Lattice points close to a smooth curve and squarefull numbers in short intervals

O. Trifonov
LATTICE POINTS CLOSE TO A SMOOTH CURVE AND
SQUAREFULL NUMBERS IN SHORT INTERVALS

OGNIAN TRIFONOVA

Abstract. We extend an approach of Swinnerton-Dyer and obtain new upper bounds for the number of lattice points close to a smooth curve. One consequence of these bounds is a new asymptotic result for the distribution of squarefull numbers in short intervals.

1. Introduction

A squarefull number is a positive integer \( n \) such that if \( p \) is a prime dividing \( n \), then \( p^2 \) divides \( n \). Let \( Q(x) \) be the number of squarefull numbers which are \( \leq x \). Bateman and Grosswald [1] proved that

\[
Q(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + o\left(x^{1/6}\right).
\]

Concerning the gaps between squarefull numbers, P. Shi [17] proved that there exist infinitely many positive integers \( n \) such that there is no squarefull number between \( n^2 \) and \( (n+1)^2 \). On the other hand, since each perfect square is a squarefull number, Shi’s result answers completely the question of how large the gaps between squarefull numbers can be.

Another question is what is the distribution of squarefull numbers in short intervals. The result of Bateman and Grosswald implies

\[
Q\left(x + x^{1/2+\theta}\right) - Q(x) \sim \frac{\zeta(3/2)}{2\zeta(3)} x^\theta
\]

when \( 1/6 \leq \theta < 1/2 \). Clearly (1) does not hold with \( \theta = 0 \). We would like to determine the smallest value of \( \theta \in (0, 1/2) \) for which (1) holds. Smaller and smaller admissible values of \( \theta \) were obtained by P. Shi [18], P.G. Schmidt [16], C.-H. Jia [13], P.G. Schmidt [16], H. Liu [14], D.R. Heath-Brown [5], M. Filaseta and O. Trifonov [2],

2000 Mathematics Subject Classification. 11J54, 11P21, 11N25.
and M.Huxley and O.Trifonov [10], with the latest paper showing that (1) holds for any $\theta \in (1/8, 1/2)$. We improve on these results by proving

**Theorem 1.** The asymptotic formula (1) holds for all $19/154 = 0.123\ldots < \theta < 1/2$.

The proof of Theorem 1 reduces to estimating the number of lattice points close to the curve $a^2b^3 = x$. Elementary estimates of this type were obtained in [7], [3], [9], and [8]. We improve on these estimates by extending an approach of Swinnerton-Dyer. In [20] Swinnerton-Dyer obtained upper bounds for the number of lattice points on a curve, provided the curve satisfies some simple analytic conditions. One way to get these bounds is to consider second divided differences of triples of lattice points on the curve. We extend the approach of Swinnerton-Dyer by working with third divided differences of quadruples of lattice points close to the smooth curve. This enables us to obtain

**Theorem 2.** Let $1 \leq C \leq M \leq T \leq M^2$, $f: [M, 2M] \mapsto R$ have a continuous third derivative and

$$0 < \frac{T}{CM^2} \leq |f^{(j)}(x)| \leq \frac{CT}{M^j}$$

for $j = 2, 3$ and all $x \in [M, 2M]$. Let $0 < \delta < 1/2$ and $\delta \leq (CM)^{-\frac{1}{2}}$. Define $S(f, \delta)$ to be the set of all integer points $(x, y)$ such that $x \in [M, 2M]$ and $|f(x) - y| < \delta$. Then for every $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that

$$|S(f, \delta)| \ll c(\epsilon)M^4C\left(M^{1/2}T^{4/27} + M^{12/25}T^{4/25} + M^{9/10}T^{4/25} + M^{12/13}T^{4/13} + M^{2/7}T^{2/7} + T/M + M^{1/2}T^{1/4} + M^{1/2}T^{5/4}\right) + C^5T M^{1/2}T^{5/2}.$$

**Theorem 2** is of interest in itself. We envision various applications of it in number theory problems. Theorem 2 gives best results when $T$ is close to $M^{3/2}$. For instance one can get the following corollary.
Corollary Let $f$ and $S(f, \delta)$ be as in Theorem 2 with $C$ an absolute constant, $0 < \delta < (CM)^{-\frac{1}{5}}$, and let $T = M^{\frac{5}{3}}$. Then

$$|S(f, \delta)| \ll c(\epsilon)M^{\frac{5}{3}+\epsilon}.$$ 

The best previous result, even when $\delta = 0$, was $|S| \ll M^{\frac{5}{3}}$ and $13/18 = .7222\ldots < .75$.

Since the proof of Theorem 2 consists of several independent steps, we present it in five sections. In Section 2 we state some well-known basic properties of divided differences; we prove some preliminary lemmas in Section 3; in Section 4 we investigate lattice points on quadratic major arcs; we estimate the number of integer solutions of a certain system of two equations in Section 5 and we prove Theorem 2 in Section 6. Section 7 is dedicated to the proof of Theorem 1.

Notation

We use the following notation throughout the whole paper:

$C$, $T$, and $M$ are real numbers with $1 \leq C \leq M \leq T \leq M^2$.

$\delta$ is a real number in $(0,1/8)$.

Let $r$ be a positive integer. Define $F_r$ to be the class of all real-valued functions which are defined on $I = [M, 2M]$, have a continuous $r$-th derivative on $I$ and

$$\frac{T}{CM^r} \leq |f^{(r)}(x)| \leq \frac{C}{M^r}$$

for all $x \in I$.

Define $S(f, \delta)$ to be the set of all integer points $(x, y)$ such that $x \in [M, 2M]$ and $|f(x) - y| < \delta$.

For any finite set $S$ of points in the plane we denote by $P(S)$ its projection on the $x$-axis, and by $|S|$ its cardinality.

We call the graph of any quadratic function $ax^2 + bx + c$ with $a$, $b$, and $c$ real numbers a parabola.

$f(u) \ll g(u)$ will mean that there exists an absolute constant $c$ such that $|f(u)| \leq cg(u)$ for all $u \geq 1$. $f(u) \ll_{\epsilon} g(u)$ will mean that there exists a constant $c(\epsilon)$ such that $|f(u)| \leq c(\epsilon)g(u)$ for all $u \geq 1$. Similarly, $f(u) \ll_{C} g(u)$ will mean that the implied constant in $\ll$ depends on $C$ only.

For $r$ and $n$ positive integers, we denote by $d_r(n)$ the number of distinct solutions in positive integers of the equation $x_1 x_2 \cdots x_r = n$. 
For $x$ a real number we denote by $\|x\|$ the distance from $x$ to the closest integer number.

We denote by $R$ the set of real numbers, and by $Z$ the set of integers.

To simplify the notation we adopt the convention $\epsilon \pm \epsilon = \epsilon$ and $\epsilon \epsilon = \epsilon$ whenever $\epsilon$ is an absolute constant. There will be no division by $\epsilon$ or comparison of $\epsilon$’s in this paper.

2. Divided differences

Divided differences are in the foundation of our estimates. In this section we state some definitions and properties of divided differences which are well-known and can be found for example in [11].

Let $f$ be a function defined on the reals and $x_0, x_1, \ldots, x_n$ be distinct real numbers. Then there is a unique polynomial $P(x)$ of degree $n$ such that $P(x_j) = f(x_j)$ for $j = 0, 1, \ldots, n$. The coefficient of $x^n$ in $P(x)$ is defined to be the $n$-th divided difference for $f$, and we denote it $f[x_0, x_1, \ldots, x_n]$. Denote $V = V(x_0, x_1, \ldots, x_n) = \prod_{0 \leq k < j \leq n} (x_j - x_k)$ and let $V_i = \prod (x_j - x_k)$ where the product is over all $k$ and $j$ such that $0 \leq k < j \leq n$, $k \neq i$, and $j \neq i$.

**Lemma 1.** $f[x_0, x_1, \ldots, x_n] = \frac{\sum_{i=0}^{n} (-1)^{n-i} V_i f(x_i)}{V}$.

**Lemma 2.** Let $m = \min(x_0, x_1, \ldots, x_n)$ and $M = \max(x_0, x_1, \ldots, x_n)$. If $f$ has continuous $n$-th derivative on $[m, M]$ then

$$f[x_0, x_1, \ldots, x_n] = f^{(n)}(\xi)/n!$$

for some $\xi \in (m, M)$.

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\hline
x_0 & x_1 & \cdots & x_n \\
\vdots & \vdots & \ddots & \vdots \\
x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1}
\end{pmatrix}
\]

**Lemma 3.** $f[x_0, x_1, \ldots, x_n] = \frac{f(x_0) f(x_1) \cdots f(x_n)}{V}$.

3. Preliminary Lemmas
This section contains three lemmas. The idea of the first two lemmas is that if some mild conditions on δ hold then one can find a subset of \( S(f, \delta) \) which has the following properties:

(i) it essentially has the same size as \( S(f, \delta) \);
(ii) it is either strictly convex or strictly concave depending on the sign of \( f'' \);
(iii) no two elements of the set are "too close".

The first lemma of this section is contained in Huxley [7, pp. 204-206].

**Lemma 4.** Let \( f \in \mathcal{F}_2 \), \( T \geq M \) and \( 0 < \delta^2 < \frac{1}{4} \min_{x \in I} |f''(x)|. \) Then there exist non-intersecting sets \( S_1 \) and \( S_2 \) such that

(i) \( S(f, \delta) = S_1 \cup S_2 \);
(ii) \( S_1 \) is either strictly convex or strictly concave set, depending on the sign of \( f'' \);
(iii) \( |S_2| \ll C \delta M + 1 \).

The next lemma is a modification of Theorem 3 in [22].

**Lemma 5.** Let \( f \in \mathcal{F}_2 \cap \mathcal{F}_3 \) and \( S_1 \subset S(f, \delta) \). Assume that \( S_1 \) is either strictly convex or strictly concave set. Let \( \frac{1}{100} > E > 0 \). Define \( S_3 \) to be the set of integer points \((x_j, y_j)\) in \( S_1 \) such that there exist elements of \( S_1 \), say \((x_i, y_i)\) and \((x_k, y_k)\), such that \( x_i < x_j < x_k, \ x_j - x_i < E, \) and \( x_k - x_j < E \). Then

\[
|S_3| \ll \frac{C^3 \delta E^3}{M} + \frac{CTE^5}{M^2}.
\]

**Proof of Lemma 5.**

Let \( a_1 \) and \( a_2 \) be positive integers. Define \( T(a_1, a_2) \) to be the set of all positive integers \( n \) such that \( n, n + a_1, \) and \( n + a_1 + a_2 \) are consecutive elements in \( \mathcal{P}(S_1) \).

Then by the definition of \( S_3 \) we have

\[
|S_3| = \sum_{a_1 < E, a_2 < E} |T(a_1, a_2)|
\]

We will derive an estimate for \( |T(a_1, a_2)| \). Fix \( a_1 < E \) and \( a_2 < E \), and let \( d = \gcd(a_1, a_2) \). Let \( x_0 \) and \( x_0 + u \) be two consecutive elements of \( T(a_1, a_2) \). Then \( x_0, x_0 + a_1, \ x_0 + a_1 + a_2, \ x_0 + u, \ x_0 + u + a_1, \) and \( x_0 + u + a_1 + a_2 \) are all in \( \mathcal{P}(S_1) \). Denote \( x_1 = x_0 + a_1, \ x_2 = x_0 + a_1 + a_2 \). Therefore \( f(x_0) = y_0 + \theta_0 \delta, \ f(x_1) = y_1 + \theta_1 \delta, \) and \( f(x_2) = y_2 + \theta_2 \delta \) for some integers \( y_0, y_1, y_2 \) and some real numbers \( \theta_i \) with \( |\theta_i| < 1 \) for \( i = 0, 1, 2 \). Denote \( x'_0 = x_0 + u, \ x'_1 = x_1 + u, \) and \( x'_2 = x_2 + u \). Again,
we have \( f(x'_0) = y'_0 + \theta'_0 \delta, f(x'_1) = y'_1 + \theta'_1 \delta, \) and \( f(x'_2) = y'_2 + \theta'_2 \delta \) for some integers \( y'_0, y'_1, y'_2 \) and some real numbers \( \theta'_i \) with \( |\theta'_i| < 1 \) for \( i = 0, 1, 2 \). By Lemma 1

(2) \( a_1 a_2 (a_1 + a_2) f[x_0, x_1, x_2] = a_2 f(x_0) - (a_1 + a_2) f(x_1) + a_1 f(x_2) = m + \theta \delta (a_1 + a_2) \)

where \( m = a_2 y_0 - (a_1 + a_2) y_1 + a_1 y_2 \) and \( \theta = \frac{1}{a_1 + a_2} (a_2 \theta_0 - (a_1 + a_2) \theta_1 + a_1 \theta_2) \). Obviously \( |\theta| < 2 \). Also, \( m \) is an integer and \( d \mid m \). Since \( S_1 \) is either strictly convex or strictly concave then \( m \neq 0 \). (If \( m = 0 \) then the lattice points \( (x_0, y_0), (x_1, y_1), \) and \( (x_2, y_2) \) are in \( S_1 \) and lie on a straight line.)

Note that the right-hand-side of (2) is \( > d/2 \) in absolute value since \( |m| \geq d \) and \( |\delta \delta (a_1 + a_2)| < 4\delta E < 1/4 \). Furthermore, Lemma 2 implies \( |f[x_0, x_1, x_2]| = |f'(\xi)|/2 < \frac{CT}{2M^2} \) with \( \xi \in I \). Therefore, if \( |T(a_1, a_2)| > 0 \), then

(3) \( a_1 a_2 (a_1 + a_2) \geq \frac{M^2 d}{CT} \).

Similarly we have

(4) \( a_1 a_2 (a_1 + a_2) f[x'_0, x'_1, x'_2] = m' + \theta' \delta (a_1 + a_2) \)

where \( m' = a_2 y'_0 - (a_1 + a_2) y'_1 + a_1 y'_2 \) and \( \theta' = \frac{1}{a_1 + a_2} (a_2 \theta'_0 - (a_1 + a_2) \theta'_1 + a_1 \theta'_2) \). Again \( |\theta'| < 2 \), \( m' \) is a nonzero integer and \( d \mid m' \). Define a function \( h (x) = f(x + u) - f(x) \). Subtracting (2) from (4) and taking into account Lemma 1 we get

(5) \( a_1 a_2 (a_1 + a_2) h[x_0, x_1, x_2] = (m' - m) + (\theta' - \theta) \delta (a_1 + a_2) \).

From Lemma 2, \( h[x_0, x_1, x_2] = \frac{h'(\eta)}{2} = \frac{f'(\eta + \eta) - f'(\eta)}{2} = \frac{w f''(\rho)}{2} \) for some \( \eta \) and \( \rho \) in \( I \). We consider two cases.

**Case I: \( m \neq m' \).**

Since \( d \mid m' - m \) we have \( |m' - m| \geq d \) in this case. Also \( |(\theta' - \theta) \delta (a_1 + a_2)| < 8E\delta < 1/2 \). Therefore the right-hand-side of (5) is \( > d/2 \) in absolute value. We get

\[
\frac{d}{|f''(\rho)| a_1 a_2 (a_1 + a_2)} \geq \frac{M^2 d}{CT a_1 a_2 (a_1 + a_2)} := P \text{ in case I.}
\]

**Case II: \( m = m' \).**

In this case \( m = a_2 y_0 - (a_1 + a_2) y_1 + a_1 y_2 = a_2 y'_0 - (a_1 + a_2) y'_1 + a_1 y'_2 = m' \), so \( a_2 (y_0 - y_1 - y'_0 + y'_1) = a_1 (y_1 - y_2 - y'_1 + y'_2) \). Note that \( y_0 - y_1 - y'_0 + y'_1 \neq 0 \). If not, the slope of the straight line through \((x_0, y_0)\) and \((x_1, y_1)\) equals the slope of the straight line through \((x'_0, y'_0)\) and \((x'_1, y'_1)\) which contradicts the strict convexity (concavity) of \( S_1 \). Therefore \( |y_0 - y_1 - y'_0 + y'_1| \geq \frac{a_1}{d} \).
Furthermore \( f(x_0) - f(x_1) - f(x_0') + f(x_1') = h(x_1) - h(x_0) = a_1 h'(\xi_1) = a_1 u f''(\eta_1) \) with \( \xi_1, \eta_1 \in I \). Also, \( f(x_0) - f(x_1) - f(x_0') + f(x_1') = y_0 - y_1 - y_0' + y_1' + (\theta_0 - \theta_1 - \theta_0' + \theta_1')\delta. \) Thus \( a_1 u f''(\eta_1) \geq \frac{9}{4} - 4\delta \geq \frac{9}{4} \) (\( \delta < 1/8 \)). We obtain

\[
u \geq \frac{M^2}{2CTd} := R.
\]

On the other hand, from (5) it follows that \( a_1 a_2 (a_1 + a_2) |h[x_0, x_1, x_2]| \leq 2\delta (a_1 + a_2) \), so \( a_1 a_2 \frac{u f''(\eta_1)}{2} \leq 2\delta \), and we get

\[
u \leq \frac{4C\delta M^3}{a_1 a_2 T} := Q.
\]

Thus, we proved that the elements of \( T(a_1, a_2) \) reside in at most \( \frac{M}{P} + 1 \) bunches, each of length \( \leq Q \). The distance between any two elements of \( T(a_1, a_2) \) which are in the same bunch is \( \geq R \). Therefore \( |T(a_1, a_2)| \leq \left( \frac{M}{P} + 1 \right) \left( \frac{Q}{R} + 1 \right) \). Note that (3) implies \( \frac{M}{P} = \frac{CTa_1 a_2 (a_1 + a_2)}{M^2 d} \geq 1 \). Therefore

\[
|T(a_1, a_2)| \leq 2 \frac{M}{P} \left( \frac{Q}{R} + 1 \right) = \frac{16C^3 \delta (a_1 + a_2) T}{M} + \frac{2CTa_1 a_2 (a_1 + a_2)}{M^2 d}.
\]

Using (6) and \( |S_3| = \sum_{a_1 \leq E, a_2 \leq E} |T(a_1, a_2)| \), the proof of the lemma follows. \( \square \)

An essential component of Swinnerton-Dyer’s approach is the strict convexity (concavity) of the set \( S(f, \delta) \). Since we are working with third divided differences rather than second, we need some sort of three-convexity condition. To ensure strict three-convexity (concavity) of \( S(f, \delta) \) one needs to impose too rigid (for the applications we envision) conditions on \( \delta \). Since we need only local three-convexity (concavity) we circumvent the above obstacle by using the following combinatorial lemma and the estimates from Section 4.

**Lemma 6.** Let \( k \) be a positive integer and let \( T = \{(x_1, y_1), \ldots, (x_{4k}, y_{4k})\} \) be a set of points in the plane with \( x_1 < x_2 < \cdots < x_{4k} \). Then either there exists a parabola that passes through at least \( k \) elements of \( T \), or there exists a subset of 5 distinct points of \( T \) such that no parabola passes through any 4 of these points.

**Proof of Lemma 6.**

Denote by \( P_{123} \) the parabola which passes through the points \((x_1, y_1), (x_2, y_2), \) and \((x_3, y_3)\). If \( k \) or more elements of \( T \) lie on \( P_{123} \) then we are done. So, assume
that \(|P_{123} \cap T| \leq k - 1\). Then there are elements of \(T\) which are not on \(P_{123}\). Let \((x_l, y_l)\) be one of them.

Let \(P_{12}\) be the parabola through the points \((x_1, y_1), (x_2, y_2)\), and \((x_l, y_l)\), \(P_{13}\) - the parabola through \((x_1, y_1), (x_3, y_3)\), and \((x_l, y_l)\), and \(P_{23}\) - the parabola through \((x_2, y_2), (x_3, y_3)\), and \((x_l, y_l)\).

If \(|P_{12} \cap T| \geq k\), or \(|P_{13} \cap T| \geq k\), or \(|P_{23} \cap T| \geq k\), then we are done. Assume otherwise. Then \(|(P_{123} \cup P_{12} \cup P_{13} \cup P_{23}) \cap T| \leq 4k - 4\) (actually one can easily replace \(4k - 4\) by \(4k - 12\)). Thus there exists an element of \(T\), say \((x_m, y_m)\), which is not on \(P_{123} \cup P_{12} \cup P_{13} \cup P_{23}\).

Let \(U = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_l, y_l), (x_m, y_m)\}\). Then \(U\) is a 5 element subset of \(T\) and no parabola passes through any 4 elements of \(U\).

\[\square\]

4. Lattice points on quadratic major arcs

To be able to use Lemma 6 we need a tool to deal with parabolas that contain lots of lattice points. To this end we use the concept of quadratic major arcs which was introduced by M. Huxley in [8]. Most of the material in this section is based on the papers [9] and [8].

Let \(S = S(f, \delta)\), and let \(W = \max_{q \leq M^3} 2^{\omega(q)}\) where \(\omega(q)\) is the number of distinct prime divisors of \(q\). We call the set of points \(\mathcal{A} = \{(x, P(x)) | x \in [\alpha, \beta]\}\) a quadratic major arc if \(P(x)\) is a quadratic polynomial in \(x\), \((\alpha, P(\alpha)) \in S\), \((\beta, P(\beta)) \in S\), and \(\mathcal{A}\) contains at least \(10W\) elements of \(S\). In other words, a quadratic major arc is a part of parabola that contains at least \(10W\) lattice points which are close to the graph of \(f\). Let \(\mathcal{Q}\) be a union of quadratic major arcs whose projections on the \(x\)-axis do not intersect. The aim of this section is to obtain an upper bound for \(|\mathcal{Q} \cap S\). We assume that \(f \in \mathcal{F}_2 \cap \mathcal{F}_3\) and \(\delta^2 \leq \frac{1}{4} \min_{x \in [M, 2M]} |f''(x)|\).

First, we use the following Lemma which was proved in [9].

**Lemma 7.** Let \(P(x)\) be a quadratic trinomial and \(f(x) \in \mathcal{F}_2 \cap \mathcal{F}_3\). Then the set \(\{x : |f(x) - P(x)| < \delta\}\) is a union of at most three disjoint open intervals and the length of each of these intervals does not exceed \(\frac{M}{4} \left(\frac{\delta}{\sqrt{T}}\right)^{1/3}\).
Let \( A = \{(x, P(x)) | x \in [\alpha, \beta]\} \) be a quadratic major arc in \( \mathcal{Q} \). Let \( J_1, J_2, \) and \( J_3 \) be disjoint open intervals such that \( \{x : |f(x) - P(x)| < \delta\} = J_1 \cup J_2 \cup J_3 \) (one or two of these intervals could be the empty set). Denote \( \{(x, P(x)) | x \in J_k\} \) by \( R_k \) for \( k = 1, 2, 3 \). Clearly, \( A \cap S \subset R_1 \cup R_2 \cup R_3 \). Let \( s \in \{1, 2, 3\} \) be such that \( |R_s \cap S| = \max_{k=1,2,3} |R_k \cap S| \). Since \( |A \cap S| \geq 10W \) we have \( |J_s \cap S| \geq 3W + 1 \). W.l.o.g. assume \( s = 1 \).

Note that if a parabola passes through the distinct lattice points \((x_1, y_1), (x_2, y_2), \) and \((x_k, y_k)\), then from Lagrange’s interpolation formula follows \( P(x) = \frac{ax^2 + bx + c}{q} \) where \( a, b, c, \) and \( q \) are integers, \( \gcd(a, b, c) = 1 \), and \( q|(x_j - x_1)(x_k - x_1)(x_k - x_j) \).

Then the lattice points on the graph of \( y = P(x) \) are all points of the form \((n, P(n))\) with \( n \) an integer solution of the congruence \( ax^2 + bx + c \equiv 0 \pmod{q} \). Next we make use of a standard lemma on congruences, stated in [8].

**Lemma 8.** The solutions of the quadratic congruence \( ax^2 + bx + c \equiv 0 \pmod{q} \) form a union of residue classes to a modulus \( q' \) with \( q'|q, q|q^2 \). The number of such residue classes modulo \( q' \) is at most \( 2^{\omega(q^2/q)} \).

Note that since \( q^2/q < q < M^2 \), \( W \) is an upper bound for the number of residue classes in the above lemma. Let \( D \) be the set of lattice points in \( R_1 \cap S \) which are in the residue class modulo \( q' \) containing the most elements of \( R_1 \cap S \). Then \( |R_1 \cap S| \leq W|D| \) and \( |D| \geq |R_1 \cap S|/W > 3 \).

Suppose \( \mathcal{Q} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_l \) with \( \mathcal{P}(\mathcal{A}_i) \cup \mathcal{P}(\mathcal{A}_j) = \emptyset \) when \( i \neq j \). For each \( \mathcal{A}_i, i = 1, 2, \ldots, l \) we perform the procedure we did on \( M \) and obtain a corresponding set of lattice points \( D_i \). The sets we get have the following properties:

(i) \( |\mathcal{A}_i \cap S| \leq 3W|D_i| \) and \( |D_i| \geq 4 \) for \( i = 1, 2, \ldots, l \);

(ii) All points in \( D_i \) lie on a parabola \( P_i(x) = \frac{a_i x^2 + b_i x + c_i}{q_i} \) where \( a_i, b_i, c_i, q_i \) are integers with \( \gcd(a_i, b_i, c_i) = 1 \), and \( q_i > 0 \);

(iii) We have \( D_i = \{(x_1^{(i)}, P_1(x_1^{(i)})), \ldots, (x_k^{(i)}, P_1(x_k^{(i)}))\} \) where \( x_1^{(i)}, x_2^{(i)}, \ldots, x_k^{(i)} \) form an arithmetic progression to some modulus \( q_i' \) with \( q_i' | q_i^2 \);

(iv) \( |P(x) - f(x)| < \delta \) for all \( x \in [x_i', x_k'], \) and \( x_k' - x_i' \leq M \left( \frac{q_i}{q_i'} \right)^{1/3} \).

Now we make use of Lemma 22 from [8] which provides an upper bound for the number of lattice points on the \( D_i \)’s with a fixed leading coefficient \( a/q \).
Lemma 9. Let \( q \leq 1/(2\delta) \) be a positive integer. Denote by \( q'' \) the least positive integer such that \( q \mid q''^2 \). Then
\[
\sum_{q_i \leq A} |D_i| \ll \frac{C^{1/3} \delta^{1/3} M}{T^{1/3} s^3} + \frac{C^{4/3} \delta M^2 q^{1/3}}{T^{2/3} q''^2} + \frac{C \delta^{3/2} M^3 q''^{1/2}}{T q''^3}.
\]
The only difference between Lemma 9 and Lemma 22 in [8] is the notation (we have \( \bigcup D_i \) instead of \( S'' \), and \( T/M^2 \) instead of \( \Delta \)).

Let \( A \leq 1/(2\delta) \) be a parameter whose value we specify latter on. Property (iii) of the \( D_i \)'s implies
\[
\sum_{q_i \geq A} |D_i| \leq \frac{M}{A} + 1.
\]
The next lemma is a slight variation of Lemma 23 in [8].

Lemma 10. We have
\[
\sum_{q_i \leq A} |D_i| \ll C^{4/3} \delta^{1/3} T^{2/3} A M^{-1} + C^{7/3} \delta T^{1/3} A^{5/3} + C^2 \delta^{3/2} A M + C \delta M + 1.
\]

Proof of Lemma 10.

First, if \( \frac{a_i}{q_i} = 0 \) then the elements of \( D_i \) lie on a straight line. In [7, pp.205-206] M. Huxley proved that there are \( \ll C \delta M + 1 \) elements of \( S(f, \delta) \) which lie on straight lines each of which contains at least three elements of \( S(f, \delta) \). This estimate holds provided \( f \in \mathcal{F}_2 \cap \mathcal{F}_3 \), \( T \geq M \), and \( \delta^2 \leq \frac{1}{4} \min_{x \in [M, 2M]} |f''(x)| \) (and we assumed \( f \) satisfies these conditions in the beginning of the section).

Now, let \( \frac{a_i}{q_i} \neq 0 \). Consider the divided difference \( f[\{x_1^{(i)}, x_2^{(i)}, x_3^{(i)}\}] \). From Lemma 1, the properties of \( D_i \), and \( x_j^{(i)} \in S(f, \delta) \) for \( j = 1, 2, 3 \), we get \( f[\{x_1^{(i)}, x_2^{(i)}, x_3^{(i)}\}] = \frac{n_i + \theta_i \delta}{2q_i} \) where \( |\theta_i| < 4 \), \( n_i \) is an integer, and \( \frac{n_i}{2q_i} = \frac{a_i}{q_i} \). Since \( a_i \neq 0 \), \( n_i \neq 0 \). Therefore \( |n_i + \theta_i \delta| \geq |n_i|/2 \geq 1/2 \). From Lemma 2 \( f[\{x_1^{(i)}, x_2^{(i)}, x_3^{(i)}\}] = f''(\xi_i)/2 \) for some \( \xi_i \in [M, 2M] \). Thus
\[
\frac{CT}{M^2} \geq \frac{1}{2} \left( \frac{1}{2} \right) \geq \frac{h(\xi_i)}{2q_i^2} = \frac{k(\xi_i)}{2M^2},
\]
and \( \frac{a_i}{q_i} \geq \frac{M^2}{2C} \). Therefore \( \frac{a_i}{q_i} \) is within \( \frac{2C}{M^2} \leq 2\delta CT_{/M^2} \) of \( f''(\xi_i)/2 \). Since the range of \( f'' \) is an interval of length \( < \frac{CT}{M^2} \), the range of the non-zero \( \frac{a_i}{q_i} \)'s is an interval of length at most \( < \frac{2CT}{M^2} \). Therefore for each \( q < A \) there are \( < \frac{2CT}{M^2} + 1 < \frac{2CT}{M^2} \) admissible values of \( a \). Let \( q = st^2 \), where \( s \) is squarefree. Then \( q'' = st \), and \( q'' \leq q_i < A \). Using Lemma 9 we get
\[
\sum_{q_i \leq A} |D_i| \ll \sum_{st \leq A} \frac{st^2 CT}{M^2} \left( \frac{C^{1/3} \delta^{1/3} M}{T^{1/3} s^3} + \frac{C^{4/3} \delta M^2 (st^2)^{1/3}}{T^{2/3} (st)^2} + \frac{C \delta^{3/2} M^3 (st^2)^{1/2}}{T (st)^3} \right).
\]
Since \( \sum_{t < A} t \ll A^2, \sum_{t < A} s^{-2/3}t^{2/3} \ll A^{5/3}, \) and \( \sum_{t < A} s^{-3/2} \ll A, \) we get the statement of the lemma. \( \square \)

Therefore we have

\[
\sum |D_i| \ll C^{4/3} \delta^{1/3} T^{2/3} A M^{-1} + C^{7/3} \delta T^{1/3} A^{5/3} + C^{2} \delta^{3/2} A M + C \delta M + 1 + \frac{M}{A}
\]

for any \( 0 < A \leq 1/(2\delta). \) Optimizing with respect to \( A \) (e.g. Lemma 2.4 of Graham and Kolesnik [4]) we obtain

\[
\mathcal{Q} \cap S \ll 3W \sum |D_i| \ll W \left( C^{2/3} \delta^{1/6} T^{1/3} + C^{7/8} \delta^{3/8} T^{1/8} M^{5/8} + C \delta^{3/4} M + 1 \right).
\]

Note that we omit the term \( C \delta M \) since it is absorbed by the term \( C \delta^{3/4} M. \)

5. On the number of integer solutions of a system of two equations

Let \( A \) and \( B \) be positive real numbers. One of the key elements in the proof of Theorem 2 is an upper bound for the number of integer solutions of the system

\[
D_1 + D_3 \neq D_2 + D_4
\]

\[
\begin{align*}
z_1 D_1 + z_3 D_3 &= z_2 D_2 + z_4 D_4 \\
z_1^2 D_1 + z_3^2 D_3 &= z_2^2 D_2 + z_4^2 D_4
\end{align*}
\]

which satisfy the conditions

\[
0 < z_1 < z_2 < z_3 < z_4 \leq A, \quad 0 < |D_j| \leq B \text{ for } j = 1, 2, 3, 4
\]

and

\[
gcd(z_1, z_2, z_3) | gcd(D_1, D_2, D_3, D_4).
\]

Our main tool will be Corollary 2 of Heath-Brown [6].

Lemma 11. (Heath-Brown) Let \( q \) be a nonsingular integral ternary quadratic form with matrix \( M. \) Let \( \Delta = |\det(M)|, \) and assume that \( \Delta \neq 0. \) Write \( \Delta_0 \) for the highest common factor of the \( 2 \times 2 \) minors of \( M. \) Then the number of primitive
integer solutions of $q(x) = 0$ in the box $|x_i| \leq R_i$ is

$$
\ll_{\epsilon} \left( 1 + \left( \frac{R_1 R_2 R_3 \Delta_0^2}{\Delta} \right)^{1/3 + \epsilon} \right) d_3(\Delta)
$$

for any $\epsilon > 0$.

Now we prove

**Lemma 12.** The number of integer solutions of the system (8) which satisfy (9), (10), and $\gcd(z_1, z_2, z_3) = d$ is

$$
\ll_{\epsilon} B^r \left( \frac{A_1 + \epsilon B^{8/3}}{d^{17/3}} + \frac{B^4}{d^4} \right).
$$

**Proof of Lemma 12.**

From (8) $z_4 = \frac{z_1 D_1 - z_2 D_2 + z_3 D_3}{D_4}$. Substituting in the last equation of (8) we get

$$
(D_1^2 - D_1 D_4) z_1^2 + (D_2^2 + D_2 D_4) z_2^2 + (D_3^2 - D_3 D_4) z_3^2
$$

$$
-2D_1 D_2 z_1 z_2 + 2D_1 D_3 z_1 z_3 - 2D_2 D_3 z_2 z_3 = 0
$$

which can be treated as a ternary quadratic form in $z_1$, $z_2$, and $z_3$. The absolute value of its determinant is $\Delta = | - D_0 D_1 D_2 D_3 D_4^2 | \neq 0$ where we have denoted $D_0 = (D_1 + D_3) - (D_2 + D_4)$. From (8) and (9) we get $0 < |D_0| \leq 4B$. Also,

$$
\Delta_0 = \gcd(D_1 D_2 D_3 D_4, D_1 D_2 D_4 (D_1 - D_2 - D_4), D_1 D_3 D_4 (D_4 - D_1 - D_3), D_2 D_3 D_4 (D_3 - D_2 - D_4)).
$$

Therefore $\Delta_0 = |D_4| \Delta_1$ where $\Delta_1 = \gcd(D_1 D_2 D_3, D_0 D_1 D_2, D_0 D_1 D_3, D_0 D_2 D_3)$, and

$$
\frac{\Delta_0^2}{\Delta} = \frac{\Delta_1^2}{|D_0 D_1 D_3 D_3|}.
$$

Let $p$ be a prime dividing $D_0 D_1 D_2 D_3$. Let $p^{\alpha_j} \mid D_j$ for $j = 0, 1, 2, 3$. Then $p^{\alpha} \mid D_0 D_1 D_2 D_3$ where $\alpha = \sum_{j=0}^{3} \alpha_j$. Let $\beta = \alpha - \max(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. Clearly $p^{\beta} \mid \Delta_1$. From the definition of $\alpha$ and $\beta$ we have $\beta \leq \frac{3 \alpha}{4}$. Thus, if $\alpha = 1$, $\beta = 0$, and if $\alpha = 2, \beta \leq 1$.

Each nonzero integer $t$ can be represented in the form $t = \pm a(t) b^2(t) c(t)$ where $a(t)$ and $b(t)$ are squarefree numbers, $c(t)$ is a cubeful number, and $\gcd(a(t), b(t)) = \gcd(b(t), c(t)) = \gcd(a(t), c(t)) = 1$. Hence, if $|D_0 D_1 D_2 D_3| = t$ then $\Delta_1 \leq b(t) c^{3/4}(t)$ and $\frac{\Delta_0^2}{\Delta} \leq \frac{c^{1/4}(t)}{a(t)}$. 


We estimate the number of solutions \( N(G) \) of (12) which satisfy (9), \( G \leq |D_0D_1D_2D_3| < 2G \), and \( \gcd(z_1, z_2, z_3) = 1 \). Lemma 11 holds. Also, for a fixed integer \( t \in [G, 2G) \) there are \( 16d_4(t) \ll \epsilon t^\epsilon \ll \epsilon G^\epsilon \ll \epsilon B^{4\epsilon} \) quadruples \((D_0, D_1, D_2, D_3)\) with \( |D_0D_1D_2D_3| = t \). Applying Lemma 11 we get

\[
N(G) \ll \epsilon \sum_{G \leq t < 2G} \left( 1 + A^{1+\epsilon} \frac{c^{1/6}(t)}{a^{1/3}(t)} \right),
\]

where we have used \( c(t) \leq t \leq G \leq B^4 \) and \( d_3(\Delta) \ll \epsilon \Delta^\epsilon \ll \epsilon B^{6\epsilon} \).

Now we prove that for each \( G \geq 1 \),

\[
\Sigma := \sum_{G \leq t < 2G} c^{1/6}(t) \ll G^{2/3}.
\]

For \( k \) a non-negative integer define \( I_k = [2^k, 2^{k+1}) \) and let \( C_3 \) be the set of cubefull numbers. We have \( \Sigma \leq \sum_k \sum_{b \in I_k} \sum_{c \in C_3} U(G, b, c) \) where the first summation is over all non-negative integers \( k \) and \( l \) with \( 2k + l \leq \log_2 16G \) and \( U(G, b, c) = \sum_a \frac{c^{1/6}}{a^{1/3}} \) where the summation is over all squarefree integers \( a \) such that \( a \geq G/(b^2 \epsilon) \).

Clearly, there are at most \( \frac{2G}{b^2} \) integers \( a \) with the above property and for these \( a \geq G/(b^2 \epsilon) \). Thus \( U(G, b, c) \leq \frac{2G^{2/3}}{b^{4/3} \epsilon^{1/2}} \leq \frac{2G^{2/3}}{24b^{9/2} \epsilon^{3/2}} \). Since there are \( 2^k \) integers in \( I_k \), and \( \ll 2^{l/3} \) cubefull integers in \( I_l \) (e.g. see [12]) we get \( \Sigma \ll \sum_{k, l} \frac{G^{2/3}}{2^{k/3} 2^{l/6}} \ll G^{2/3} \).

Using (13) we obtain \( N(G) \ll \epsilon \sum_{G \leq t < 2G} \left( 1 + A^{1+\epsilon} G^{2/3} \right) \). Summing the above estimate with \( G = 2^m \) for \( 0 \leq m \leq 4 \log_2 B \) we get that the number of primitive solutions of (12) which satisfy (9), \( \gcd(z_1, z_2, z_3) = 1 \), and \( 0 < |D_0D_1D_2D_3| \leq B^4 \) is \( \ll \epsilon \sum_{G \leq t < 2G} \left( 1 + A^{1+\epsilon} G^{2/3} \right) \). Thus, we proved the lemma when \( \gcd(z_1, z_2, z_3) = 1 \).

Now, let \( \gcd(z_1, z_2, z_3) = d > 1 \). From (10) \( d | D_j \) for \( j = 0, \ldots, 4 \). Let \( D_j' = D_j/d \) and \( z_j' = z_j/d \). Then \( \gcd(z_1', z_2', z_3') = 1 \), \( 0 < z_j' \leq A/d \) for \( j = 1, 2, 3 \) and \( 0 < |D_j'| \leq B/d \) for \( j = 1, 2, 3, 4 \). Equation (12) holds with the \( z_j \) replaced by \( z_j' \) and the \( D_j \) replaced by \( D_j' \). Applying the estimate for the case \( d = 1 \) we get the statement of the lemma.

\[ \square \]

6. Proof of Theorem 2

First, by Lemma 4 \( S(f, \delta) = S_1 \cup S_2 \), where \( S_1 \) is either strictly convex or strictly concave set, depending on the sign of \( f'' \), and \( |S_2| \ll C \delta M + 1 \). Next, let \( E <
1/(165) be a positive parameter whose value we specify later on. We apply Lemma 5 with \(E\) to \(S_1\) and obtain a subset \(S_3\) that contains the elements of \(S_1\) which are “close” to both of their neighbors. (The point \((x_i, y_i)\) ∈ \(S_3\) if \((x_i, y_i)\) ∈ \(S_1\) and there exist \((x_j, y_j), (x_k, y_k)\) ∈ \(S_1\) with \(x_j < x_i < x_k\) and \(x_i - x_j < E, x_k - x_i < E\).

We discard all elements of \(S_3\) from \(S_1\), and then discard every other element of the remaining set. We denote what is left by \(S_4\). Clearly \(|S_1| \leq 2|S_4| + |S_3|\). Furthermore, \(S_4\) has the property that if \((x_i, y_i)\) and \((x_j, y_j)\) are distinct elements of \(S_4\) then \(|x_j - x_i| \geq E\).

Next, we split the set \(S_4\) into disjoint subsets, each containing \(40W\) consecutive elements of \(S_4\) (the last subset may contain less elements). According to Lemma 6 from each subset we can either select \(10W\) points which are on one and the same parabola (a quadratic major arc), or we can select a quintuple of points such that no four of them lie on the same parabola. Therefore there exists disjoint sets \(S_5\) and \(S_6\) such that:

(i) \(S_4 = S_5 \cup S_6\);

(ii) \(S_5 = \bigcup_{j=1}^{k} \{(x_1^{(j)}, y_1^{(j)}), (x_2^{(j)}, y_2^{(j)}), (x_3^{(j)}, y_3^{(j)}), (x_4^{(j)}, y_4^{(j)}), (x_5^{(j)}, y_5^{(j)})\}\)

with \(x_1^{(1)} < x_2^{(1)} < \cdots < x_5^{(1)} < x_1^{(2)} < \cdots < x_5^{(2)} < \cdots < x_1^{(k)} < \cdots < x_5^{(k)}\).

We call the set \(\{(x_1^{(j)}, y_1^{(j)}), (x_2^{(j)}, y_2^{(j)}), (x_3^{(j)}, y_3^{(j)}), (x_4^{(j)}, y_4^{(j)}), (x_5^{(j)}, y_5^{(j)})\}\) the \(j\)-th quintuple, and each quintuple has the property that no four of its elements lie on the same parabola;

(iii) \(S_5\) is a union of quadratic major arcs whose projections on the \(x\)-axis do not intersect;

(iv) \(|S_4| \leq 8W|S_5| + 4|S_6| + 40W\).

Therefore

\[
|S(f, \delta)| \ll C\delta M + |S_5| + W + |S_4| + |S_6|.
\]

Note that from (7) we have

\[
|S_6| \ll W \left(C^{2/3}\delta^{1/6}T^{1/3} + C^{7/8}\delta^{3/8}T^{1/8}M^{5/8} + C\delta^{3/4}M + 1\right).
\]

Using Lemma 5 one can get an estimate for \(|S_6|\). What is left is to get an estimate for \(|S_5|\). Consider the \(j\)-th quintuple of consecutive integer points in \(S_5\), \(\{x_i^{(j)}, y_i^{(j)}\} i = 1, \ldots, 5\). Let \(A\) be a positive fixed parameter whose value we
specify later on. There are \( \leq \frac{4}{A} + 1 \) quintuples with \( x_2^{(j)} - x_1^{(j)} > A \). Now, let us estimate the number of quintuples with \( x_5^{(j)} - x_1^{(j)} \leq A \). We will concentrate on just one quintuple and to simplify the notation denote \( x_i^{(j)} \) by \( x_i \) and \( y_i^{(j)} \) by \( y_i \) for \( i = 1, \ldots, 5 \). Define \( a_i = x_{i+2} - x_{i+1} \) and \( b_i = y_{i+2} - y_{i+1} \) for \( i = 0, 1, 2, 3 \). We have \( a_0 + a_1 + a_2 + a_3 \leq A \). Also \( a_i \geq E \) for \( i = 0, 1, 2, 3 \).

Consider the following matrix

\[
\mathcal{R} = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_5 \\
x_1^2 & x_2^2 & \cdots & x_5^2 \\
x_1^3 & x_2^3 & \cdots & x_5^3 \\
y_1 & y_2 & \cdots & y_5 
\end{bmatrix}.
\]

Denote \( D_{s-1} = \text{det}(\mathcal{R}_{4,s}) \) for \( s = 1, 2, 3, 4, 5 \) (here \( \mathcal{R}_{4,s} \) is the \( (4,s) \) minor of the matrix \( \mathcal{R} \)). Thus, we defined a mapping \( \mathcal{H} \) from the set of quintuples in \( S_5 \) to the set of thirteentuples \( (a_0, \ldots, a_3, b_0, \ldots, b_3, D_0, \ldots, D_4) \). Note, that the mapping is one-to-one. Indeed, if the \( i \)-th and the \( j \)-th quintuples map to the same thirteentuple \( (a_0, \ldots, b_0, \ldots, D_4) \), then \( \frac{y_2^{(i)} - y_1^{(i)}}{x_2^{(i)} - x_1^{(i)}} = \frac{y_2^{(j)} - y_1^{(j)}}{x_2^{(j)} - x_1^{(j)}} \) which contradicts the strong convexity (concavity) of \( S_5 \). (We have \( S_5 \subset S_4 \subset S_1 \) and \( S_1 \) is strictly convex (concave).) Thus, \( |S_5| = |\mathcal{H}(S_5)| \). So, instead of estimating the size of \( S_5 \), we estimate the number of thirteentuples in the image of \( S_5 \).

Next, note that the following identities hold.

\[
\begin{align*}
D_0 & - D_1 + D_2 - D_3 + D_4 = 0 \\
x_1 D_0 & - x_2 D_1 + x_3 D_2 - x_4 D_3 + x_5 D_4 = 0 \\
x_1^2 D_0 & - x_2^2 D_1 + x_3^2 D_2 - x_4^2 D_3 + x_5^2 D_4 = 0 \\
x_1^3 D_0 & - x_2^3 D_1 + x_3^3 D_2 - x_4^3 D_3 + x_5^3 D_4 = -\text{det}(\mathcal{R})
\end{align*}
\]

Also, if \( (x_1, x_2, x_3, x_4, x_5) \) satisfies the above system of equations so does \( (x_1 + c, x_2 + c, x_3 + c, x_4 + c, x_5 + c) \) where \( c \) is any real number. Take \( c = -x_1 \) and denote \( z_j = x_{j+1} - x_1, \ j = 1, 2, 3, 4 \). (We have \( z_1 = a_0, z_2 = a_0 + a_1, z_3 = a_0 + a_1 + a_2, z_4 = a_0 + a_1 + a_2 + a_3 \leq A, E \leq z_1 < z_2 < z_3 < z_4, \) and \( E \leq z_j - z_{j-1} \) for \( j = 2, 3, 4 \).)

Then
\[-z_1D_1 + z_2D_2 -z_3D_3 + z_4D_4 = 0\]
\[-z_1^2D_1 + z_2^2D_2 -z_3^2D_3 + z_4^2D_4 = 0\]
\[-z_1^3D_1 + z_2^3D_2 -z_3^3D_3 + z_4^3D_4 = -\det(R)\]

Lemma 3 implies \(D_j = 0\) if and only if the four points of the quintuple minus \((x_j, y_j)\) are on the same parabola. Since no four points of the quintuple are on the same parabola we have \(D_j \neq 0\) for \(j = 0, \ldots, 4\). We can also bound from above the absolute values of the \(D_j\)'s. For instance, from Lemma 3 follows

\[f[x_1, x_2, x_3, x_4]V(x_1, x_2, x_3, x_4) = D_4 + \theta A^3\]

where \(|\theta| < 4\). Therefore

\[|D_4| \leq A^3 CT + 4\delta A^3 := B.\]

Working similarly, one can prove \(0 < |D_j| \leq B\) for each \(j = 0, \ldots, 4\). Let \(d = \gcd(z_1, z_2, z_3)\). From the definition of \(z_1, z_2, z_3, \gcd(z_1, z_2, z_3) = \gcd(a_0, a_1, a_2)\).

Next, we prove that \(d | D_j\) for \(j = 0, \ldots, 4\).

By definition \(D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 & x_1 & x_2 & x_3 & x_4 & x_1^2 & x_2^2 & x_3^2 & x_4^2 & y_1 & y_2 & y_3 & y_4 \end{vmatrix}\). Subtracting the third column from the fourth, the second from the third, the first from the second, and expanding with respect to the first row we get \(D_4 = \begin{vmatrix} a_0 & a_1 & a_2 & a_0(x_1 + x_2) & a_1(x_2 + x_3) & a_2(x_3 + x_4) & b_0 & b_1 & b_2 \end{vmatrix}\).

Subtracting the first row multiplied by \(x_2 + x_3\) from the second row, and expanding with respect to the second row we get

\[(16) \quad D_4 = a_0(a_0 + a_1)(a_1 b_2 - a_3 b_4) - a_2(a_1 + a_2)(a_0 b_1 - a_1 b_0).\]

Clearly, \(d^3 | D_4\).

Applying the same column operations and similar row operations to the determinants defining \(D_1, D_2,\) and \(D_3\) we get \(D_1 = (a_0 + a_1)(a_0 + a_1 + a_2)((a_0 + a_1)b_3 - (b_0 + b_1)a_3) - a_3(a_2 + a_3)((a_0 + a_1)b_2 - a_2(b_0 + b_1))\),

\[D_2 = a_0(a_0 + a_1 + a_2)((a_1 + a_2)b_3 - a_3(b_1 + b_2)) - a_3(a_1 + a_2 + a_3)((a_1 + a_2)b_0 - a_0(b_1 + b_2)).\]
and $D_3 = a_0(a_0 + a_1)(a_1(b_2 + b_3) - (a_2 + a_3)b_1) - (a_2 + a_3)(a_1 + a_2 + a_3)(a_0b_1 - a_1b_0)$.

It follows that $d|D_1$, $d|D_2$, and $d|D_3$. Since $0 \neq D_0 = D_1 - D_2 + D_3 - D_4$, then $d|D_0$ as well, and $D_1 + D_3 \neq D_2 + D_4$.

Consider the projection

$$(a_0, \ldots, a_3, b_0, \ldots, b_3, D_0, \ldots, D_4) \rightarrow (a_0, \ldots, a_3, D_0, \ldots, D_4)$$

defined on $\mathcal{H}(S_5)$. Note that all ninetuples we get from the projection satisfy the conditions (8), (9), (10). Assume that

$$(17) \quad \gcd(a_0, a_1, a_2) = d.$$

It follows from Lemma 12, the number of ninetuples in the projection of $\mathcal{H}(S_5)$ which satisfy $a_0 + a_1 + a_2 + a_3 \leq A$ is

$$\ll \varepsilon B^\varepsilon \left( \frac{A^{1+\varepsilon}B^{8/3}}{d^{1/3}} + \frac{B^4}{d^4} \right).$$

Next, we estimate the number of distinct thirteentuples in $\mathcal{H}(S_5)$ which have the same $a_i$'s and $D_i$'s (project to the same ninetuple).

Claim: If $\gcd(a_0, a_1, a_2) = d$ and

$$(18) \quad A \leq \min \left( \frac{1}{8\delta}, \frac{1}{2} M^{5/7}(CT)^{-2/7}, \frac{1}{3} M^{1/2}(\delta CT)^{-1/4}, M^2/(CT) \right)$$

then there are $\ll d$ elements of $\mathcal{H}(S_5)$ of the form

$$(a_0, a_1, a_2, a_3, \ldots, D_0, D_1, D_2, D_3, D_4).$$

Suppose the $i$-th quintuple maps to $(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, D_0, D_1, D_2, D_3, D_4)$ and the $j$-th quintuple $(j > i)$, to $(a_0, a_1, a_2, a_3, b'_0, b'_1, b'_2, b'_3, D_0, D_1, D_2, D_3, D_4)$.

From (16) follows that $D_4 = a_2(a_1 + a_2)d_1 - a_0(a_0 + a_1)d_2 = a_2(a_1 + a_2)d'_1 - a_0(a_0 + a_1)d'_2$ where $d_1 = a_1b_0 - a_0b_1$, $d_2 = a_2b_1 - a_1b_2$, $d'_1 = a_1b'_0 - a_0b'_1$, and $d'_2 = a_2b'_1 - a_1b'_2$.

The first step in proving the claim will be to show $d_1 = d'_1$ and $d_2 = d'_2$. We have $(d_1 - d'_1)a_0(a_0 + a_1) = (d_2 - d'_2)a_0(a_0 + a_1)$. Let $a = \gcd(a_2(a_1 + a_2), a_0(a_0 + a_1))$. Then $\frac{a_0(a_0 + a_1)}{a}[d_1 - d'_1].$ Using Lemma 1 we obtain

$$(19) \quad \left| f[x_1^{(i)}, x_2^{(i)}, x_3^{(i)}] - \frac{d_1}{a_0a_1(a_0 + a_1)} \right| \leq \frac{2\delta}{a_0a_1}.$$
So, \( |d_1| \leq 2\delta(a_0 + a_1) + a_0a_1(a_0 + a_1)\frac{CT}{M^2} \) and using the same argument one obtains the same upper bound for \( |d'_1| \). If we assume \( d_1 \neq d'_1 \) we get
\[
\frac{a_0(a_0 + a_1)}{a} \leq 4\delta(a_0 + a_1) + 2a_0a_1(a_0 + a_1)\frac{CT}{M^2}, \quad \text{and} \quad 1 \leq \frac{4a\delta}{a_0} + 2a_1\frac{CT}{M^2}.
\]

Since \( A\delta < \frac{1}{8} \), and \( a \leq a_0(a_0 + a_1) \leq a_0A \), we obtain \( a \geq \frac{M^2}{4CTA} \). From (16) follows that \( a \mid D_4 \). Using that
\[
B = A^6\frac{CT}{M^3} + 4\delta A^3 \geq |D_4| \geq a \geq \frac{M^2}{4CTA}
\]
we obtain a contradiction with (18). Therefore \( d_1 = d'_1 \) and \( d_2 = d'_2 \).

Similarly, we prove that \( D_4 \) determines uniquely \( d_3 = a_2(b_0 + b_1) - b_2(a_0 + a_1) \) and \( d_4 = b_0(a_1 + a_2) - a_0(b_1 + b_2) \). Note that \( D_4 \) also equals
\[
(a_0 + a_1)a_2(b_0(a_1 + a_2) - a_0(b_0 + b_1)) - a_0(a_1 + a_2)(a_2(b_0 + b_1) - b_2(a_0 + a_1)).
\]

Therefore \( D_4 \) determines uniquely \( d_1, d_2, \) and \( a_0b_2 - a_2b_0 \). Thus
\[
\frac{b_i}{a_i} - \frac{b_j}{a_j} = \frac{b'_i}{a_i} - \frac{b'_j}{a_j}
\]
for \( 0 \leq i < j \leq 2 \), so \( b'_i = b_i + a_it/d \) for \( i = 0, 1, 2 \) and some integer \( t \). Also, since \( D_1, b_0, b_1, b_2, a_0, a_1, a_2, \) and \( a_3 \) determine uniquely \( b_3 \) (see the formula for \( D_1 \)), the claim will be proved if we can show that there are \( \ll d \) possible choices for \( t \).

Lemma 2 implies that there exists \( \xi_i \in (x^{(i)}_1, x^{(i)}_3) \) such that \( f[x^{(i)}_1, x^{(i)}_2, x^{(i)}_3] = f''(\xi_i)/2 \). Using (19) we get
\[
|f''(\xi_i) - \frac{2d_i}{a_0a_1(a_0 + a_1)}| \leq \frac{4\delta}{a_0a_1}.
\]

Similarly, there exists \( \xi_j \in (x^{(j)}_1, x^{(j)}_3) \) such that \( |f''(\xi_j) - \frac{2d_j}{a_0a_1(a_0 + a_1)}| \leq \frac{4\delta}{a_0a_1} \).
\[
|\xi_i - \xi_j| = |(\xi_i - \xi_j) f''(\eta)| \leq \frac{8\delta}{a_0a_1} \quad \text{where} \quad \eta \in (x^{(i)}_1, x^{(j)}_3), \quad \text{so}
\]
\[
|\xi_i - \xi_j| \leq \frac{8CM^3\delta}{T_0a_0a_1}, \quad \text{and} \quad |x^{(i)}_1 - x^{(j)}_3| \leq 2A + \frac{8CM^3\delta}{T_0a_0a_1}.
\]

Therefore if a quintuple maps to thirteenthuple of the type
\[
(a_0, a_1, a_2, a_3, \ldots, D_0, D_1, D_2, D_3, D_4)
\]
then all \( x \)-coordinates of the points in the quintuple are in an interval \( J \) of length
\[
\leq 2A + \frac{8CM^3\delta}{T_0a_0a_1}.
\]
Since \( \frac{y_2^{(i)} - y_1^{(i)}}{x_2^{(i)} - x_1^{(i)}} = \frac{b_0}{a_0} = f'(\eta_i) + \frac{2\theta_j \delta}{a_0} \) with \( |\theta_i| < 1, \eta_i \in J \), and \( \frac{y_0 - b_0}{a_0} = f'(\eta_j) + \frac{2\theta_j \delta}{a_0} \) with \( |\theta_j| < 1, \eta_j \in J \), we get

\[
\frac{|y_0 - b_0|}{a_0} = \frac{|\theta|}{d} \leq |f'(\eta_j) - f'(\eta_i)| + \frac{2|\theta_j - \theta_i| \delta}{a_0} \leq |J| \frac{CT}{M^2} + \frac{4\delta}{a_0}.
\]

Therefore there are \( \leq 1 + \frac{4d\delta}{a_0} + \frac{2CTAd}{M^2} + \frac{8C^2M\delta}{a_0a_1} \) choices for \( t \). Note that \( d \leq a_0 \) and from (18) \( CTa \leq M^2 \). Pick \( E = \min \left( \frac{1}{16\delta}, C(M\delta)^{1/2} \right) \). Since \( 1 \leq C \leq M \) and \( \delta < (CM)^{-1/2} \), we have \( E \ll C(M\delta)^{1/2} \). Also, since \( a_0 \geq E \) and \( a_1 \geq E \), we get that there are \( \ll d \) choices for \( t \) which proves the claim.

Now, using the Claim, Lemma 12, and summing over \( d \) we get

\[
|S_3| \ll \epsilon B^\epsilon \left( A^{1+\epsilon}B^{8/3} + B^4 \right) + M/A + 1,
\]

and substituting for \( B \),

(20)

\[
|S_3| \ll M^\epsilon \left( C^{8/3}A^{17/3}M^{-8} + A^9\delta^{8/3} + C^4A^{24}T^4M^{-12} + A^{12}\delta^4 \right) + M/A + 1.
\]

Note that the above estimate holds for any positive \( A \) which satisfies (18). Optimizing with respect to \( A \) we get

\[
|S_3| \ll CM^\epsilon \left( M^{1/2}T^{4/27} + M^{9/10}\delta^{4/15} + M^{12/25}T^{4/25} + M^{12/13}\delta^{4/13} \right)
\]

\[
+ M\delta + (CT)^{2/7} + M^{1/2}(C\delta T)^{1/4} + CT/M.
\]

Recall (14). Since \( E < 1/(16\delta) \), Lemma 5 holds and we obtain \( |S_3| \ll C^{5T}M^{1/2}\delta^{5/2} \).

Recall (15). Using \( 1 \leq M \leq T \leq M^2 \) we get

\[
|S_3| \ll C \left( T^{1/3}M^{-1/12} + T^{1/8}M^{7/16} + M^{5/8} \right) \ll CM^{1/2}T^{4/27}.
\]

Substituting the upper bounds for \( |S_3|, |S_3|, \) and \( |S_3| \) in (14) and noting that \( C\delta M \) and \( W \) get absorbed by larger terms we obtain the theorem.

\section{7. Proof of Theorem 1.}

\textit{W.L.O.G} we can assume \( 19/154 < \theta \leq .129 \) since the theorem has been already established in [10] when \( 1/8 < \theta < 1/2 \). We also assume that \( x \) is sufficiently large \( (x \geq 10^7 \) will do).
Next, we reduce the problem to estimating the number of lattice points close to certain curves by using Lemma 5 from [10].

**Lemma 13.** Let \( x > 1, \epsilon > 0, \sqrt{\epsilon} \leq c^h \leq \epsilon^5 x, \text{ and } h = x^{1/2+\theta}. \) Then

\[
Q(x+h) - Q(x) = \frac{\zeta(3)}{2\zeta(3)} x^e + O(e^x) + O(R_1 + R_2)
\]

where \( R_1 \) is the number of pairs of positive integers \((m,k)\) with \( ex^e < m \leq x^{1/5}, \) \( m^2k^3 \in (x,x+h], \) \( \) and \( R_2 \) is the number of pairs of positive integers \((m,k)\) with \( ex^e < k \leq x^{1/5}, \) \( m^2k^3 \in (x,x+h]. \)

We use Lemma 13 with \( h = x^{1/2+\theta}, \) \( \) and \( \epsilon = x^{-0.1}. \) First, we estimate \( R_1. \) Let \( S_1(A,B) \) be the number of integers \( m \) in the interval \((A,B]\) for which there exists an integer \( k \) with \( m^2k^3 \in (x,x+h]. \) Then \( R_1 = S_1(x^{0.01}, x^{1.5}). \)

It is easy to check that \( S_1(M,2M) \leq \{ m \in (M,2M] \cap Z : \|f(m)\| < \delta \} \) where \( f(m) = m^{1/3}m^{-2/3} \) and \( \delta = x^{0.1/6}M^{-2/3}. \) Indeed, if \( m \in S_1(M,2M) \) then \( f(m) = (m^2)^{1/3} < k \leq ((x+h)/m^2)^{1/3} \) for some integer \( k, \) \( \) and \( \|f(m)\| < ((x+h)^{1/3} - x^{1/3})m^{-2/3}. \)

To estimate the size of \( S_1(M,2M) \) we use Theorem 2 from [10].

**Lemma 14.** Let \( f \in \mathcal{F}_2 \cap \mathcal{F}_3, \) \( 1 \leq M \leq T, \) \( \) and \( 0 \leq \delta \leq \frac{1}{2} C^{-1/2}T^{1/2}M^{-1}. \) Then

\[
|S(f,\delta)| \ll 1 + T^{3/10}M^{3/10}(\log M)^{1/2} + T^{4/11}M^{2/11}(\log M)^{5/11}
\]

\[
+\delta^{1/8}T^{3/8}M^{1/4}(\log M)^{5/8} + \delta^{1/7}T^{1/7}M^{4/7}(\log M)^{5/7} + \delta^{2/5}T^{1/5}M^{3/5}\log M + \delta M.
\]

From Lemma 14 with \( f(m) = (m^2)^{1/3}, \) \( \delta = x^{0.1/6}M^{-2/3}, \) \( \) and \( T = x^{1/3}M^{-2/3} \) we get

\[
S_1(M,2M) \ll x^{12}\log x \quad \text{for all } M \in [x^{0.01},x^{1.5}].
\]

Summing over \( O(\log x) \) intervals of the form \((M,2M]\) we obtain

\[
R_1 \ll x^{12}\log^2 x.
\]

Next, we estimate \( R_2. \) Let \( S_2(A,B) \) be the number of integers \( k \) in the interval \((A,B]\) for which there exists an integer \( m \) with \( m^2k^3 \in (x,x+h]. \) Then \( R_2 = S_2(x^{0.01}, x^{1.5}).\)
Again, it is easy to check that $S_2(M, 2M) \leq \{k \in (M, 2M] \cap Z : \|f_1(k)\| < \delta_1\}$ where $f_1(k) = x^{1/2-k^{-3/2}}$ and $\delta_1 = x^\theta M^{-3/2}$.

We estimate $S_2(M, 2M)$ for $M \in (x^{θ^{-0.01}}, x^{1.38}]$ via exponential sums. We use a handy lemma of Steckin [19] (one can find a proof of the lemma in [21, pp. 290–292]).

**Lemma 15.** Let $g : R \rightarrow R$ be a 1-periodic function of bounded variation on the interval $[0, 1]$. Let $f$ be any real-valued function defined on $R$, $a < b$ be real numbers with $b - a > 1$, and $N$ be a positive real number. Then

$$\left| \sum_{a < k \leq b} g(f(k)) - (b - a) \int_a^b g(t)dt \right| \ll \sum_{l=1}^N \frac{|S_l|}{l} + \frac{b - a}{N},$$

where $S_l = \sum_{a < k \leq b} e^{2\pi i f(k)}$, the summation is over all integers $k \in (a, b]$, and the constant in $\ll$ depends only on the variation of $g$. Furthermore, if the measure of the support of $g$ in $[0, 1]$ is $\ll 1/N$ then

$$\left| \sum_{a < k \leq b} g(f(k)) - (b - a) \int_a^b g(t)dt \right| \ll \frac{1}{N} \sum_{l=1}^N |S_l| + \frac{b - a}{N}.$$

The proof of the above lemma is via $L_1$, one-sided approximation by trigonometric polynomials. We apply Lemma 15 with $f(k) = x^{1/2-k^{-3/2}}$,

$$g(x) = \begin{cases} 1 & \text{if } \|x\| < \delta_1 \\ 0 & \text{if } \|x\| \geq \delta_1 \end{cases},$$

$a = M$, and $b = 2M$. Then $S_2(M, 2M) \ll \sum_{l=1}^N \frac{|S_l|}{l} + \frac{M}{N}$. To estimate $S_l$ we use the exponent pair $(\frac{1}{30}, \frac{33}{30}) = AABAAB(0, 1)$ (e.g. see [4]) and obtain $|S_l| \ll (lx^{1/2} M^{-5/2})^{1/20} M^{33/40}$. Set $N = M^{2/7} x^{-1/42}$. We get $S_2(M, 2M) \ll x^{1/42} M^{5/7}$. Splitting the interval $(x^{θ^{-0.01}}, x^{1.38}]$ to subintervals of the form $(M, 2M]$ and applying the last estimate to each of these subintervals we obtain

$$S_2(x^{θ^{-0.01}}, x^{1.38}] \ll x^{-123}.$$

When $x^{0.138} \leq M \leq x^{0.158}$ we use Corollary 1 of [22].

**Lemma 16.** Let $f \in \mathcal{F}_2 \cap \mathcal{F}_3$ and $0 \leq \delta \leq \frac{1}{4} C^{-1/2} T^{1/2} M^{-1}$. Then

$$|S(f, \delta)| \ll C M^{1/2} T^{1/6} + \delta M + \delta^{1/4} T^{1/4} M^{1/2}.$$
Applying Lemma 16 with \( f_1, \delta_1, \) and \( T = x^{1/2}M^{-3/2} \) we get
\[
S_2(M, 2M) \leq x^{1/12} M^{1/4} + x^{\theta} M^{-1/2} + x^{1/8 + \theta/4} M^{-1/4}.
\]
This implies
\[
S_2(x^{0.138}, x^{0.158}) \lesssim x^{123}.
\]

The next range is \( M \in (x^{1/5}, x^{41/231}) \) (41/231 = 0.177...). This is the critical range. All conditions of Theorem 2 hold with \( f_1, \delta_1, \) and \( T = x^{1/2}M^{-3/2} \). We get
\[
S_2(M, 2M) \leq x^{\theta} M^{(x^{2/27} M^{5/18} + x^{2/25} M^{6/25} + x^{4/15} M^{1/2} + x^{4/13} M^{6/13} + x^{1/7} M^{-1/7} + x^{1/2} M^{-5/2} + x^{(26+1)/8} M^{-1/4}) + x^{(1+59)/2} M^{-19/4}}.
\]
This implies
\[
S_2(x^{158}, x^{41/231}) \leq x^{10^{19} + \epsilon}.
\]

Finally, we estimate \( S_2(M, 2M) \) when \( M \in (x^{41/231}, x^{1/5}) \) by using Lemma 14 with \( f_1 \) and \( \delta_1 \). We obtain
\[
S_2(M, 2M) \leq \log M(x^{3/20} M^{-3/20} + x^{2/11} M^{-4/11} + x^{(3+20)/16} M^{-1/2} + x^{(1+20)/14} M^{-1/7} + x^{(1+40)/10} M^{-3/10} + x^{\theta} M^{-1/2}),
\]
which in turn implies
\[
S_2(x^{41/231}, x^{1/5}) \leq x^{19/154} \log x.
\]

So, \( R_2 = S_2(x^{0.01}, x^{0.138}) + S_2(x^{0.138}, x^{0.158}) + S_2(x^{0.158}, x^{41/231}) + S_2(x^{41/231}, x^{1/5}) \)
\[
\leq \epsilon x^{19/154}. \quad \text{Combining Lemma 13 and the estimates for } R_1 \text{ and } R_2 \text{ we get}
\]
\[
Q\left(x + x^{1/2+\theta}\right) - Q(x) = \frac{\zeta(3)}{2\zeta(3)} x^{\theta} + O\left(x^{0.01}\right) + O_{\epsilon}\left(x^{19/154+\epsilon}\right).
\]

Setting \( \epsilon = \frac{1}{2}(\theta - 0.10) \) we complete the proof of the theorem.

Acknowledgments. The author was supported by grant DMS-9970455 from the National Science Foundation. The author is grateful to S.Konyagin. The idea to use local three-convexity condition and Lemma 8 belong to him. Also, special thanks are due to M.Filaseta for the numerous insightful suggestions and comments.

References


Mathematics Department
University of South Carolina
Columbia, SC 29208
trifonov@math.sc.edu