estimate for $n(\varepsilon_0 + \varepsilon_1)$.

We may use estimate (3.37) and (3.40) in the proof of Theorem 3.9, with $\varepsilon = \varepsilon_0 + \varepsilon_1$, together with the above estimates for $\#\Sigma^\nu$ and L^ν to obtain

$$\mathbb{A}_{n(\varepsilon_0 + \varepsilon_1)}^{TC}(f)_p \leq c(\varepsilon_0 + \varepsilon_1) (\#\Sigma)_{n(\varepsilon_0 + \varepsilon_1)}^{1/p}, \quad (3.46)$$

from which the desired estimate (3.42) follows. The final estimate (3.44) is proved by selecting $\varepsilon_j = (2c/n)^{1/\tau_j} \mathcal{N}_j$, which by our result (3.43), gives that $n(\varepsilon_0 + \varepsilon_1) \leq n$ and so

$$\mathbb{A}_n^{TC}(f)_p \leq \mathbb{A}_{n(\varepsilon_0 + \varepsilon_1)}^{TC}(f)_p \leq c n^{1/p} (\varepsilon_0 + \varepsilon_1) \leq c \left(n^{-\alpha_0} \mathcal{N}_0 + n^{-\alpha_1} \mathcal{N}_1 \right)$$

where we have used estimate (3.46) in the second inequality. \square

3.3 “Push the Error” algorithm

The idea of this algorithm to our knowledge first appeared in [5]. Our goal is to adapt this algorithm for nonlinear n -term Courant element approximation in the uniform norm and perfect it so that the resulting algorithm achieves the rate of convergence of the best approximation.

In §3.3.1, we describe the “Push the Error” algorithm in its most simple and naive form. We follow with three examples which illustrate deficiencies of the simple algorithm and the types of traps to which it may fall prey. In §3.3.2, we give our refined version of that algorithm. Throughout this section, we assume that $\mathcal{T} = \bigcup_{m=0}^{\infty} \mathcal{T}_m$ is an LR-triangulation of some compact polygonal domain E in \mathbb{R}^2 , where the approximation takes place (see §2.1), and $f \in C(E)$.

3.3.1 A naive “Push the Error” algorithm ($p = \infty$)

We begin by outlining the basic elements of the algorithm.

Step 1. (*Decompose*) In this subsection we denote by $Q_j(f)$ the piecewise linear continuous function which interpolates f at the vertices V_j of all triangles from \mathcal{T}_j . Clearly $f \in C(E)$ can be represented as follows

$$f = Q_0(f) + \sum_{j=1}^{\infty} (Q_j(f) - Q_{j-1}(f)) =: \sum_{\theta \in \Theta} c_{\theta} \varphi_{\theta}, \quad (3.47)$$

where the series converges uniformly. In practice the series terminates at some finest level Θ_J ($J > 1$), so that

$$f = \sum_{j=0}^J \sum_{\theta \in \Theta_j} c_{\theta} \varphi_{\theta}.$$

Assuming that initially $f = \sum_{\theta \in \Theta_J} c_{\theta} \varphi_{\theta}$, there exists a fast and efficient procedure for obtaining (3.47).

Step 2. (*Threshold & “Push the Error”*) Fix $\varepsilon > 0$. We shall begin at the coarsest level Θ_0 and proceed consecutively through to higher resolution levels $\Theta_1, \Theta_2, \dots, \Theta_J$. We define Λ_0 as the set of all cells $\theta \in \Theta_0$ such that $|c_{\theta}| > \varepsilon$ ($\|\varphi_{\theta}\| = 1$) and set

$$A_0 := \sum_{\theta \in \Lambda_0} c_{\theta} \varphi_{\theta} =: \sum_{\theta \in \Lambda_0} b_{\theta} \varphi_{\theta}.$$

Next we rewrite all remaining terms $c_{\theta} \varphi_{\theta}$ ($\theta \in \Theta_0 \setminus \Lambda_0$) at the next finer level and add the resulting terms to the corresponding terms from $(c_{\theta} \varphi_{\theta})_{\theta \in \Theta_1}$. Thus we obtain a representation of f in the form

$$f = A_0 + \sum_{\theta \in \Theta_1} b_{\theta} \varphi_{\theta} + \sum_{j=2}^J \sum_{\theta \in \Theta_j} c_{\theta} \varphi_{\theta}.$$

We next process the Courant elements at level Θ_1 . We define Λ_1 as the set of all $\theta \in \Theta_1$ such that $|b_\theta| > \varepsilon$ and set $A_1 := \sum_{\theta \in \Lambda_1} b_\theta \varphi_\theta$. All remaining terms $b_\theta \varphi_\theta$, $\theta \in \Theta_1 \setminus \Lambda_1$, we rewrite at the finer level Θ_2 and add the resulting terms to the corresponding terms $(c_\theta \varphi_\theta)_{\theta \in \Theta_2}$. The representation of f at this stage is written as

$$f = A_0 + A_1 + \sum_{\theta \in \Theta_2} b_\theta \varphi_\theta + \sum_{j=3}^J \sum_{\theta \in \Theta_j} c_\theta \varphi_\theta. \quad (3.48)$$

We continue in this way until we reach the finest (*i.e.*, highest resolution) level Θ_J . The only modification at this finest level is that we discard all terms whose coefficients in absolute value do not exceed our threshold parameter ε . In this way we obtain our approximation

$$A := A_\varepsilon(f) := \sum_{j=0}^J A_j = \sum_{\theta \in \Lambda} b_\theta \varphi_\theta, \quad \Lambda := \bigcup_{j=0}^J \Lambda_j.$$

Since only small terms ($|b_\theta| \leq \varepsilon$) at a single (in this case, finest) level are discarded, they cannot stack up and we have

$$\|f - A_\varepsilon(f)\|_\infty \leq \varepsilon.$$

Some modifications must be made however to insure that this simple and efficient algorithm will achieve sparse representations in an asymptotically optimal sense and avoid hidden traps that will result in using too many terms in the approximation.

We indicate briefly each of the possible pitfalls to keep in mind before developing the algorithm in full in the next section.

Trap 1. The interpolation scheme we used to represent f in (3.47) leads to difficulties since it does not always lead to sparse representations. We give here a univariate example which may be easily extended to two dimensions.

Let $E := [-1, 1]$ and f be the hat function on $[-\frac{1}{2^N}, \frac{1}{2^N}]$ for N sufficiently large, *i.e.*, $f(x) = \varphi(2^N x)$ with $\varphi(x) := (1 - |x|)\mathbb{1}_{[-1, 1]}(x)$, $x \in \mathbb{R}$. We assume that \mathcal{T} consists of all dyadic subintervals of $[-1, 1]$. Using the interpolation scheme described in Step 1 at the coarsest level, we must interpolate the extremes at $-1, 0, 1$ in order to decrease the L^∞ error. The resulting error after this stage however is $1 - \frac{1}{2^N}$. Proceeding with the “naive” Push-the-Error algorithm with any $\varepsilon < \frac{1}{2}$ results in an index set Λ with $\#\Lambda \sim N$. However, the best approximation is achieved using the single fine scale element $\varphi(2^N x)$. Therefore, any reasonable algorithm which retains n terms in the approximant should give a rate of convergence $\mathcal{O}(n^{-\gamma})$ for any $\gamma > 0$.

Trap 2. For a given $\varepsilon > 0$ the algorithm as currently described may produce a great number of undesired terms due to the superposition of a large number of fine level nonintersecting terms $(c_\theta \varphi_\theta)$ with a single coarse level term (φ_{θ_0})

$$f = \varepsilon \left(\varphi_{[-1, 1]} + \sum_{\theta \in \mathcal{M}} \delta \varphi_\theta \right). \quad (3.49)$$

We set \mathcal{M} as a set of disjoint cells θ from level 2^{2N} with $\theta \subset (-\delta, \delta)$, where $\delta = 2^{-N}$. It is clear that we can choose these cells for \mathcal{M} so that $\#\mathcal{M} = 2^N$. At the central vertex x_θ of each cell θ we have $f(x_\theta) > \varepsilon(1 - \delta) + \delta\varepsilon = \varepsilon$. The “Push-the-Error algorithm” will produce an inefficient approximation since it will not select the coarse first term in (3.49) as one might hope. Instead, no such element will be chosen at the coarsest level and the error will be pushed. At each successive stage the coefficients of the rewritten descendant Courant elements for θ_0 will all again lie beneath the threshold and be further rewritten until all cells are on level 2^{2N} . At that stage they will be combined with the remaining terms in (3.49). The corresponding cells will now have coefficients that exceed the threshold and must be selected producing at least 2^N terms in the approximant. As indicated above, a desirable algorithm should have anticipated the trap of many small, finely supported elements that may come at a late stage, and would have chosen for this function the approximation (with threshold ε) which consists of a single element, namely $\varepsilon \varphi_{[-1,1]}$.

Trap 3. The final example is one that outmaneuvers a quick remedy to Trap 2, *i.e.*, merely thresholding all small terms at the finest level. For a given $\varepsilon > 0$, we define

$$f = \varepsilon \varphi_{[-1,1]} + \sum_{j=1}^N \delta_j \varphi_{[0,2^{-m_j}]} + \varepsilon \varphi_{[0,2^{-M}]},$$

where $m_j = j^2$, $\delta_j = 2^{-j}\varepsilon$, and $M = 2^{N^2}$. In this example, elements are again building near the origin, but now appear at many levels with small amplitudes. The “Push-the-Error” algorithm will again take no elements at the coarsest level and push the error to the next level. Continuing with the algorithm, we are forced to take essentially all terms as the approximation to the given function when, optimally, only two terms need be taken.

It is obvious that we can take each of these template examples as building blocks and build functions to cause these problems for all ε , at all locations and scales.

3.3.2 “Push the Error” algorithm in the uniform norm ($p = \infty$)

In this section we indicate the refinements needed in order to guarantee that the “Push the Error” algorithm will achieve optimal rates of approximation. As with the “Trim and Cut” algorithm we break it down into manageable steps.

- **Description of the algorithm.**

Step 1. (*Decompose*) For $f \in C(E)$ initially represented by (3.1), we may assume without loss of generality that there exists a finest level Θ_J ($J > 0$) such that f is written as

$$f = \sum_{j=0}^J \sum_{\theta \in \Theta_j} b_\theta \varphi_\theta. \tag{3.50}$$

Step 2. (*“Prune the shrubs”*) In the current algorithm we are not able to organize the cells of Θ into trees as we did in the “Trim and Cut” method, since, once we rewrite the

error on a finer level, adjacent trees are immediately affected and we lose their intended organization properties. This step of our algorithm, however, is analogous to Step 3 of the “Trim and Cut” algorithm. We fix $\varepsilon > 0$ and let $\varepsilon^* := \varepsilon/2$. Our goal is, by discarding small insignificant terms $b_\theta\varphi_\theta$ in the representation of f from (3.50), to prevent our refined algorithm from being trapped by a situation such as that described in “Trap 2” (see the *naive* “Push the Error” algorithm of §3.3.1). We shall remove such terms but insuring that the resulting uniform error is at most ε^* and denote by Γ the set of all retained cells. In addition, we shall construct a set $\Gamma_f \subset \Gamma$, consisting of “final cells” in Γ .

First, we need to introduce a organizational concept as a replacement for the tree structures of §3.2. We shall say (figuratively) that a cell $\theta \in \Theta$ *sits* on another cell $\theta^\diamond \in \Theta$, if θ is at least as fine as θ^\diamond and its interior (denoted by θ°) intersects the interior of θ^\diamond . Furthermore, for $\theta^\diamond \in \Theta$, we denote the collection of all cells which sit on θ^\diamond by

$$\mathcal{Y}_{\theta^\diamond} := \{\theta \in \Theta : \theta^\circ \cap \theta^\diamond \neq \emptyset \text{ and } \text{level}(\theta) \geq \text{level}(\theta^\diamond)\}. \quad (3.51)$$

The procedure of Step 2 will begin at the finest level and proceed to the coarsest, level by level, constructing sets Γ_f and Γ .

To initialize the procedure we put into Γ_f all *significant* cells $\theta \in \Theta_J$, *i.e.* such that $|b_\theta| > \varepsilon^*$. We place in Γ any cell from Θ_J which sits on a cell from Γ_f .

The inductive step proceeds as follows. Suppose that all cells from Θ_j with levels $j > m$ ($0 \leq m < J$), have already been processed. We now describe how to process Θ_m . We place into Γ_f all cells $\theta^\diamond \in \Theta_m$ which satisfy

$$\sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta| > \varepsilon^*, \quad (3.52)$$

and for which there is no $\theta \in \Gamma_f$ from a higher level (*i.e.* $> m$) which sits on θ^\diamond . A cell θ^\diamond from Θ_m is placed in Γ if there is a cell θ in the current Γ_f which sits on θ^\diamond . We may consider the current version of Γ_f as an intermediate (m -th) version of a final set for Γ . Obviously, a cell θ^\diamond from Θ_m is discarded and not placed in Γ if

$$\sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta| \leq \varepsilon^*, \quad (3.53)$$

and there is no $\theta \in \Gamma_f$ from level m or finer which sits on θ^\diamond .

The procedure is terminated after Θ_0 is processed and *Step 2* of the algorithm is completed.

The two sets of cells Γ and Γ_f ($\Gamma_f \subset \Gamma \subset \Theta$) produced by Step 2 have the following properties which follow directly from their construction:

- (i) if $\theta_1, \theta_2 \in \Gamma_f$ and $\text{level}(\theta_1) \neq \text{level}(\theta_2)$, then $\theta_1^\circ \cap \theta_2^\circ = \emptyset$;
- (ii) for each $\theta^\diamond \in \Gamma_f$, the inequality (3.52) holds;
- (iii) for each $\theta^\diamond \in \Gamma$, there exists $\theta \in \Gamma_f$ which sits on θ^\diamond .

We set $f_\Gamma := \sum_{\theta \in \Gamma} b_\theta \varphi_\theta$ and define

$$a_\theta := \begin{cases} b_\theta, & \text{if } \theta \in \Gamma, \\ 0, & \text{if } \theta \in \Theta \setminus \Gamma, \end{cases} \quad (3.54)$$

then obviously

$$f_\Gamma = \sum_{\theta \in \Theta} a_\theta \varphi_\theta. \quad (3.55)$$

It follows from the construction that

$$\|f - f_\Gamma\|_\infty \leq \varepsilon^*. \quad (3.56)$$

Indeed, to see that this estimate holds, we let \mathcal{D} denote the set of all cells $\theta \in \Theta$ that were discarded during the implementation of Step 2, *i.e.*, $\mathcal{D} = \Theta \setminus \Gamma$. Let $x \in E$ be arbitrary. If $x \notin \bigcup_{\theta \in \mathcal{D}} \theta$, then x does not belong to any cell which was discarded and so $f_\Gamma(x) = f(x)$. On the other hand, if $x \in \bigcup_{\theta \in \mathcal{D}} \theta$, then there exists a cell $\theta^\diamond \in \mathcal{D}$ which contains x and has coarsest level. Since θ^\diamond was discarded, the inequality (3.53) must hold. It follows that

$$|f(x) - f_\Gamma(x)| = \left| \sum_{\theta \in \mathcal{D}} b_\theta \varphi_\theta(x) \right| \leq \sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta| \leq \varepsilon^*,$$

where we have normalized our elements so that $\|\varphi_\theta\|_\infty = 1$. This verifies the desired inequality (3.56).

Step 3. (*Push the Error*) We now process cells of f_Γ with ε^* , starting from the coarsest level Θ_0 and continuing to finer levels. The outcome of this step will be an approximant $\mathcal{A} := \mathcal{A}_\varepsilon^P(f)$ of the form

$$\mathcal{A} = \sum_{j=0}^J \mathcal{A}_j := \sum_{j=0}^J \sum_{\theta \in \Lambda_j} d_\theta \varphi_\theta, \quad (3.57)$$

where $\Lambda_j \subset \Theta_j$ and Λ_j will depend on f .

We use the notation

$$\mathcal{X}_{\theta^\diamond} := \{\theta \in \Theta : \theta^\diamond \cap \theta \neq \emptyset \text{ and } \text{level}(\theta) = \text{level}(\theta^\diamond)\}$$

for cells from the same level as θ^\diamond which are adjacent to it.

We start from the representation of f_Γ in (3.55). We define $\tilde{\Lambda}_0$ as the set of all $\theta \in \Theta_0$ such that $|a_\theta| > \varepsilon^*$ ($\|\varphi_\theta\|_\infty = 1$) and set $\Lambda_0 := \bigcup_{\theta \in \tilde{\Lambda}_0} \mathcal{X}_\theta$. We denote

$$\mathcal{A}_0 := \sum_{\theta \in \Lambda_0} a_\theta \varphi_\theta =: \sum_{\theta \in \Lambda_0} d_\theta \varphi_\theta.$$

For each $\theta^\diamond \in \Theta_j$, $\varphi_{\theta^\diamond}$ can be represented as a linear combination of φ_θ 's with $\theta \in \Theta_{j+1}$. We use this to rewrite (represent) all remaining terms $a_\theta \varphi_\theta$, $\theta \in \Theta_0 \setminus \Lambda_0$, at the next level and add the resulting terms to the corresponding terms $a_\theta \varphi_\theta$, $\theta \in \Theta_1$. We denote by $d_\theta \varphi_\theta$, $\theta \in \Theta_1$, the new terms and therefore obtain a representation of f in the form

$$f = \mathcal{A}_0 + \sum_{\theta \in \Theta_1} d_\theta \varphi_\theta + \sum_{j=2}^J \sum_{\theta \in \Theta_j} a_\theta \varphi_\theta.$$

Continuing with the next level, we define $\tilde{\Lambda}_1$ as the set of all $\theta \in \Theta_1$ such that $|d_\theta| > \varepsilon^*$, set $\Lambda_1 := \bigcup_{\theta \in \tilde{\Lambda}_1} \mathcal{X}_\theta$, and define $\mathcal{A}_1 := \sum_{\theta \in \Lambda_1} d_\theta \varphi_\theta$. As for the previous level, we rewrite the remaining terms $d_\theta \varphi_\theta$, $\theta \in \Theta_1 \setminus \Lambda_1$, at the next level and add the resulting terms to the corresponding terms $a_\theta \varphi_\theta$, $\theta \in \Theta_2$. We obtain the following representation of f :

$$f = \mathcal{A}_0 + \mathcal{A}_1 + \sum_{\theta \in \Theta_2} d_\theta \varphi_\theta + \sum_{j=3}^J \sum_{\theta \in \Theta_j} a_\theta \varphi_\theta.$$

We continue in this way until we reach the highest level of cells Θ_J . At level Θ_J , we define $\tilde{\Lambda}_J$, Λ_J , and \mathcal{A}_J as above and discard all terms $d_\theta \varphi_\theta$, $\theta \in \Theta_J \setminus \Lambda_J$. We finally obtain our approximant $\mathcal{A} = \mathcal{A}_\varepsilon^P(f)$ in the form (3.57). We denote $\Lambda := \Lambda_\varepsilon := \bigcup_{j=0}^J \Lambda_j$ and $\tilde{\Lambda} := \tilde{\Lambda}_\varepsilon := \bigcup_{j=0}^J \tilde{\Lambda}_j$ and so $\mathcal{A} = \sum_{\theta \in \Lambda} d_\theta \varphi_\theta$.

Since we throw away only elements $d_\theta \varphi_\theta$ with $|d_\theta| \leq \varepsilon^*$ at the finest level Θ_J , we have the estimate

$$\|f_\Gamma - \mathcal{A}\|_\infty \leq \left\| \sum_{\theta \in \Theta_J \setminus \Lambda_J} d_\theta \varphi_\theta \right\|_\infty \leq \varepsilon^*$$

and hence, using (3.56),

$$\|f - \mathcal{A}\|_\infty \leq 2\varepsilon^* = \varepsilon. \quad (3.58)$$

This completes *Step 3* and with that the description of the algorithm.

We want to point out an important distinction between the ‘‘Push the Error’’ steps in the above algorithm and the ‘‘naive’’ algorithm described in § 3.3.1. The difference is that each time we put a significant term $d_\theta \varphi_\theta$ ($|d_\theta| > \varepsilon^*$) into \mathcal{A} we also include the neighboring terms (*i.e.*, from the index collection \mathcal{X}_θ). This prevents our algorithm from being defeated by a situation as that described in ‘‘Trap 3’’ in § 3.3.1.

• **Error estimation for the ‘‘Push the Error’’ algorithm.** Suppose ‘‘Push the Error’’ is applied to a function f with $\varepsilon > 0$ and $\mathcal{A}_\varepsilon^P(f)$ is the approximant obtained: $\mathcal{A}_\varepsilon^P(f) := \sum_{\theta \in \Lambda_\varepsilon} d_\theta \varphi_\theta$. As in the ‘‘Trim & Cut’’ method, we use the corresponding notation

$$n(\varepsilon) := \#\Lambda_\varepsilon, \quad \mathbb{A}_{n(\varepsilon)}^P(f)_\infty := \mathbb{A}_{n(\varepsilon)}^P(f, \mathcal{T})_\infty := \|f - \mathcal{A}_\varepsilon^P(f)\|_\infty,$$

and

$$\mathbb{A}_n^P(f)_\infty := \mathbb{A}_n^P(f, \mathcal{T})_\infty := \inf\{\mathbb{A}_{n(\varepsilon)}^P(f)_\infty : n(\varepsilon) \leq n\}.$$

We remark that if $f \in B_\tau^\alpha(\mathcal{T})$, then by the Embedding Theorem 2.7 it follows that f is continuous. Estimates (3.59) and (3.60), established in the following theorem, imply uniform convergence of the ‘‘Push the Error’’ approximants to f and provide the necessary rates of approximation by the method. As is well known, the embedding is actually compact and these inequalities may be regarded as the quantification of the total boundedness of bounded subsets of $B_\tau^\alpha(\mathcal{T})$ in $C(E)$.

Theorem 3.11. *If $f \in B_\tau^\alpha(\mathcal{T})$, $\alpha \geq 1$, $\tau := 1/\alpha$, then for each $\varepsilon > 0$*

$$\mathbb{A}_{n(\varepsilon)}^P(f)_\infty \leq \varepsilon \quad \text{and} \quad n(\varepsilon) \leq c\varepsilon^{-\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau, \quad (3.59)$$

where $c = 6N_0^3$. Furthermore, we have

$$\mathbb{A}_n^P(f)_\infty \leq cn^{-\alpha} \|f\|_{B_\tau^\alpha(\mathcal{T})}, \quad n = 1, 2, \dots \quad (3.60)$$

with $c = (6N_0^3)^\alpha$.

Proof. In order to prove (3.59), we first observe that the direct approximation estimate $\mathbb{A}_{n(\varepsilon)}^P(f)_\infty \leq \varepsilon$ follows from inequality (3.58) in the construction of the algorithm. Therefore it only remains to show that $\#\Lambda_\varepsilon \leq c\varepsilon^{-\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau$. Clearly,

$$\#\Lambda_\varepsilon \leq (N_0 + 1)(\#\tilde{\Lambda}_\varepsilon) \quad (3.61)$$

and we need only estimate the cardinality of $\tilde{\Lambda} := \tilde{\Lambda}_\varepsilon$. We split $\tilde{\Lambda}$ into two disjoint sets $\tilde{\Lambda}_f$ and $\tilde{\Lambda}_r$. We define $\tilde{\Lambda}_f$ as the set of all final cells in $\tilde{\Lambda}$, that is, the set of all $\theta \in \tilde{\Lambda}$ for which there is no $\theta' \in \tilde{\Lambda}$ of a higher level sitting on θ . We set $\tilde{\Lambda}_r := \tilde{\Lambda} \setminus \tilde{\Lambda}_f$.

We shall make repeated use of the following simple lemma.

Lemma 3.12. *Suppose $\mathcal{M} \subset \Theta$ satisfies the condition that cells from different levels do not have interiors that intersect. Then each $\theta \in \Theta$ may sit on at most $N_0 + 1$ cells from \mathcal{M} .*

Proof. The simple hypothesis of the lemma just states that for a cell θ_2 to sit on a cell θ_1 , it must be on the same level, but there can be at most $N_0 + 1$ such cells. \square

We first estimate the number of elements $\#\Gamma_f$ which arise as final cells in Step 2. For each $\theta^\diamond \in \Gamma_f$, we have, by (3.52),

$$\varepsilon^* < \sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta| \leq \left(\sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta|^\tau \right)^{1/\tau} \quad (\tau \leq 1). \quad (3.62)$$

Clearly, Γ_f satisfies the hypothesis of Lemma 3.12 (see Property (i) of Γ_f which is stated following (3.52)) and hence each $\theta \in \Theta$ may sit on at most $N_0 + 1$ cells from Γ_f . Using this together with (3.62), we obtain

$$\|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau := \sum_{\theta \in \Theta} |b_\theta|^\tau \geq (N_0 + 1)^{-1} \sum_{\theta^\diamond \in \Gamma_f} \sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta|^\tau \geq (N_0 + 1)^{-1} (\#\Gamma_f) (\varepsilon^*)^\tau$$

which, since $\tau \leq 1$, implies

$$\#\Gamma_f \leq 2(N_0 + 1)\varepsilon^{-\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau. \quad (3.63)$$

We next estimate $\#\tilde{\Lambda}_f$, the number of final cells for the index set $\tilde{\Lambda}$ constructed in Step 3. Clearly from that construction, a cell $\theta \in \tilde{\Lambda}$ may occur only if $\theta \in \Gamma$ and hence $\tilde{\Lambda} \subset \Gamma$. On the other hand, from Step 2, for each $\theta \in \Gamma$ there exists $\theta' \in \Gamma_f$ sitting on θ . Therefore, for each $\theta \in \tilde{\Lambda}_f$ there exists $\theta' \in \Gamma_f$ sitting on θ . But $\tilde{\Lambda}_f$ satisfies the hypothesis of Lemma 3.12 (with \mathcal{M} replaced by $\tilde{\Lambda}_f$) and hence a cell $\theta \in \Gamma_f$ may sit on at most $N_0 + 1$ cells from $\tilde{\Lambda}_f$. From this and (3.63), we have

$$\#\tilde{\Lambda}_f \leq (N_0 + 1)(\#\Gamma_f) \leq 2(N_0 + 1)^2 \varepsilon^{-\tau} \|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau. \quad (3.64)$$

To complete the estimate for $\#\tilde{\Lambda}$, we must estimate $\#\tilde{\Lambda}_r$. Suppose $\theta^\diamond \in \tilde{\Lambda}_r := \tilde{\Lambda} \setminus \tilde{\Lambda}_f$ and let $\theta' \in \tilde{\Lambda}$ be a cell sitting on θ^\diamond with level(θ') > level(θ^\diamond) and such that level(θ') is the minimum of the levels of all cells in $\tilde{\Lambda}$ sitting on θ^\diamond . Such a cell exists, by the definition of $\tilde{\Lambda}_r$, but it is possibly not unique. We denote by $\mathcal{Z}_{\theta^\diamond}$ the set of all $\theta \in \Gamma$ which, while ‘‘Pushing the Error’’ from θ^\diamond in Step 3, have contributed to the term $d_{\theta'}\varphi_{\theta'}$. Due to the minimality of θ' , we see that

$$d_{\theta'} = d_{\theta'}\varphi_{\theta'}(v_{\theta'}) = \sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} b_\theta \varphi_\theta(v_{\theta'}), \quad (3.65)$$

where $v_{\theta'}$ is the ‘‘central vertex’’ of θ' . Since $\theta' \in \tilde{\Lambda}$, then $|d_{\theta'}| > \varepsilon^*$ and hence, using (3.65),

$$\varepsilon^* < |d_{\theta'}| \leq \left\| \sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} b_\theta \varphi_\theta \right\|_\infty \leq \sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta| \leq \left(\sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta|^\tau \right)^{1/\tau} \quad (\tau \leq 1). \quad (3.66)$$

It is easily seen that each $\theta \in \mathcal{Z}_{\theta^\diamond}$ satisfies the following properties:

- (a) $\theta \supset \theta'$,
- (b) level(θ^\diamond) < level(θ) \leq level(θ'),
- (c) the ‘‘central vertex’’ of θ lies on θ^\diamond and hence θ sits on θ^\diamond .

Property (a) follows by observing that the support of an element which is rewritten at a finer level always contains the supports of the contributing finer elements. Property (b) holds since $\mathcal{X}_{\theta^\diamond} \subset \Lambda$ and hence no terms $b_\theta \varphi_\theta$ with level(θ) \leq level(θ^\diamond) may contribute to $d_{\theta'}\varphi_{\theta'}$. Note that it is possible that there are θ which satisfy properties (a)-(c) above but do not belong to $\mathcal{Z}_{\theta^\diamond}$.

Next, we show that each $\theta \in \Gamma$ may belong to at most $N_0 + 1$ sets \mathcal{Z}_{θ^*} with $\theta^* \in \tilde{\Lambda}_r$. Indeed, let $\theta \in \Gamma$ and suppose $\theta^\diamond \in \tilde{\Lambda}_r$ is such that $\theta \in \mathcal{Z}_{\theta^\diamond}$. In the following, we shall use the notation from above which involves θ^\diamond , but we will consider such θ^\diamond as arbitrary in $\tilde{\Lambda}$. Let \mathcal{M}_θ denote the set of all $\theta^\sharp \in \tilde{\Lambda}$ such that $\theta \in \mathcal{Z}_{\theta^\sharp}$. In particular $\theta^\diamond \in \mathcal{M}_\theta$. We fix \mathcal{M}_θ and show that it satisfies the hypothesis of Lemma 3.12. Indeed, let $\theta_1, \theta_2 \in \mathcal{M}_\theta$ from different levels. But this implies $\theta \in \mathcal{Z}_{\theta_j}$, ($j = 1, 2$) and we may as well consider $\theta_1 = \theta^\diamond$ and say $\theta_2 = \theta^\sharp$, where level(θ^\sharp) \neq level(θ^\diamond). Evidently, level(θ^\sharp) < level(θ') from property (b) applied to θ^\sharp and θ .

By symmetry, we may assume level(θ^\sharp) < level(θ^\diamond). If $(\theta^\sharp)^\diamond \cap (\theta^\diamond)^\diamond \neq \emptyset$, then θ^\diamond sits on θ^\sharp and hence, since level(θ) > level(θ^\diamond), θ cannot be in $\mathcal{Z}_{\theta^\sharp}$, which is a contradiction. Therefore, $(\theta^\sharp)^\diamond \cap (\theta^\diamond)^\diamond = \emptyset$ which verifies the hypothesis of Lemma 3.12.

Now that Lemma 3.12 can be applied to \mathcal{M}_θ , then θ (as any other cell from Θ) may sit on at most $N_0 + 1$ cells $\theta^* \in \mathcal{M}_\theta$. Therefore, θ may belong to at most $N_0 + 1$ such sets \mathcal{Z}_{θ^*} with $\theta^* \in \tilde{\Lambda}_r$. Using this and (3.66), we obtain

$$\|f\|_{B_\varphi^\tau}^\tau \geq \sum_{\theta \in \Gamma} |b_\theta|^\tau \geq (N_0 + 1)^{-1} \sum_{\theta^\diamond \in \tilde{\Lambda}_r} \sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta|^\tau \geq (N_0 + 1)^{-1} (\#\tilde{\Lambda}_r) (\varepsilon^*)^\tau.$$

Therefore, it follows (recall that $\tau < 1$) that

$$\#\tilde{\Lambda}_r \leq 2(N_0 + 1)\varepsilon^{-\tau} \|f\|_{B_\varphi^\tau}^\tau.$$

We combine this estimate with (3.61) and (3.64) to obtain the desired estimate of $\#\Lambda_\varepsilon$ in (3.59). Estimate (3.60) follows immediately from (3.59). \square

The following lemma will be needed in §5.

Lemma 3.13. *Let $f = f^0 + f^1$, where $f^j = \sum_{\theta \in \Theta} b_\theta^j \varphi_\theta$, $f^j = \sum_{\theta \in \Theta} b_\theta^j \varphi_\theta$ ($j = 0, 1$), and $b_\theta = b_\theta^0 + b_\theta^1$ (all $\theta \in \Theta$), and suppose*

$$\mathcal{N}_j := \left(\sum_{\theta \in \Theta} |b_\theta^j|^{\tau_j} \right)^{1/\tau_j} < \infty \quad (j = 0, 1),$$

where $\alpha_0, \alpha_1 \geq 1$ and $\tau_0 := 1/\alpha_0$, $\tau_1 := 1/\alpha_1$. Furthermore, suppose that “Push the Error” is applied using the above representation of f , with $\varepsilon := \varepsilon_0 + \varepsilon_1$, where $\varepsilon_0, \varepsilon_1 > 0$, then we have

$$\mathbb{A}_{n(\varepsilon_0 + \varepsilon_1)}^P(f)_\infty \leq \varepsilon_0 + \varepsilon_1 \quad \text{and} \quad (3.67)$$

$$n(\varepsilon_0 + \varepsilon_1) \leq c\varepsilon_0^{-\tau_0} \mathcal{N}_0^{\tau_0} + c\varepsilon_1^{-\tau_1} \mathcal{N}_1^{\tau_1}, \quad (3.68)$$

where $c = 6N_0^3$. Consequently, the estimate

$$\mathbb{A}_n^P(f)_\infty \leq c_0 n^{-\alpha_0} \mathcal{N}_0 + c_1 n^{-\alpha_1} \mathcal{N}_1, \quad n = 1, 2, \dots \quad (3.69)$$

holds, with $c_j = (12N_0^3)^{\alpha_j}$.

Proof. We follow in the foot steps of the proof of Theorem 3.11. We shall use the notation from there and only indicate the differences as they arise. We denote $\varepsilon^* := \varepsilon_0^* + \varepsilon_1^*$ with $\varepsilon_j^* := \varepsilon_j/2$, $j = 0, 1$. Estimate (3.67) is immediate from the description of the algorithm.

It remains to provide estimate (3.68) for the number of terms used in the approximation. As in (3.61), we have

$$n(\varepsilon_0 + \varepsilon_1) := \#\Lambda_\varepsilon \leq (N_0 + 1)(\#\tilde{\Lambda}_\varepsilon), \quad (3.70)$$

where we denote $\tilde{\Lambda} := \tilde{\Lambda}_\varepsilon$ and $\tilde{\Lambda}_f$ and $\tilde{\Lambda}_r$ have the same definitions, proceeding exactly as in the proof of Theorem 3.11. Continuing as there, we have to estimate $\#\Gamma_f$. For each $\theta^\diamond \in \Gamma_f$, we have, by (3.52) and the fact that $0 < \tau_j \leq 1$ ($j = 0, 1$), that

$$\begin{aligned} \varepsilon_0^* + \varepsilon_1^* = \varepsilon^* &< \sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta| \leq \sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta^0| + \sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta^1| \\ &\leq \left(\sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta^0|^{\tau_0} \right)^{1/\tau_0} + \left(\sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta^1|^{\tau_1} \right)^{1/\tau_1}. \end{aligned}$$

From this, it follows that, for each $\theta^\diamond \in \Gamma_f$, at least one of

$$\varepsilon_0^* < \left(\sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta^0|^{\tau_0} \right)^{1/\tau_0} \quad \text{or} \quad \varepsilon_1^* < \left(\sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta^1|^{\tau_1} \right)^{1/\tau_1} \quad (3.71)$$

must hold. We denote by Γ_f^0 and Γ_f^1 the sets of all $\theta^\diamond \in \Gamma_f$ such that the respective condition from (3.71) holds for either $j = 0$ or $j = 1$. For $j = 0, 1$, we have similarly, as in the proof of Theorem 3.11,

$$\mathcal{N}_j^{\tau_j} := \sum_{\theta \in \Theta} |b_\theta^j|^{\tau_j} \geq (N_0 + 1)^{-1} \sum_{\theta^\diamond \in \Gamma_f^j} \sum_{\theta \in \mathcal{Y}_{\theta^\diamond}} |b_\theta^j|^{\tau_j} \geq (N_0 + 1)^{-1} (\#\Gamma_f^j) (\varepsilon_j^*)^{\tau_j}$$

and hence ($\tau_j \leq 1$)

$$\#\Gamma_f^j \leq 2(N_0 + 1)\varepsilon_j^{-\tau_j}\mathcal{N}_j^{\tau_j}.$$

Therefore,

$$\begin{aligned} \#\tilde{\Lambda}_f &\leq (N_0 + 1)(\#\Gamma_f) \leq (N_0 + 1)(\#\Gamma_f^0 + \#\Gamma_f^1) \\ &\leq 2(N_0 + 1)^2 \left(\varepsilon_0^{-\tau_0}\mathcal{N}_0^{\tau_0} + \varepsilon_1^{-\tau_1}\mathcal{N}_1^{\tau_1} \right). \end{aligned} \quad (3.72)$$

To complete the proof, we must next estimate $\#\tilde{\Lambda}_r$. For each $\theta^\diamond \in \tilde{\Lambda}_r$, we define $\theta' \in \tilde{\Lambda}$ and $\mathcal{Z}_{\theta^\diamond}$ exactly as in the proof of Theorem 3.11. Similarly as in (3.66), we have

$$\begin{aligned} \varepsilon_0^* + \varepsilon_1^* = \varepsilon^* &< \sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta| \leq \sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta^0| + \sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta^1| \\ &\leq \left(\sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta^0|^{\tau_0} \right)^{1/\tau_0} + \left(\sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta^1|^{\tau_1} \right)^{1/\tau_1}. \end{aligned}$$

From this, it follows that, for each $\theta^\diamond \in \tilde{\Lambda}_r$, at least one of

$$\varepsilon_0^* < \left(\sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta^0|^{\tau_0} \right)^{1/\tau_0} \quad (3.73)$$

or

$$\varepsilon_1^* < \left(\sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta^1|^{\tau_1} \right)^{1/\tau_1} \quad (3.74)$$

must hold. We denote by $\tilde{\Lambda}_r^0$ and $\tilde{\Lambda}_r^1$ the sets of all $\theta^\diamond \in \tilde{\Lambda}_r$ for which (3.73) and (3.74) hold, respectively. As in the proof of Theorem 3.11, each $\theta \in \Theta$ may belong to at most $N_0 + 1$ sets $\mathcal{Z}_{\theta^\diamond}$, $\theta^\diamond \in \tilde{\Lambda}_r$. Therefore, for $j = 0, 1$,

$$\mathcal{N}_j^{\tau_j} \geq \sum_{\theta \in \Gamma} |b_\theta^j|^{\tau_j} \geq (N_0 + 1)^{-1} \sum_{\theta^\diamond \in \tilde{\Lambda}_r^j} \sum_{\theta \in \mathcal{Z}_{\theta^\diamond}} |b_\theta^j|^{\tau_j} \geq (N_0 + 1)^{-1} (\#\tilde{\Lambda}_r^j) (\varepsilon_j^*)^{\tau_j}$$

and hence

$$\#\tilde{\Lambda}_r^j \leq 2(N_0 + 1)\varepsilon_j^{-\tau_j}\mathcal{N}_j^{\tau_j}, \quad j = 0, 1.$$

Therefore,

$$\#\tilde{\Lambda}_r \leq \#\tilde{\Lambda}_r^0 + \#\tilde{\Lambda}_r^1 \leq 2(N_0 + 1) \left(\varepsilon_0^{-\tau_0}\mathcal{N}_0^{\tau_0} + \varepsilon_1^{-\tau_1}\mathcal{N}_1^{\tau_1} \right).$$

This estimate, together with (3.70), and (3.72) imply (3.68) (since $N_0 \geq 3$). Estimate (3.69) follows by using $\varepsilon_j := (2c)^{\alpha_j} n^{-\alpha_j} \mathcal{N}_j$ ($j = 0, 1$) in (3.67) and (3.68) to obtain $n(\varepsilon_0 + \varepsilon_1) \leq n$ and so $\mathbb{A}_n^P(f)_\infty \leq \mathbb{A}_{n(\varepsilon_0 + \varepsilon_1)}^P(f)_\infty \leq \varepsilon_0 + \varepsilon_1$. \square

4 Best n-term Courant element approximation

In this section, we assume that \mathcal{T} is a locally regular triangulation of a bounded polygonal domain E with parameters $N_0, M_0, r, \rho, \delta$, and $\#\mathcal{T}_0$ (see §2.1). We denote by $\Phi(\mathcal{T})$ the collection of all Courant elements φ_θ generated by \mathcal{T} . Notice that $\Phi(\mathcal{T})$ is not a basis; $\Phi(\mathcal{T})$ is redundant. We consider nonlinear n -term approximation in $L_p(E)$ ($0 < p \leq \infty$) from $\Phi(\mathcal{T})$, where we identify $L_\infty(E)$ as $C(E)$. Our main goal is to characterize the approximation spaces generated by this approximation with emphasis on the case $p = \infty$. We let $\Sigma_n(\mathcal{T})$ denote the nonlinear set consisting of all continuous piecewise linear functions S of the form

$$S = \sum_{\theta \in \mathcal{M}} a_\theta \varphi_\theta,$$

where $\mathcal{M} \subset \Theta(\mathcal{T})$, $\#\mathcal{M} \leq n$, and \mathcal{M} may vary with S . We denote by $\sigma_n(f, \mathcal{T})_p$ the best L_p -approximation of $f \in L_p(E)$ from $\Sigma_n(\mathcal{T})$:

$$\sigma_n(f, \mathcal{T})_p := \inf_{S \in \Sigma_n(\mathcal{T})} \|f - S\|_p.$$

In order to characterize the approximation spaces generated by $(\sigma_n(f, \mathcal{T})_p)$, we begin in this section by first proving a companion pair of Jackson and Bernstein inequalities and then follow with the usual techniques of interpolation of operators (see for example [6, 15, 13]).

In the following, we assume in general that $0 < p \leq \infty$ and that $\alpha \geq 1$ for $p = \infty$ and $\alpha > 0$ if $p < \infty$; in either case we set $1/\tau := \alpha + 1/p$.

Theorem 4.1. [Jackson estimate] *If $f \in B_\tau^\alpha(\mathcal{T})$, then*

$$\sigma_n(f, \mathcal{T})_p \leq cn^{-\alpha} \|f\|_{B_\tau^\alpha(\mathcal{T})} \quad (4.1)$$

where c depends only on α, p and the parameters of \mathcal{T} .

Proof. Estimate (4.1) follows from any of our constructive algorithms as formulated in the corresponding Theorems 3.1, 3.7, 3.9, or 3.11. \square

Theorem 4.2. [Bernstein estimate] *If $S \in \Sigma_n(\mathcal{T})$, then*

$$\|S\|_{B_\tau^\alpha(\mathcal{T})} \leq cn^\alpha \|S\|_p \quad (4.2)$$

where c depends only on α, p , and the parameters of \mathcal{T} .

Proof. We shall prove estimate (4.2) only in the case $p = \infty$. For the proof when $p < \infty$, see [11]. Suppose $S \in \Sigma_n^k(\mathcal{T})$ and $S =: \sum_{\theta \in \mathcal{M}} c_\theta \varphi_\theta$, where $\mathcal{M} \subset \Theta(\mathcal{T})$ and $\#\mathcal{M} \leq n$. Let Λ be the set of all triangles $\Delta \in \mathcal{T}$ which are involved in all cells $\theta \in \mathcal{M}$. Then $S = \sum_{\Delta \in \Lambda} S_\Delta$, where $S_\Delta =: \mathbb{1}_\Delta \cdot P_\Delta$, P_Δ a linear polynomial. Evidently, $\#\Lambda \leq N_0 \#\mathcal{M} \leq cn$.

We shall utilize the natural tree structure in \mathcal{T} induced by the inclusion relation: Each triangle $\Delta \in \mathcal{T}_m$ has (contains) $\leq M_0$ children in \mathcal{T}_{m+1} and one parent in \mathcal{T}_{m-1} , if $m \geq 1$. Let Γ_0 be the set of all $\Delta \in \mathcal{T}$ such that $\Delta \supset \Delta'$ for some $\Delta' \in \Lambda$. We denote by Γ_b the set of all *branching triangles* in Γ_0 (triangles with more than one child in Γ_0) and by Γ'_b the set of all *children of branching triangles* in \mathcal{T} (which may or may not belong to Γ_0). Now, we

extend Γ_0 to $\Gamma := \Gamma_0 \cup \Gamma'_b$. We also extend Λ to $\tilde{\Lambda} := \Lambda \cup \Gamma_b \cup \Gamma'_b$. In addition, we introduce the following subsets of Γ : Γ_f the set of all *final triangles* in Γ (triangles in Γ containing no other triangles in Γ), $(\Gamma_0)_f$ the final triangles in Γ_0 , and $\Gamma_{ch} := \Gamma \setminus \tilde{\Lambda}$ the set of all *chain triangles*. Note that each triangle $\Delta \in \Gamma_{ch}$ has exactly one child in Γ . We may argue as we did for trees of cells in (3.9) that the number of branching triangles does not exceed the number of final triangles, $\#\Gamma_b \leq \#(\Gamma_0)_f$, and since $(\Gamma_0)_f \subset \Lambda$, then $\#\Gamma_b \leq cn$. Using this, we have $\#\Gamma'_b \leq M_0 \#\Gamma_b \leq cn$, $\#\Gamma_f \leq \#\Lambda + \#\Gamma'_b \leq cn$, and $\#\tilde{\Lambda} \leq \#\Lambda + \#\Gamma_b + \#\Gamma'_b \leq cn$. Keep in mind, however, that $\#\Gamma_{ch}$ can be much larger than n .

We next estimate $|S|_{B_\tau^\tau(\mathcal{T})} := \sum_{\Delta \in \mathcal{T}} |\Delta|^{-1} \mathbb{S}_\Delta(S)_\tau^\tau$, where $\tau := 1/\alpha$ (see (2.5) for the notation). We denote, for $m \geq 0$, $S_m := \sum_{\theta \in \mathcal{M}, \text{level}(\theta) \leq m} c_\theta \varphi_\theta$. We shall use that, for $\Delta \in \mathcal{T}_m$,

$$\mathbb{S}_\Delta(S)_\tau = \mathbb{S}_\Delta(S - S_m)_\tau \leq \|S - S_m\|_{L_\tau(\Omega_\Delta)} \quad (4.3)$$

and, also, $\mathbb{S}_\Delta(S)_\tau \leq \|S\|_{L_\tau(\Omega_\Delta)}$. Recall that Ω_Δ is the collection of triangles from the same level as Δ and which share a vertex. We denote

$$\mathcal{H}_m := \{\Delta \in \mathcal{T}_m : \Delta \subset \Omega_{\Delta'} \text{ for some } \Delta' \in \tilde{\Lambda} \cap \mathcal{T}_m\} \quad \text{and} \quad \mathcal{H} := \bigcup_{m \geq 0} \mathcal{H}_m.$$

Evidently, $\#\mathcal{H}_m \leq 3N_0 \#\tilde{\Lambda} \leq cn$ (the valence of each vertex is $\leq N_0$). We consider two possibilities for each $\Delta \in \mathcal{T}$: (a) $\Delta \in \mathcal{H}$, or (b) $\Delta \in \mathcal{T} \setminus \mathcal{H}$:

(a) If $\Delta \in \mathcal{H}_m$, then $\Omega_\Delta \supset \Delta'$ for some $\Delta' \in \tilde{\Lambda} \cap \mathcal{T}_m$. Using (2.3), we obtain

$$|\Delta|^{-1} \mathbb{S}_\Delta(S)_\tau^\tau \leq |\Delta|^{-1} \|S\|_{L_\tau(\Omega_\Delta)}^\tau \leq |\Delta|^{-1} |\Omega_\Delta| \|S\|_\infty^\tau \leq c \|S\|_\infty^\tau.$$

Therefore, by summing over all $m \geq 0$, we obtain in this case

$$\begin{aligned} \sum_{\Delta \in \mathcal{H}} |\Delta|^{-1} \mathbb{S}_\Delta(S)_\tau^\tau &= \sum_{m \geq 0} \sum_{\Delta \in \mathcal{H}_m} |\Delta|^{-1} \mathbb{S}_\Delta(S)_\tau^\tau \\ &\leq c \|S\|_\infty^\tau \sum_{m \geq 0} \#\mathcal{H}_m \\ &= c \|S\|_\infty^\tau \#\mathcal{H} \leq cn \|S\|_\infty^\tau. \end{aligned} \quad (4.4)$$

(b) Let $\Delta \in \mathcal{T}_m \setminus \mathcal{H}_m$. Then $\Omega_\Delta =: \bigcup_{j=1}^{n_\Delta} \Delta_j$ for some $\Delta_j \in (\Gamma_{ch} \cap \mathcal{T}_m) \cup (\mathcal{T}_m \setminus \Gamma)$, $j = 1, \dots, n_\Delta$, with $n_\Delta \leq 3N_0$. We have, using (4.3),

$$\mathbb{S}_\Delta(S)_\tau^\tau = \mathbb{S}_\Delta(S - S_m)_\tau^\tau \leq \sum_{j=1}^{n_\Delta} \|S - S_m\|_{L_\tau(\Delta_j)}^\tau.$$

Note that if $\Delta_j \in \mathcal{T}_m \setminus \Gamma$, then $S|_{\Delta_j} = S_m|_{\Delta_j}$ and hence $\|S - S_m\|_{L_\tau(\Delta_j)} = 0$. Suppose $\Delta_j \in \Gamma_{ch} \cap \mathcal{T}_m$. For each $\Delta \in \Gamma_{ch}$, we shall denote by $\tilde{\Delta}$ ($\tilde{\Delta} \neq \Delta$) the unique largest triangle of $\tilde{\Lambda}$ contained in Δ . Clearly, we have $S|_{\Delta_j \setminus \tilde{\Delta}_j} = S_m|_{\Delta_j \setminus \tilde{\Delta}_j} = \mathbb{1}_{\Delta_j \setminus \tilde{\Delta}_j} \cdot P_{\Delta_j}$ and $S_m|_{\Delta_j} = \mathbb{1}_{\Delta_j} \cdot P_{\Delta_j}$, where P_{Δ_j} is a linear polynomial. Therefore,

$$\begin{aligned} \|S - S_m\|_{L_\tau(\Delta_j)}^\tau &= \|S - S_m\|_{L_\tau(\tilde{\Delta}_j)}^\tau \\ &\leq c |\tilde{\Delta}_j| (\|S\|_\infty^\tau + \|P_{\Delta_j}\|_{L_\infty(\tilde{\Delta}_j)}^\tau) \leq c |\tilde{\Delta}_j| \|S\|_\infty^\tau, \end{aligned}$$

where we used that $\|P_{\Delta_j}\|_{L_\infty(\tilde{\Delta}_j)} \leq \|P_{\Delta_j}\|_{L_\infty(\Delta_j)} \leq c\|P_{\Delta_j}\|_{L_\infty(\Delta_j \setminus \tilde{\Delta}_j)} \leq c\|S\|_\infty$, applying Lemma 2.1. From above, it follows that

$$|\Delta|^{-1}\mathbb{S}_\Delta(S)_\tau^\tau \leq c\|S\|_\infty^\tau \sum_{1 \leq j \leq n_\Delta, \Delta_j \in \Gamma_{ch} \cap \mathcal{T}_m} |\tilde{\Delta}_j|/|\Delta_j|$$

and hence

$$\sum_{\Delta \in \mathcal{T}_m \setminus \mathcal{H}_m} |\Delta|^{-1}\mathbb{S}_\Delta(S)_\tau^\tau \leq c\|S\|_\infty^\tau \sum_{\Delta \in \Gamma_{ch} \cap \mathcal{T}_m} |\tilde{\Delta}|/|\Delta|.$$

Summing over $m \geq 0$ in this case as well, we find

$$\begin{aligned} \sum_{\Delta \in \mathcal{T} \setminus \mathcal{H}} |\Delta|^{-1}\mathbb{S}_\Delta(S)_\tau^\tau &\leq c\|S\|_\infty^\tau \sum_{\Delta \in \Gamma_{ch}} |\tilde{\Delta}|/|\Delta| \\ &\leq c\|S\|_\infty^\tau \sum_{\Delta' \in \tilde{\Lambda}} \sum_{\Delta \in \mathcal{T}, \Delta \supset \Delta'} |\Delta'|/|\Delta| \\ &\leq c\|S\|_\infty^\tau \sum_{\Delta' \in \tilde{\Lambda}} \sum_{j=0}^{\infty} \rho^j \leq c\|S\|_\infty^\tau \#\tilde{\Lambda} \leq cn\|S\|_\infty^\tau, \end{aligned} \quad (4.5)$$

where we have once switched the order of summation and used that $|\Delta'| \leq \rho|\Delta|$ if Δ' is a child of Δ (see (2.2)).

Combining inequalities (4.4) and (4.5), we obtain $|S|_{B_{\mathcal{T}}^\tau} \leq cn\|S\|_\infty^\tau$ which is equivalent to (4.2). \square

We define the approximation space $A_q^\gamma(L_p) := A_q^\gamma(L_p, \mathcal{T})$ generated by the n -term Courant element approximation to be the set of all functions $f \in L_p(E)$ such that

$$\|f\|_{A_q^\gamma(L_p)} := \|f\|_p + \left(\sum_{n=1}^{\infty} (n^\gamma \sigma_n(f, \mathcal{T})_p)^q \frac{1}{n} \right)^{1/q} < \infty \quad (4.6)$$

with the usual modification when $q = \infty$.

For a fixed LR-triangulation \mathcal{T} , we denote by $K(f, t) := K(f, t; L_p, B_\tau^\alpha(\mathcal{T}))$ the K -functional as defined in (2.30). The Jackson and Bernstein estimates from Theorem 4.1 and Theorem 4.2 yield (see, e.g., Theorem 3.16 of [15] and its proof) the following direct and inverse estimates:

$$\sigma_n(f, \mathcal{T})_p \leq cK(f, n^{-\alpha}) \quad (4.7)$$

and

$$K(f, n^{-\alpha}) \leq cn^{-\alpha} \left(\|f\|_p + \left(\sum_{k=1}^n \frac{1}{k} (k^\alpha \sigma_k(f, \mathcal{T})_p)^{p^*} \right)^{1/p^*} \right), \quad p^* := \min\{p, 1\}, \quad (4.8)$$

where c depends only on α , p , and the parameters of \mathcal{T} .

The following characterization of the approximation spaces $A_q^\gamma(L_p, \mathcal{T})$ is immediate from the inequalities (4.7) and (4.8), using the observation (2.31):

Theorem 4.3. *If $0 < \gamma < \alpha$, and $0 < q \leq \infty$, then*

$$A_q^\gamma(L_p, \mathcal{T}) = (L_p, B_\tau^\alpha(\mathcal{T}))_{\frac{\gamma}{\alpha}, q}$$

with equivalent norms.

The next result establishes an important (continuous) embedding which will be needed in §5 in order to identify the approximation spaces (the ones determined by the algorithms, as well as best n -term Courant element approximation) as B -spaces.

Theorem 4.4. *Suppose our standing assumptions hold, i.e., $\alpha > 1$ if $p = \infty$ and $\alpha > 0$ if $p < \infty$. If we let $1/\tau := \alpha + 1/p$, then $A_\tau^\alpha(L_p, \mathcal{T}) \subset B_\tau^\alpha(\mathcal{T})$ and*

$$\|f\|_{B_\tau^\alpha(\mathcal{T})} \leq c\|f\|_{A_\tau^\alpha(L_p, \mathcal{T})} \quad (4.9)$$

where c depends only on α , p , and the parameters of \mathcal{T} .

Proof. We shall prove (4.9) only in the case $p = \infty$, proceeding similarly as in [7]. For a proof in the case $0 < p < \infty$, see [4]. Suppose $f \in A_\tau^\alpha(L_\infty, \mathcal{T})$ and let $S_m \in \Sigma_m(\mathcal{T})$ be such that

$$\|f - S_m\|_\infty \leq 2\sigma_m(f, \mathcal{T})_\infty. \quad (4.10)$$

Since $\sigma_m(f, \mathcal{T})_\infty \rightarrow 0$, we have $f = S_1 + \sum_{\nu=1}^\infty (S_{2^\nu} - S_{2^{\nu-1}})$ with the series converging uniformly and hence ($\tau < 1$)

$$\|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau \leq \|S_1\|_{B_\tau^\alpha(\mathcal{T})}^\tau + \sum_{\nu=1}^\infty \|S_{2^\nu} - S_{2^{\nu-1}}\|_{B_\tau^\alpha(\mathcal{T})}^\tau. \quad (4.11)$$

We apply the Bernstein estimate from Theorem 4.2 to $S_{2^\nu} - S_{2^{\nu-1}} \in \Sigma_{2^{\nu+1}}(\mathcal{T})$ to obtain

$$\|S_{2^\nu} - S_{2^{\nu-1}}\|_{B_\tau^\alpha(\mathcal{T})} \leq c2^{\nu\alpha} \|S_{2^\nu} - S_{2^{\nu-1}}\|_\infty \leq c2^{\nu\alpha} (\sigma_{2^\nu}(f, \mathcal{T})_\infty + \sigma_{2^{\nu-1}}(f, \mathcal{T})_\infty)$$

and similarly

$$\|S_1\|_{B_\tau^\alpha(\mathcal{T})} \leq c(\|f\|_\infty + \sigma_1(f, \mathcal{T})_\infty).$$

Substituting the above in (4.11), we find

$$\|f\|_{B_\tau^\alpha(\mathcal{T})}^\tau \leq c\|f\|_\infty^\tau + c \sum_{\nu=1}^\infty (2^{\nu\alpha} \sigma_{2^\nu}(f, \mathcal{T})_\infty)^\tau \leq c\|f\|_{A_\tau^\alpha(L_\infty, \mathcal{T})}^\tau. \quad \square$$

5 Approximation spaces for algorithms

Our goal in this section is to show that the algorithms that we developed and explored in §3 achieve (in a certain sense) the rate of convergence of the best n -term Courant element approximation. We shall utilize the characterization of the approximation spaces

$$A_q^\gamma(L_p, \mathcal{T}; \sigma) := A_q^\gamma(L_p, \mathcal{T}).$$

from the previous section (see Theorems 4.3-4.4). We shall denote by $A_q^\gamma(L_p, \mathcal{T}; \mathbb{A}^T)$, $A_q^\gamma(L_p, \mathcal{T}; \mathbb{A}^{TC})$, and $A_q^\gamma(L_p, \mathcal{T}; \mathbb{A}^P)$ the approximation spaces generated by the “Threshold”, “Trim & Cut”, and “Push the Error” algorithms, respectively. Namely, $f \in A_q^\gamma(L_p, \mathcal{T}; \mathbb{A})$, where \mathbb{A} is \mathbb{A}^T , \mathbb{A}^{TC} or \mathbb{A}^P , if $f \in L_p(E)$ and

$$\|f\|_{A_q^\gamma(L_p, \mathcal{T}; \mathbb{A})} := \|f\|_p + \left(\sum_{n=1}^{\infty} (n^\gamma \mathbb{A}_n(f, \mathcal{T})_p)^q \frac{1}{n} \right)^{1/q} < \infty$$

with the usual modification when $q = \infty$ (it is not quite a norm).

Theorem 5.1. *Let \mathcal{T} be an LR-triangulation of a bounded polygonal domain $E \subset \mathbb{R}^2$.*

(a) *If $p = \infty$, $\alpha > 1$, and $\tau := 1/\alpha$, then*

$$A_\tau^\alpha(L_\infty, \mathcal{T}; \mathbb{A}^P) = A_\tau^\alpha(L_\infty, \mathcal{T}; \mathbb{A}^{TC}) = A_\tau^\alpha(L_\infty, \mathcal{T}; \sigma) = B_\tau^\alpha(\mathcal{T}) \quad (5.1)$$

with equivalent “norms”.

(b) *If $0 < p < \infty$, $\alpha > 0$, and $\tau := (\alpha + 1/p)^{-1}$, then*

$$A_\tau^\alpha(L_p, \mathcal{T}; \mathbb{A}^{TC}) = A_\tau^\alpha(L_p, \mathcal{T}; \mathbb{A}^T) = A_\tau^\alpha(L_p, \mathcal{T}; \sigma) = B_\tau^\alpha(\mathcal{T}) \quad (5.2)$$

with equivalent “norms”, where “Trim & Cut” is applied with parameter $\tau \leq \varrho < p$.

Proof. (a) Let $p = \infty$. We let $\mathbb{A}_n(f)_\infty$ denote $\mathbb{A}_n^P(f)_\infty$ or $\mathbb{A}_n^{TC}(f)_\infty$, and $A_\tau^\alpha(L_\infty; \mathbb{A})$ denote the approximation space generated by the corresponding algorithm. Suppose $\|f\|_{A_\tau^\alpha(L_\infty; \mathbb{A})} < \infty$. Evidently, $\sigma_n(f)_\infty \leq \mathbb{A}_n(f)_\infty$ and hence, using Theorem 4.4,

$$\|f\|_{B_\tau^\alpha} \leq c \|f\|_{A_\tau^\alpha(L_\infty; \sigma)} \leq c \|f\|_{A_\tau^\alpha(L_\infty; \mathbb{A})}.$$

It remains to show that if $\|f\|_{B_\tau^\alpha} < \infty$, then

$$\|f\|_{A_\tau^\alpha(L_\infty; \mathbb{A})} \leq c \|f\|_{B_\tau^\alpha}. \quad (5.3)$$

For the proof of this estimate, we shall employ Lemma 3.8 and Lemma 3.13. Since they are identical, it does not matter if we prove (5.3) for “Push the Error” or for “Trim & Cut”.

Suppose $f = \sum_{\theta \in \Theta} b_\theta \varphi_\theta$ is the representation of f which is used while “Push the Error” or “Trim & Cut” is applied. We have

$$\|f\|_{B_\tau^\alpha} := \left(\sum_{\theta \in \Theta} |b_\theta|^\tau \right)^{1/\tau}, \quad \tau := 1/\alpha, \quad \alpha > 1.$$

Next, we use a well known interpolation technique. We choose α_0 , α_1 , τ_0 , and τ_1 as follows: $1 = \alpha_1 < \alpha < \alpha_0$ and $\tau_0 := 1/\alpha_0$, $\tau_1 := 1/\alpha_1$. Hence $0 < \tau_0 < \tau < \tau_1 = 1$. Let now $(|b_{\theta_j}|)_{j=1}^\infty$ be the decreasing rearrangement of the sequence $(|b_\theta|)_{\theta \in \Theta}$, *i.e.*, indexed so that

$$|b_{\theta_1}| \geq |b_{\theta_2}| \geq \dots \quad (5.4)$$

We fix $\nu \geq 0$ and denote $f^0 := \sum_{j=1}^{2^\nu} b_{\theta_j} \varphi_{\theta_j}$ and $f^1 := \sum_{j=2^{\nu+1}}^\infty b_{\theta_j} \varphi_{\theta_j}$. In going further, we apply Lemma 3.8 or Lemma 3.13, to $f = f^0 + f^1$, from above, to obtain

$$\mathbb{A}_{2^\nu}(f)_\infty \leq c 2^{-\nu \alpha_0} \left(\sum_{j=1}^{2^\nu} |b_{\theta_j}|^{\tau_0} \right)^{1/\tau_0} + c 2^{-\nu} \sum_{j=2^{\nu+1}}^\infty |b_{\theta_j}|.$$

Using property (5.4), the facts that $\tau = 1/\alpha$, $1 < \alpha < \alpha_0$, and $\tau_0 = 1/\alpha_0$, we infer

$$\begin{aligned} \sum_{\nu=0}^{\infty} (2^{\nu\alpha} \mathbb{A}_{2^\nu}(f)_\infty)^\tau &\leq c \sum_{\nu=0}^{\infty} \left[2^{-\nu(\alpha_0-\alpha)\tau_0} \sum_{k=0}^{\nu} 2^k |b_{\theta_{2^k}}|^{\tau_0} \right]^{\tau/\tau_0} \\ &\quad + c \sum_{\nu=0}^{\infty} \left[2^{\nu(\alpha-1)} \sum_{k=\nu}^{\infty} 2^k |b_{\theta_{2^k}}| \right]^\tau \\ &\leq c \sum_{k=0}^{\infty} 2^k |b_{\theta_{2^k}}|^\tau \leq c \sum_{\theta \in \Theta} |b_\theta|^\tau = c \|f\|_{B_\tau^\alpha}^\tau, \end{aligned}$$

where we used the well known Hardy inequalities, namely, we applied the inequality from Lemma 3.10 in [15] to estimate the first sum and Lemma 3.4 from [6] to the second term.

(b) For $0 < p < \infty$, the proof of (5.2) is similar to the proof of (5.1). The only difference is that the appropriate roles of Lemma 3.8 or Lemma 3.13 are now played by Lemma 3.2 or Lemma 3.10. We omit the details. \square

6 Concluding remarks

Our primary goal in the present article is to quantify the nonlinear n -term approximation from Courant elements and use it to develop algorithms capable of achieving the rate of the best approximation. This is closely related to the fundamental question in nonlinear approximation of how to measure the smoothness of the functions. As we show in this article, for n -term Courant element approximation when the triangulation \mathcal{T} is fixed, it is natural to measure the smoothness via the scale of the B -spaces $B_\tau^\alpha(\mathcal{T})$. The use of these spaces allows one to characterize the approximation spaces for any rate of convergence $\alpha > 0$. It also enables us to develop algorithms which attain the rate of the best approximation.

It is natural to add another degree of nonlinearity to the approximation by allowing the triangulation \mathcal{T} to vary. Thus a function f should be considered smooth of order $\alpha > 0$ if $\inf_{\mathcal{T}} \|f\|_{B_\tau^\alpha(\mathcal{T})} < \infty$, where the infimum is taken over all LR-triangulations \mathcal{T} (with fixed parameters). Therefore the rate of n -term Courant element approximation to f is roughly $O(n^{-\alpha})$. Summarizing, our approximation scheme proceeds as follows: (i) for a given function f , find a triangulation \mathcal{T}_f and a B -space $B_\tau^\alpha(\mathcal{T}_f)$ in which f exhibits the most smoothness, (ii) find an optimal representation of f in terms of Courant elements from $\Phi_{\mathcal{T}}$, (iii) run an algorithm which achieves the rate of the best n -term Courant element approximation. The first step in this scheme is the most complicated one. We do not have an efficient solution for this as yet. In the simpler case of nonlinear approximation from piecewise polynomial over dyadic partitions, this problem, however, has a complete and efficient solution [14]. As we show, once the triangulation \mathcal{T} is determined, the remaining two steps are now well understood and have efficient solutions in both theoretical and practical senses.

The three algorithms that we develop and explore in this article provide solutions of the problem under appropriate conditions. A common feature of these algorithms is the first step, a nontrivial decomposition from the redundant collection of all Courant elements from $\Phi_{\mathcal{T}}$. After this initial step, however, they take three different routes. The ‘‘Threshold’’ algorithm is completely unstructured but easy to implement. The drawback of this procedure is that

it is not valid in the case of the uniform norm and as a consequence it does not perform well in L_p for p large. The “Trim & Cut” algorithm is valid for L_p , $0 < p \leq \infty$, but it is over structured and as a result the performance suffers. The “Push the error” algorithm appears to be the preferred approximation method.

The algorithms that we develop in this article are not restricted to n -term Courant element approximation. They can be applied immediately to the approximation from (discontinuous) piecewise approximation over multilevel triangulations (for the precise setting, see [11]). In this case the role of the B -spaces $B_\tau^\alpha(\mathcal{T})$ should be played by the skinny B -spaces $\mathcal{B}_\tau^\alpha(\mathcal{T})$, introduced in (2.37). The results are similar, but simplify considerably and we omit the details. Upon simplifying the ideas to the one dimensional case, the spaces (fat, thin, and skinny) all coincide with the B spaces and the algorithms studied here provide effective and efficient approximation schemes.

Furthermore, these algorithms can easily be adapted to nonlinear n -term approximation from smooth piecewise polynomial basis functions such as these considered in [4] and, in particular, from box splines. The main difference would be that one should use the corresponding B -spaces, developed in [4], but proceeding in a similar manner to this paper.

It is natural to use (wavelet or prewavelet) bases in nonlinear approximation, and specifically for approximation in L_p ($1 < p < \infty$). We are not aware of compactly supported wavelets (prewavelets) generated by Courant elements or smoother piecewise polynomials on general multilevel triangulations. It is clear to us that such wavelet bases would be very “expensive” to construct and hence are of limited practical value. However, in the case of uniform triangulations, compactly supported prewavelets and wavelet frames generated by Courant elements, or box splines, do exist and have been implemented in practice. Obviously, the n -term approximation from such bases or frames cannot surpass the rate of the best n -term Courant (or box spline) approximation, but they may give better constants and hence better performance results in practical situations.

It is also an important observation that, even in the case of uniform triangulations, the B -spaces used here are different from the Besov spaces used in nonlinear approximation. For more complete discussion of this issue, see [11] and [4].

Finally, we remark that in a related paper [12] we extend the arguments of this paper to develop a corresponding approach in the Hausdorff metric which is natural for approximating surfaces. There we also consider various practical aspects for decompositions, numerical approximation, and data structures.

Appendix. Coloring Lemma

In order to keep focus on the main analytical results of the paper, we have postponed the proof of the coloring lemma used in Section 3.2 to this appendix. This decomposition result was used to create a manageable collection of tree structures for both estimating the error and the number of elements used in our constructed approximant. Since this is a general purpose result which may prove useful in similar settings we give its proof in full in this appendix. For clarity we have broken down the proof into a series of lemmas. Since the coloring is done in several refinement stages, it is helpful to think of the coloring as an ordered triple of *primary*, *secondary*, and *shade* colors. The primary coloring will sort the elements periodically by resolution level, the secondary coloring will insure there is spatial color separation, and the third coloring (shading) is a more delicate adjustment to insure that tree structures are formed. We begin by repeating the statement of the coloring lemma for the reader's convenience.

Coloring Lemma [see Lemma 3.2.] *For any LR-triangulation \mathcal{T} of E , the set $\Theta := \Theta(\mathcal{T})$ of all cells generated by \mathcal{T} can be represented as a finite disjoint union of its subsets $(\Theta^\nu)_{\nu=1}^K$ with $K = K(N_0, M_0)$ (N_0 is the maximal valence and M_0 is the maximal number of children of a triangle in \mathcal{T}), such that each Θ^ν has a tree structure with respect to the inclusion relation, i.e., if $\theta', \theta'' \in \Theta^\nu$ then $(\theta')^\circ \cap (\theta'')^\circ \neq \emptyset$, or $\theta' \subset \theta''$, or $\theta'' \subset \theta'$.*

To begin the proof, we show, without loss of generality, that for the purposes of coloring we may assume that the multiresolution triangulation provides sufficient resolution with each refinement step. We argue below that after a certain fixed number of increments of the level there will be a guaranteed refinement of each edge and triangle, which by hypothesis is controlled from above, i.e. uniformly bounded valences and max number of sub-triangles for each refinement. Consequently, we may separate the levels of Θ into L ($L := \lceil 12N_0^4 \ln_2 M_0 \rceil$) disjoint classes (primary colors) by placing two levels in the same class iff their indices are the same (mod L). Thus a class $\tilde{\Theta}$ is of the form $\tilde{\Theta} = \cup_{j=0}^{\infty} \tilde{\Theta}_j$, where $\tilde{\Theta}_0 := \Theta_{j_0}$ for some $0 \leq j_0 < L$ and $\tilde{\Theta}_j := \Theta_{j_0+jL}$. Since each such class $\tilde{\Theta}$ has a different primary color, it will suffice to show how to designate the secondary colors of the members of a single $\tilde{\Theta}$. Therefore to simplify the notation and wording of arguments, we will simply refer to (secondary) coloring the classes $\tilde{\Theta}$ instead of Θ . In Lemma A.1 below we show however that these classes have additional useful properties. Loosely speaking, part (a) shows that the old vertices on a given level are far apart in terms of the graph metric. In part (b) a similar statement is given for the “central parts” of non-overlapping edges of Courant elements from different levels of $\tilde{\Theta}$.

For $D \subset \mathbb{R}^2$ and $m \geq 0$, we define the *star* $St_m^k(D)$ inductively by $St_m^0(D) := D$ and $St_m^k(D) := \cup\{\theta \in \Theta : \text{level } \theta = m, \theta^\circ \cap St_m^{k-1}(D) \neq \emptyset\}$. For the vertices in resolution level m , this is just the neighborhood of radius k in the graph metric. For an edge e with vertices v' and v'' and an integer $m > \text{level } e$, we define the “central part” of the edge to be $st(e, m) := St_m^2(e \setminus St_m^{R-1}(\{v', v''\}))$, where $R := M_0^{4N_0^2} + 4$. This selection for R has been made sufficiently large so that part (b) of the following lemma holds.

Lemma A.1. *The Courant collection $\tilde{\Theta}$, which is based on the multilevel triangulation described above, satisfies:*

- (a) for each edge $[v, v']$ the distance between v and v' , measured in the graph metric on the next finer level of $\tilde{\Theta}$, is at least $4R$.
- (b) if e and e' are edges from cells in $\tilde{\Theta}$, m is an integer with $m - L \geq \text{level } e \geq \text{level } e'$, and $e \not\subseteq e'$, then $st(e, m) \cap e' = \emptyset$.

Proof. (a) Note that each edge in Θ gets subdivided at least once after $2N_0$ levels. Further, observe that after $\tilde{N}_0 := 2N_0^2$ refinements of any triangle, none of its vertices can be connected to their opposite edge by a single edge at the finer level. Using this observation repeatedly, one can verify that after L refinements, the graph metric distance between v and v' will be at least $2^{L/\tilde{N}_0} = M_0^{6N_0^2} > 4R$.

(b) Let v and v' be the vertices of e . Using twice the observation from the proof of part (a), we conclude that the distance from each of the vertices in $e \setminus St_{m-2\tilde{N}_0}^1(\{v, v'\})$ to e' is at least 4 when measured in the graph metric on level m . Therefore, on the m -th level $e \setminus St_{m-2\tilde{N}_0}^1(\{v, v'\})$ has a buffer of at least three layers of triangles that separates it from e' . On the other hand, the existence of M_0 and the choice of R guarantee that $St_m^{R-1}(\{v, v'\}) \supset St_{m-2\tilde{N}_0}^1(\{v, v'\})$ and establishes the claim. \square

This completes the **primary coloring** and from this point on we only need work with a particular $\tilde{\Theta}$ (i.e., a fixed primary color). In this case ‘level θ ’ will now refer to the level of θ in $\tilde{\Theta}$ rather than in Θ as will the star $St_m^k(\theta)$ and $st(e, m)$. Also, when referring to the color of a cell we will now mean the **secondary color**, unless otherwise specified. For $\theta \in \tilde{\Theta}$ we denote by $\partial\theta$ the boundary of θ , and by x_θ the central point of θ . We say that the cells in $\Theta' \subset \tilde{\Theta}_m$ are R -disjoint ($R \geq 1$) if $\theta^\circ \cap St_m^R(\theta') = \emptyset$ for any $\theta, \theta' \in \Theta'$.

The next result is used for the (secondary) coloring of cells of $\tilde{\Theta}$, proceeding from coarse to fine levels, and uses M colors so that same color cells are R -disjoint.

Lemma A.2. *Suppose some of the cells on a given level are colored in $M := N_0^{R+1} + 1$ ($R \geq 1$) colors so that the same color cells are R -disjoint. Then the rest of the cells on that level can be colored in the same M colors so that the same color cells are R -disjoint.*

Proof. To complete the coloring on the given level, we first use the Color #1 to paint as many cells as possible so that the same color cells are R -disjoint. Next, we use the Color #2 as much as possible, followed by the third and so on until either all cells get painted or we run out of colors. The latter case, however, never occurs. Indeed, assume to the contrary and let θ be the first cell which can not be colored by this algorithm with the M colors. The cell θ has the property that within its $R + 1$ star $St_m^{R+1}(\theta)$ there must be at least one cell painted with each of the M colors. But this contradicts the fact that M was selected to be at least as large as the number of cells within $St_m^{R+1}(\theta)$. \square

For the **secondary** coloring we proceed inductively, beginning at the coarsest level $\tilde{\Theta}_0$, and color cells in M colors so that same color cells are R -disjoint. Suppose then that all levels up to $\tilde{\Theta}_k$ ($k > 0$) have been colored. We color $\tilde{\Theta}_k$ as follows.

Step a) (*Color corner cells.*) First we define the notion of corner cell. A cell θ of level k is called a *corner cell* for a coarser cell θ' if θ' has an adjacent cell θ'' (at the same level of course) so that x_θ lies on edge $[x_{\theta'}, x_{\theta''}]$ and x_θ is adjacent to $x_{\theta''}$ on the level k (see

Figure 2). Given a cell $\theta' \in \tilde{\Theta}_{k-1}$, we color each of its corner cells $\theta \in \tilde{\Theta}_k$ the same color as θ' . This insures that a cell's color is propagated through all finer levels to its corner cells.

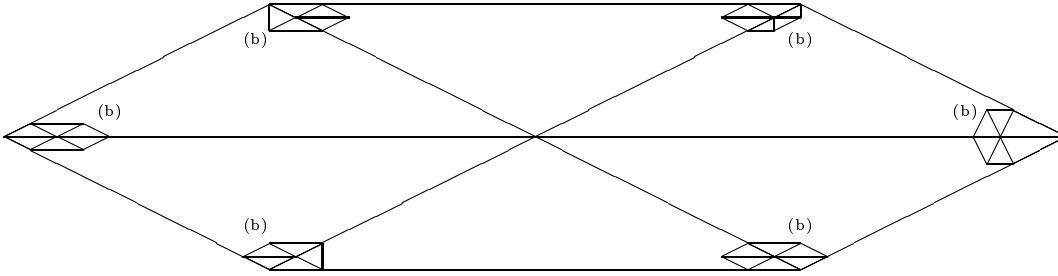


Figure 2: Corner cells from Step a).

Step b) (*Extend the coloring to R -Stars of the vertices on level $(k - 1)$.*) For each vertex v on level $(k - 1)$, we paint the cells contained in $St_k^{R+2}(v)$ using M colors so that the coloring done in Step a) is preserved and each color is used at most once. This is always possible since M was selected sufficiently large. Note that after this step the same color cells are R -disjoint since part (a) of Lemma A.1 guarantees that the stars are sufficiently separated.

Step c) (*Complete the secondary coloring of $\tilde{\Theta}_k$.*) Accounting for the cells previously painted in Steps a) and b), we color the remaining cells from $\tilde{\Theta}_k$ as described in Lemma A.2.

This procedure specifies the secondary coloring of $\tilde{\Theta}$ and we have thus represented it as a finite disjoint union $\cup_{\nu=1}^M \tilde{\Theta}^\nu$, where $\tilde{\Theta}^\nu$ are all cells (secondarily) colored in the ν -th color. Thus the primary color skips levels until sufficient refinement is guaranteed, while the secondary color insures sufficient spatial separation on each level to control cell overlaps. Unfortunately, the collection of same primary-secondary colored cells ($\tilde{\Theta}^\nu$) might not form a tree structure, i.e., there might be two cells in $\tilde{\Theta}^\nu$ whose interiors meet but neither of them contains the other. This may only happen when a finer cell lies on the edge of a given cell. To fix this defect we will set for each fixed $\tilde{\Theta}^\nu$ the third coloring component, the **Shade** of the cells, from two possible choices. First, we say that θ' and θ'' ($\theta', \theta'' \in \Theta$) *touch* if an edge of the finer of the cells is contained in an edge of the coarser.

We now restrict our cells to be of fixed primary and secondary colors (i.e., fix $\tilde{\Theta}^\nu$) and inductively determine the *Shade* of these cells. On the coarsest level $\tilde{\Theta}_0^\nu$ of $\tilde{\Theta}^\nu$ all cells are disjoint and we assign them Shade #1. For the induction step, we suppose cells of all levels of $\tilde{\Theta}^\nu$ up to level k have been shaded and each shaded collection satisfies the desired tree properties. We say that a cell θ is *Shade-consistent* with $\tilde{\theta}$ if x_θ does not lie on an edge of any cell which has the same Shade as $\tilde{\theta}$. Hence it is possible to place θ in this Shade collection and preserve the tree structure. In this case we will also use the terminology that θ is consistent with that particular Shade. We now proceed to shade the cells belonging to level k , i.e., $\theta \in \tilde{\Theta}_k^\nu$, according to:

Case i. If θ both touches and is Shade-consistent with some coarser cell $\tilde{\theta}$, then we assign to θ the same Shade as that of the finest such $\tilde{\theta}$. Recall that this finest cell is unique by the construction of $\tilde{\Theta}^\nu$.

Case ii. Otherwise, we assign to θ the first numbered Shade for which θ is consistent. If no such shade exists, we introduce a new shade for θ .

By the construction in the induction step, it is obvious that each Shade sub-collection has the desired tree structure. We will show that these criteria introduce at most two Shades. For this we need a couple of technical facts, reminding the reader that all cells belong to a fixed $\tilde{\Theta}^\nu$, i.e., they have a fixed primary and secondary color.

Lemma A.3. *If θ intersects an edge e' of a coarser cell θ' but is not one of its corner cells, then $\theta \subset st(e', m)$, where $m := \text{level}\theta$.*

Proof. Let e' be an edge of θ' which intersects θ , and let v be a vertex of e' . By Step b) of our coloring procedure (for secondary colors), $St_m^1(v)$ contains a corner cell θ'' in $\tilde{\Theta}_m^\nu$ which is shaded the same as θ' . By Step c) in the construction of $\tilde{\Theta}^\nu$, $St_m^R(v)$ does not contain any other cells from $\tilde{\Theta}_m^\nu$. Since θ is not a corner cell of θ' , then $\theta \neq \theta''$. Therefore $\theta \cap St_m^{R-1}(v) = \emptyset$ and so θ must meet $e' \setminus St_m^{R-1}(\{v, v'\})$, where v' is the remaining vertex of e' . Therefore, $\theta \subset st(e', m)$. \square

Lemma A.4. *Cells of $\tilde{\Theta}^\nu$ with different shades do not touch.*

Proof. Suppose to the contrary that cells $\theta_j, \theta_k \in \tilde{\Theta}^\nu$ of different shades (Shade #j, Shade #k, respectively) do touch. We may first assume that θ_j is a maximal (i.e., coarsest level) cell of Shade #j that touches θ_k , and conversely, that θ_k is a maximal cell in Shade #k that touches θ_j . This follows by iteration and the fact that there are only finitely many coarser levels so the iteration must terminate.

We may assume without loss of generality that $\text{level}\theta_j < \text{level}\theta_k =: m_k$ and let e_j, e_k denote the edges of θ_j, θ_k respectively, such that $e_k \subset e_j$. We consider the two cases under which the finer cell θ_k could have been shaded and show that each one leads to a contradiction.

For Case i

In this event there would be a coarser cell $\tilde{\theta}_k \in \tilde{\Theta}^\nu$ of Shade #k which touches θ_k and to which θ_k would be Shade-consistent. Let \tilde{e}_k be an edge of $\tilde{\theta}_k$ where it is touched by θ_k . We consider two possible subcases depending upon the relative level of $\tilde{\theta}_k$ to that of θ_j .

sub-Case i.a $\tilde{\theta}_k$ is finer than θ_j

Since $\text{level}\theta_j \leq \text{level}\tilde{\theta}_k < \text{level}\theta_k$, then by part (b) of Lemma A.1 either $\tilde{e}_k \subset e_j$ or $st(\tilde{e}_k, m_k) \cap e_j = \emptyset$. The first possibility may be ruled out since it would imply that the coarser cell $\tilde{\theta}_k$ would touch θ_j , but θ_k is the maximal such cell of Shade #k. Hence $st(\tilde{e}_k, m_k)$ must be disjoint from e_j . Note that θ_k is not a corner cell of $\tilde{\theta}_k$. If that were the case, then θ_k would be disjoint from the interiors of all edges on level $\tilde{\theta}_k$ except the edge on which x_{θ_k} lies and the edges (at most two, possibly one) where $\tilde{\theta}_k$ is touched by θ_k . Hence, e_j must overlie one of the these edges since it contains e_k . This, however, contradicts the fact that θ_j touches θ_k in the former case and contradicts the maximality of θ_k in the later. Therefore θ_k cannot be a corner cell of $\tilde{\theta}_k$ and so by Lemma A.3, $\theta_k \subset st(\tilde{e}_k, m_k)$. But we already have proved that $st(\tilde{e}_k, m_k) \cap e_j = \emptyset$ which is impossible since θ_k touches θ_j on e_j .

sub-Case i.b $\tilde{\theta}_k$ is coarser than θ_j

Since $\text{level } \tilde{\theta}_k < \text{level } \theta_j < \text{level } \theta_k$ then again by Part (b) of Lemma A.1 either $e_j \subset \tilde{e}_k$ or $st(e_j, m_k) \cap \tilde{e}_k = \emptyset$. The former case contradicts maximality of θ_k relative to θ_j . For the latter case, note that θ_k cannot be a corner cell of θ_j because θ_k and θ_j have different Shades. Therefore, by Lemma A.3, $\theta_k \subset st(e_j, m_k)$ and so we obtain $\theta_k \cap \tilde{e}_k = \emptyset$ which is impossible since θ_k touches $\tilde{\theta}_k$ on \tilde{e}_k .

For Case ii

If this case occurred for the shading of θ_k , then since θ_j is both coarser than and touches θ_k , θ_k must not have been Shade $\#j$ consistent. Hence there must be a $\tilde{\theta}_j \in \tilde{\Theta}^\nu$ of Shade $\#j$ which is coarser than θ_k and x_{θ_k} belongs to some edge \tilde{e}_j of $\tilde{\theta}_j$. We consider two possible subcases depending upon the relative level of $\tilde{\theta}_j$ to that of θ_j .

sub-Case ii.a θ_j is coarser than $\tilde{\theta}_j$

Since $\text{level } \theta_j \leq \text{level } \tilde{\theta}_j < \text{level } \theta_k$, then compare edges e_j, \tilde{e}_j using Part (b) of Lemma A.1 to infer either $st(\tilde{e}_j, m_k) \cap e_j = \emptyset$ or $\tilde{e}_j \subset e_j$. In the latter case, it follows that both the edge e_k (recall θ_k touches the coarser θ_j on e_k) and the opposite vertex x_{θ_k} (since $x_{\theta_k} \in \tilde{e}_j$) of a triangle in \mathcal{T}_k are contained in e_j , which is clearly impossible. If the former case holds, i.e., $st(\tilde{e}_j, m_k) \cap e_j = \emptyset$, then a contradiction also results. To see this observe that θ_k cannot be a corner cell for $\tilde{\theta}_j$ due to the fact that they have different shades. But Lemma A.3 implies that $\theta_k \subset st(\tilde{e}_j, m_k)$ holds which contradicts the fact that $\theta_k \cap e_j \neq \emptyset$.

sub-Case ii.b θ_j is finer than $\tilde{\theta}_j$

Since $\text{level } \tilde{\theta}_j < \text{level } \theta_j < \text{level } \theta_k$, we again compare edges \tilde{e}_j, e_j using Part (b) of Lemma A.1 to imply either $st(e_j, m_k) \cap \tilde{e}_j = \emptyset$ or $e_j \subset \tilde{e}_j$. By quite similar arguments to the previous subcase we can prove contradictions are reached. Specifically, the latter statement implies that both the central vertex x_{θ_k} and its opposite edge e_k belong to the edge \tilde{e}_j . On the other hand, the fact that θ_k cannot be a corner cell for θ_j will imply that $\theta_k \subset st(e_j, m_k)$ which will show that x_{θ_k} belongs to the intersection $st(e_j, m_k) \cap \tilde{e}_j$ and contradict the former statement above.

By our assumption that different shaded cells could touch we are led in all cases to contradictions, thereby completing our contrapositive proof. \square

By combining the previous results with the next lemma, it follows immediately that Θ can be colored with $K := 2ML$ colors and the proof of the Coloring Lemma will be complete.

Lemma A.5. *At most two shades are required.*

Proof. Suppose in Case ii of the shading step above that a third shade were needed for some cell θ . Then its central point $x_\theta \in e_1 \cap e_2$ for some edges e_1 of θ_1 and e_2 of θ_2 , where $\theta_1, \theta_2 \in \tilde{\Theta}^\nu$ are coarser than θ and have Shade $\#1$ and Shade $\#2$, respectively. Now, if x_θ were a vertex for e_1 , then there would be a corner cell of θ_1 in $\tilde{\Theta}^\nu$ adjacent to θ which is clearly impossible since cells at the same level are R -disjoint. The same reasoning applies

to e_2 . Therefore x_θ cannot be a vertex for either e_1 or e_2 and we conclude that $e_1^\circ \cap e_2^\circ \neq \emptyset$. Hence, θ_1 and θ_2 touch which contradicts Lemma A.4. \square

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