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M. Nielsen and D.-X. Zhou

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Department of Mathematics
University of South Carolina

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Morten Nielsen*
Department of Mathematics
University of South Carolina
Columbia, South Carolina 29208
USA
E-mail: nielsen@math.sc.edu

Ding-Xuan Zhou†
Department of Mathematics
City University of Hong Kong
Tat Chee Avenue, Kowloon
HONG KONG
E-mail: mazhou@math.cityu.edu.hk

Abstract

We study the mean size of wavelet packets in L_p. An exact formula for the mean size is given in terms of the p-norm joint spectral radius. This will be a corollary of an asymptotic formula for the L_p-norms on the subdivision trees. Then the stability and Schauder basis property of wavelet packets in L_p, and the instability of biorthogonal wavelet packets in L_2 are analyzed.

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Mean Size of Wavelet Packets

§1. Introduction

Wavelet analysis is a powerful tool for time-frequency localization. The central equation in wavelet analysis is the **refinement equation**

$$\phi(x) = \sum_{k\in\mathbb{Z}} a(k)\phi(2x-k). \tag{1.1}$$

Here $a := \{a(k)\}$ is a finitely supported sequence called **refinement mask**. A solution of (1.1) is called a **refinable function**. It is called a scaling function if its shifts $\{\phi(x-k)\}$ are orthonormal.

A necessary condition for (1.1) to have a scaling function solution is that $\sum a(k) = 2$ and

$$\sum_{k\in\mathbb{Z}} a(k)\overline{a(k+2j)} = 2\delta_{j,0}. \tag{1.2}$$

Conversely, if the mask a satisfies the above conditions, then under some easily checked conditions (see [3, 16, 15]), the refinement equation (1.1) has a solution ϕ with orthonormal shifts and $\hat{\phi}(0) = 1$. Define

$$\psi(x) = \sum_{k\in\mathbb{Z}} b(k)\phi(2x-k),$$

where $b(k) = (-1)^k\overline{a(1-k)}$. Then $\{\psi(2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis of $L_2(\mathbb{R})$, called an orthonormal wavelet basis. This basis has good localization in the time-frequency domain and has many applications in different fields.

In order to have better localization for high frequency components in the wavelet decomposition, Coifman, Meyer, and Wickerhauser introduced another kinds of bases called **wavelet packets**. Let $w_0 = \phi, w_1 = \psi$ and

$$\begin{aligned} w_{2n}(x) &= \sum a(k)w_n(2x-k), \\ w_{2n+1}(x) &= \sum b(k)w_n(2x-k). \end{aligned} \tag{1.3}$$

The family of functions $\{w_n : n \in \mathbb{Z}_+\}$ is called the wavelet packets. Their shifts $\{w_n(x-k) : n \in \mathbb{Z}_+, k \in \mathbb{Z}\}$ form an orthonormal basis of $L_2(\mathbb{R})$ called the orthonormal wavelet packet basis. This in connection with the orthonormal wavelets provides a best basis algorithm in terms of the entropy estimates, see Coifman and Wickerhauser [7].

Since the wavelet packets form orthonormal bases of L_2, it is natural to ask whether they also form bases for L_p spaces. In fact, Paley [18] showed a long time ago that for the Haar wavelet, the corresponding wavelet packets (Walsh system) constitute a Schauder basis for $L_p(\mathbb{R})(1 < p < \infty)$. However, Coifman, Meyer and Wickerhauser [6] proved that the wavelet packets associated with the Meyer wavelet are not uniformly bounded in L_p when p is large, hence their frequency localization is not very satisfactory. This result was improved by Fan in [10]. Recently, Nielsen [17] considered some of the well-known compactly supported orthogonal wavelets (Daubechies' orthogonal wavelets, least asymmetric wavelets, Coiflets) and showed that the corresponding wavelet packets do not form Schauder bases for L_p when p is sufficiently large. These negative results are all based on estimates of L_p-norms of the wavelet packets.

The orthogonal wavelet packets have been extended to a biorthogonal setting. One inconvenience of orthogonal wavelets is the lack of symmetry except the Haar wavelet. In order to have symmetry, Cohen, Daubechies and Feauveau [5] constructed biorthogonal wavelets. In this case, we have two refinable functions ϕ and $\tilde{\phi}$ associated with refinement masks a and $\tilde{a}$, respectively.

If ϕ and $\tilde{\phi}$ have biorthogonal shifts, then

$$\sum_{k\in\mathbb{Z}} a(k)\overline{\tilde{a}(k+2j)} = 2\delta_{j,0}.$$

The biorthogonal wavelets ψ and $\tilde{\psi}$ can be constructed by

$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{\tilde{a}(1-k)}\phi(2x-k),$$

and

$$\tilde{\psi}(x) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{a(1-k)}\tilde{\phi}(2x-k).$$

Their shifts constitute dual Riesz bases of $L_2(\mathbb{R})$.

The biorthogonal wavelet packets were considered by Chui and Li [2]. However, it was shown by Cohen and Daubechies [4] that the biorthogonal wavelet packets are globally instable. The essential step in their approach is to estimate the L_2-norms of the biorthogonal wavelet packets, see [4, Theorem 6.4].

The purpose of this paper is to study the mean size of the wavelet packets in L_p. This will be stated in Section 2 as a corollary of an asymptotic formula for the L_p-norms on subdivision trees. Then in Section 3 we apply the estimates for the mean size of wavelet packets to analyze the stability and Schauder basis property of wavelet packets in L_p. Section 4 is devoted to a quantitative study of the instability of biorthogonal wavelet packets in L_2.

Our general result will be stated for multivariate vector refinement equations. That is, in (1.1) ϕ is a vector of functions $(\phi_1, \cdots, \phi_r)^T$ on $\mathbb{R}^s$, the mask $a = (a(\alpha))_{\alpha \in \mathbb{Z}^s}$ is a sequence on $\mathbb{Z}^s$ and each $a(\alpha)$ is an $r \times r$ matrix. Thus, our analysis applied to wavelet packets generated by multiple wavelets.

§2. Subdivision Trees and Their Norm Estimates

In this section we study the mean size of L_p-norm of a general infinite tree called subdivision tree.

Let $1 \le p \le \infty, s, r, N$ be positive integers, and $\psi = (\psi^1, \cdots, \psi^r)^T$ be a vector of functions in $(L_p(\mathbb{R}^s))^r$ supported in $[0, N]^s$. Let $\mathcal{M} := \{a_\varepsilon : \varepsilon \in \mathcal{E}\}$ be a finite set of sequences of $r \times r$ matrices supported on $\{0, \cdots, N\}^s$. The **subdivision tree** $T(\mathcal{M}, \psi)$ associated with $\mathcal{M}$ and ψ is an infinite tree of vectors of functions $\{\psi_{\varepsilon_1, \cdots, \varepsilon_n} : n \in \mathbb{N} \cup \{0\}, \varepsilon_1, \cdots, \varepsilon_n \in \mathcal{E}\}$ and are defined by

$$\psi_{\varepsilon_1, \cdots, \varepsilon_n}(x) = \sum_{\alpha \in \mathbb{Z}^s} a_{\varepsilon_1}(\alpha) \psi_{\varepsilon_2, \cdots, \varepsilon_n}(2x - \alpha), \quad x \in \mathbb{R}^s. \tag{2.1}$$

Here $\psi_{\varepsilon_1, \cdots, \varepsilon_n} = \psi$ when $n = 0$.

We want to study the mean size of the L_p-norms of a subdivision tree. Our purpose is to investigate the limit

$$M_p(\mathcal{M}, \psi) := \lim_{n \to \infty} \left\{ \sum_{\varepsilon_1, \cdots, \varepsilon_n \in \mathcal{E}} \|\psi_{\varepsilon_1, \cdots, \varepsilon_n}\|_p^p \right\}^{1/np}. \tag{2.2}$$

Here for $f = (f^1, \cdots, f^r)^T \in (L_p(\mathbb{R}^s))^r$, we denote $\|f\|_p := \left(\sum_{j=1}^r \|f^j\|_p^p\right)^{1/p}$. If this limit exists, then asymptotically, $\|\psi_{\varepsilon_1, \cdots, \varepsilon_n}\|_p$ is almost $M_p(\mathcal{M}, \psi)(\#\mathcal{E})^{-1/p}$.

We shall show that the limit (2.2) always exists and equals the p-norm joint spectral radius of a finite set of matrices.

Let us review the concept of p-norm joint spectral radius. Let $\mathcal{A}$ be a finite collection of linear operators on a vector space V, which is not necessarily finite dimensional. A subspace of V is said to be $\mathcal{A}$-invariant if it is invariant under every operator in $\mathcal{A}$. For $v \in V$, we call the intersection of all $\mathcal{A}$-invariant subspaces of V containing v the **minimal $\mathcal{A}$-invariant subspace** generated by v, denoted as $V(v)$. If W is an $\mathcal{A}$-invariant subspace of V with $\dim W < \infty$ and $v \in W$, then $V(v)$ is spanned by

$$\{A_1 \cdots A_j v : A_1, \cdots, A_j \in \mathcal{A}, j = 0, 1, \cdots, \dim W - 1\}.$$

Suppose that $V(v)$ is finite dimensional. We choose an arbitrary norm $\|\cdot\|$ on $V(v)$. For a linear operator A on $V(v)$,

$$\|A\| = \max\{\|Au\| : u \in V(v), \|u\| = 1\}.$$

Define

$$\|\mathcal{A}^n|_{V(v)}\|_p := \begin{cases} \left(\sum_{A_1,\cdots,A_n \in \mathcal{A}} \|A_1 \cdots A_n|_{V(v)}\|^p\right)^{1/p}, & \text{if } 0 < p < \infty, \\ \max_{A_1,\cdots,A_n \in \mathcal{A}} \|A_1 \cdots A_n|_{V(v)}\|, & \text{if } p = \infty. \end{cases}$$

Then the p-norm joint spectral radius of $\mathcal{A}|_{V(v)}$ is defined to be

$$\rho_p(\mathcal{A}|_{V(v)}) = \lim_{n\to\infty} \|\mathcal{A}^n|_{V(v)}\|_p^{1/n}.$$

It is easily seen that $\rho_p(\mathcal{A}|_{V(v)})$ is independent of the choice of the vector norm $\|\cdot\|$ on $V(v)$, and

$$\lim_{n\to\infty} \|\mathcal{A}^n|_{V(v)}\|_p^{1/n} = \inf_{n\in\mathbb{N}} \|\mathcal{A}^n|_{V(v)}\|_p^{1/n}.$$

The uniform joint spectral radius ($p = \infty$) was introduced by Rota and Strang [19] and applied to the investigation of wavelets by Daubechies and Lagarias [9]. The mean joint spectral radius ($p = 1$) was studied by Wang [24]. The p-norm joint spectral radius was introduced by Jia [13] for $1 \le p \le \infty$, while for $0 < p < 1$ it appeared in [26].

The p-norm joint spectral radius is hard to compute if one uses the definition, since the limit in the definition is reached very slowly. An efficient formula provided by Zhou

[26] is to compute p-norm joint spectral radius in terms of the spectral radius of some finite matrix when p is an even integer. With this formula we can estimate the p-norm joint spectral radius for other p by the relation among the p-norm joint spectral radii presented by Strang and Zhou in [23].

Denote

$$\|\mathcal{A}^n v\|_p := \begin{cases} \left(\sum_{A_1,\cdots,A_n\in\mathcal{A}} \|A_1\cdots A_n v\|^p\right)^{1/p}, & \text{if } 0<p<\infty, \\ \max_{A_1,\cdots,A_n\in\mathcal{A}} \|A_1\cdots A_n v\|, & \text{if } p=\infty. \end{cases}$$

Then

$$\rho_p(\mathcal{A}|_{V(v)}) = \lim_{n\to\infty} \|\mathcal{A}^n v\|_p^{1/n}.$$

This relation was proved for $1 \le p \le \infty$ by Han and Jia [12]. Moreover, there exists a positive constant C such that

$$\|\mathcal{A}^n|_{V(v)}\|_p / C \le \|\mathcal{A}^n v\|_p \le C\|\mathcal{A}^n|_{V(v)}\|_p, \qquad n\in\mathbb{N}. \tag{2.3}$$

In our application of joint spectral radius, we consider the space $V = (\ell_0(\mathbb{Z}^s))^{r\times 1}$, the space of all finitely supported sequences of $r\times 1$ vectors. The collection $\mathcal{A}$ is a set of linear operators $\{A_\varepsilon^{(\eta)} : \varepsilon\in\mathcal{E}, \eta\in E\}$, where E is the set $\{0,1\}^s$. For $\varepsilon\in\mathcal{E}, \eta\in E$, the linear operator $A_\varepsilon^{(\eta)}$ is defined on $(\ell_0(\mathbb{Z}^s))^{r\times 1}$ as

$$A_\varepsilon^{(\eta)} v(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_\varepsilon(\eta+2\alpha-\beta)v(\beta), \qquad \alpha\in\mathbb{Z}^s, \quad v\in(\ell_0(\mathbb{Z}^s))^{r\times 1}. \tag{2.4}$$

Under these circumstances, for any $v\in(\ell_0(\mathbb{Z}^s))^{r\times 1}$, the minimal $\mathcal{A}$-invariant subspace $V(v)$ is always finite dimensional.

The p-norm joint spectral radius of these linear operators will be used to estimate the norms concerning the subdivision sequences appearing in (2.1).

For the set $\mathcal{M}$, we define the **subdivision sequence** $\{(a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha))_{\alpha\in\mathbb{Z}^s} : n\in\mathbb{N}, \varepsilon_1,\cdots,\varepsilon_n\in\mathcal{E}\}$ as

$$a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_{\varepsilon_1}(\beta)a_{\varepsilon_2,\cdots,\varepsilon_n}(\alpha-2^{n-1}\beta), \qquad ,\alpha\in\mathbb{Z}^s. \tag{2.5}$$

The subdivision sequence has an equivalent form as follows.

Lemma 2.1. *Let $\mathcal{M} = \{a_\varepsilon : \varepsilon \in \mathcal{E}\}$ be a finite set of finitely supported sequences of $r \times r$ matrices. Define the subdivision sequence by (2.5). Then for $n \in \mathbb{N}, \varepsilon_1, \cdots, \varepsilon_n \in \mathcal{E}$, we have*

$$a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_{\varepsilon_1,\cdots,\varepsilon_{n-1}}(\beta) a_{\varepsilon_n}(\alpha - 2\beta), \qquad \alpha \in \mathbb{Z}^s. \tag{2.6}$$

Proof. The case $n = 2$ is trivial by the definition (2.5).

Suppose (2.6) holds for n. Then by the definition (2.5), for $\alpha \in \mathbb{Z}^s$ we have

$$a_{\varepsilon_1,\cdots,\varepsilon_n,\varepsilon_{n+1}}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_{\varepsilon_1}(\beta) a_{\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{n+1}}(\alpha - 2^n\beta).$$

The induction hypothesis tells us that

$$a_{\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{n+1}}(\alpha - 2^n\beta) = \sum_{\gamma \in \mathbb{Z}^s} a_{\varepsilon_2,\cdots,\varepsilon_n}(\gamma) a_{\varepsilon_{n+1}}(\alpha - 2^n\beta - 2\gamma).$$

Therefore, by the definition (2.5) again

$$a_{\varepsilon_1,\cdots,\varepsilon_n,\varepsilon_{n+1}}(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_{\varepsilon_1}(\beta) \sum_{\gamma \in \mathbb{Z}^s} a_{\varepsilon_2,\cdots,\varepsilon_n}(\gamma) a_{\varepsilon_{n+1}}(\alpha - 2^n\beta - 2\gamma)$$

$$= \sum_{\eta \in \mathbb{Z}^s} \{ \sum_{\beta \in \mathbb{Z}^s} a_{\varepsilon_1}(\beta) a_{\varepsilon_2,\cdots,\varepsilon_n}(\eta - 2^{n-1}\beta) \} a_{\varepsilon_{n+1}}(\alpha - 2\eta) = \sum_{\eta \in \mathbb{Z}^s} a_{\varepsilon_1,\cdots,\varepsilon_n}(\eta) a_{\varepsilon_{n+1}}(\alpha - 2\eta).$$

This completes the induction procedure and the proof of Lemma 2.1. $\square$

An induction procedure shows that the subdivision tree defined by (2.1) can be written as combinations of scaled shifts of ψ with the subdivision sequence coefficients.

Lemma 2.2. *Let $T(\mathcal{M}, \psi)$ be a subdivision tree defined by (2.1), and the subdivision sequence be defined by (2.5). Then for $n \in \mathbb{N}, \varepsilon_1, \cdots, \varepsilon_n \in \mathcal{E}$, we have*

$$\psi_{\varepsilon_1,\cdots,\varepsilon_n}(x) = \sum_{\alpha \in \mathbb{Z}^s} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha) \psi(2^n x - \alpha).$$

By Lemma 2.2, the norm $\|\psi_{\varepsilon_1,\cdots,\varepsilon_n}\|_p = \|\sum a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha)\psi(2^n x - \alpha)\|_p$ can be easily expressed up to a uniform constant if the shifts $\{\psi^j(x - \alpha) : 1 \le j \le r, \alpha \in \mathbb{Z}^s\}$ are stable, that is, there exists a positive constant C such that

$$\|c\|_p / C \le \| \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^s} c_j(\alpha) \psi^j(x - \alpha) \|_p \le C \|c\|_p. \tag{2.7}$$

Here for $c = (c_1, \cdots, c_r)^T \in (\ell_p(\mathbb{Z}^s))^r$, the linear space of vectors of $\ell_p(\mathbb{Z}^s)$ sequences, the norm $\|c\|_p$ is defined by

$$\|c\|_p := \big(\sum_{j=1}^{r} \|c_j\|_p^p\big)^{1/p}.$$

In general, without assuming stability, there always exist $d \in \mathbb{N}$ and a vector of compactly supported functions $g = (g^1, \cdots, g^d)^T \in (L_p(\mathbb{R}^s))^d$ such that the shifts of g are stable and

$$\psi(x) = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) g(x - \alpha), \tag{2.8}$$

where $b := \big(b(\alpha)\big)_{\alpha \in \mathbb{Z}^s}$ is in $(\ell_0(\mathbb{Z}^s))^{r \times d}$. Such a vector g is called a **generator** of the shift-invariant space

$$S(\psi) := \big\{\sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^s} c_j(\alpha) \psi^j(x - \alpha) : c_j(\alpha) \in \mathbb{C}\big\}. \tag{2.9}$$

In fact, we may even choose generators g with linear independent shifts by taking a basis of span$\{\psi^j|_{\alpha+[0,1)^s} : 1 \le j \le r, \alpha \in \mathbb{Z}^s\}$. We denote $G(\psi)$ as the set of all generators of $S(\psi)$.

A generator g of $S(\psi)$ is called **perfect** if $S(\psi) = S(g)$. In the univariate case $s = 1$, perfect generators with $d \le r$ always exist. In the multivariate case $s > 1$, perfect generator may not exist. See [28] for detailed discussions.

Suppose that $g \in G(\psi)$ and (2.8) holds. Then

$$\|\psi_{\varepsilon_1, \cdots, \varepsilon_n}\|_p = \|\sum a_{\varepsilon_1, \cdots, \varepsilon_n}(\alpha) \psi(2^n x - \alpha)\|_p = \|\sum a_{\varepsilon_1, \cdots, \varepsilon_n} * b(\alpha) g(2^n x - \alpha)\|_p,$$

where $a_{\varepsilon_1, \cdots, \varepsilon_n} * b \in (\ell_0(\mathbb{Z}^s))^{r \times d}$ is the convolution given by

$$a_{\varepsilon_1, \cdots, \varepsilon_n} * b(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_{\varepsilon_1, \cdots, \varepsilon_n}(\beta) b(\alpha - \beta), \qquad \alpha \in \mathbb{Z}^s.$$

The norm on $(\ell_0(\mathbb{Z}^s))^{r \times d}$ is defined by

$$\|c\|_p := \big(\sum_{j=1}^{r} \sum_{k=1}^{d} \|c_{jk}\|_p^p\big)^{1/p}, \qquad c(\alpha) = (c_{jk}(\alpha))_{1 \le j \le r, 1 \le k \le d}, \quad \alpha \in \mathbb{Z}^s.$$

It follows from the stability assumption of g that for all $n \in \mathbb{N}, \varepsilon_1, \cdots, \varepsilon_n$,

$$\|a_{\varepsilon_1,\cdots,\varepsilon_n} * b\|_p / C \le \| \sum_{\alpha \in \mathbb{Z}^s} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha)\psi(x-\alpha)\|_p \le C \|a_{\varepsilon_1,\cdots,\varepsilon_n} * b\|_p. \tag{2.10}$$

Thus, we only need to understand the norm for the sequence $a_n * b$. To this end, we need the following result on the relation between the norms concerning subdivision sequences and the linear operators defined by (2.4). When the set $\mathcal{M}$ contains only one sequence, this result was proved by Goodman, Micchelli and Ward [11] for $r=1, M=2I$ (see also [1]), by Han and Jia [12] for $r=1$ and the general dilation matrix M, by Jia, Riemenschneider and Zhou [14] for $r>1, s=1$, and by Zhou [28] for $r>1$ and $s>1$.

Lemma 2.3. *Let $\varepsilon_1, \cdots, \varepsilon_n \in \mathcal{E}$ and $\alpha = \eta_n + 2\varepsilon_{n-1} + \cdots + 2^{n-1}\varepsilon_1 + 2^n\gamma$ with $\eta_1, \cdots, \eta_n, \gamma \in \mathbb{Z}^s$. Then for $v \in (\ell_0(\mathbb{Z}^s))^{r\times 1}$,*

$$a_{\varepsilon_1,\cdots,\varepsilon_n} * v(\alpha) = A_{\varepsilon_1}^{(\eta_1)} \cdots A_{\varepsilon_n}^{(\eta_n)} v(\gamma).$$

Proof. The case $n=1$ is trivial by the definition (2.4).

Suppose the statement is true for n. Let $\alpha = \eta_{n+1} + 2\alpha_1$. Then by (2.6)

$$\begin{aligned} a_{\varepsilon_1,\cdots,\varepsilon_{n+1}} * v(\alpha) &= \sum_{\beta \in \mathbb{Z}^s} \sum_{\gamma \in \mathbb{Z}^s} a_{\varepsilon_1,\cdots,\varepsilon_n}(\gamma) a_{\varepsilon_{n+1}}(\eta_{n+1} + 2\alpha_1 - \beta - 2\gamma) v(\beta) \\ &= \sum_{\gamma \in \mathbb{Z}^s} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha_1 - \gamma) \sum_{\beta \in \mathbb{Z}^s} a_{\varepsilon_{n+1}}(\eta_{n+1} + 2\gamma - \beta) v(\beta) = a_{\varepsilon_1,\cdots,\varepsilon_n} * \big(A_{\varepsilon_{n+1}}^{(\eta_{n+1})} v\big)(\alpha_1). \end{aligned}$$

Set $\alpha_1 = \eta_n + \cdots + 2^{n-1}\eta_1 + 2^n\gamma$. The induction hypothesis tells us that

$$a_{\varepsilon_1,\cdots,\varepsilon_n} * (A_{\varepsilon_{n+1}}^{(\eta_{n+1})} v)(\alpha_1) = A_{\varepsilon_1}^{(\eta_1)} \cdots A_{\varepsilon_n}^{(\eta_n)} A_{\varepsilon_{n+1}}^{(\eta_{n+1})} v(\gamma).$$

Therefore,

$$a_{\varepsilon_1,\cdots,\varepsilon_{n+1}} * v(\alpha) = A_{\varepsilon_1}^{(\eta_1)} \cdots A_{\varepsilon_n}^{(\eta_n)} A_{\varepsilon_{n+1}}^{(\eta_{n+1})} v(\gamma).$$

thereby completing the induction procedure. □

Lemma 2.3 tells us that

$$\|a_{\varepsilon_1,\cdots,\varepsilon_n} * v\|_p^p = \sum_{\eta_1,\cdots,\eta_n \in \{0,1\}^s} \|A_{\varepsilon_1}^{(\eta_1)} \cdots A_{\varepsilon_n}^{(\eta_n)} v\|_p^p.$$

This in connection with (2.10) and Lemma 2.2 implies that for $\varepsilon_1, \cdots, \varepsilon_n \in \mathcal{E}$,

$$C^{-1}2^{-n/p}\{\sum_{j=1}^{d}\sum_{\eta_1,\cdots,\eta_n\in\{0,1\}^s}\|A_{\varepsilon_1}^{(\eta_1)}\cdots A_{\varepsilon_n}^{(\eta_n)}(be_j)\|_p^p\}^{1/p} \le \|\psi_{\varepsilon_1,\cdots,\varepsilon_n}\|_p$$
$$\le C2^{-n/p}\{\sum_{j=1}^{d}\sum_{\eta_1,\cdots,\eta_n\in\{0,1\}^s}\|A_{\varepsilon_1}^{(\eta_1)}\cdots A_{\varepsilon_n}^{(\eta_n)}(be_j)\|_p^p\}^{1/p}.$$

Here e_j denotes the jth column of the $d \times d$ identity matrix, and be_j is the sequence in $(\ell_0(\mathbb{Z}^s))^{r\times 1}$ given by $be_j(\alpha) = b(\alpha)e_j$ for $\alpha \in \mathbb{Z}^s$.

Now we are in a position to state our main result.

Theorem 1. *Let $\mathcal{M} = \{a_\varepsilon : \varepsilon \in \mathcal{E}\}$ be a finite set of finitely supported sequences of $r \times r$ matrices. Define $\{\psi_{\varepsilon_1,\cdots,\varepsilon_n} : n \in \mathbb{N}, \varepsilon_1, \cdots, \varepsilon_n \in \mathcal{E}\}$ by (2.1). Suppose that $1 \le p \le \infty, g \in G(\psi)$ and (2.8) holds. If $\{A_\varepsilon^{(\eta)} : \varepsilon \in \mathcal{E}, \eta \in \{0,1\}^s\}$ is defined by (2.4), then $M_p(\mathcal{M},\psi)$ equals*

$$\lim_{n\to\infty}\left\{\sum_{\varepsilon_1,\cdots,\varepsilon_n\in\mathcal{E}}\|\psi_{\varepsilon_1,\cdots,\varepsilon_n}\|_p^p\right\}^{1/np} = \max_{1\le j\le d} 2^{-1/p}\rho_p\big(\{A_\varepsilon^{(\eta)}|_{V(be_j)} : \varepsilon\in\mathcal{E}, \eta\in\{0,1\}^s\}\big).$$

If the shifts of ψ are stable, *i.e.*, (2.7) holds, then the mean size of the subdivision tree can also be described in the following (somehow simpler) way.

Theorem 2. *Let $1 \le p \le \infty, N \in \mathbb{N}$ and $\mathcal{M} = \{a_\varepsilon : \varepsilon \in \mathcal{E}\}$ be a finite set of sequences of $r \times r$ matrices supported on $[0, N]$. Suppose ψ satisfy (2.7). Define $\{\psi_{\varepsilon_1,\cdots,\varepsilon_n} : n \in \mathbb{N}, \varepsilon_1, \cdots, \varepsilon_n \in \mathcal{E}\}$ by (2.1). Then*

$$M_p(\mathcal{M},\psi) = \lim_{n\to\infty}\left\{\sum_{\varepsilon_1,\cdots,\varepsilon_n\in\mathcal{E}}\|\psi_{\varepsilon_1,\cdots,\varepsilon_n}\|_p^p\right\}^{1/np}$$
$$= 2^{-1/p}\rho_p\left(\left\{\big(a_\varepsilon(\eta+2\alpha-\beta)\big)_{\alpha,\beta\in\{0,\cdots,N-1\}^s} : \varepsilon\in\mathcal{E}, \eta\in\{0,1\}^s\right\}\right).$$

§3. Stability of Wavelet Packets in L^p

Wavelet packets were introduced to improve the frequency localization of wavelets to be able to do a more refined analysis of signals containing both stationary and transient components. However, the price one often has to pay for better frequency resolution is less stability of the system when one looks in other function spaces than L^2. A well known example of this phenomenon is given by the trigonometric system $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$, where it is known that the dyadic partial sums associated with an expansion of an L^p-function, $1<p<\infty$, converges unconditionally, whereas the partial sums only converges conditionally; the system only forms a so-called Schauder basis for $L^p[0,1)$, $1<p<\infty$, but not an unconditional basis. The idea is exactly the same going form the wavelet expansion to a wavelet packet expansion since the wavelet expansion can be considered a dyadic type partial sum of a wavelet packet expansion. However, for certain wavelet packet systems the situation can be even worse than for the trigonometric system, it was demonstrated in [17] that there are basic wavelet packet systems that fail to be a Schauder basis for L^p when p is large. In this section we will extend the results and estimates in [17] to subdivision trees and give necessary conditions for certain subsystems of the tree to be a Schauder basis for L^p.

3.1. The growth of branches in subdivision trees

We will use the methods of the previous section to obtain more information on the asymptotic behavior in $L^p(\mathbb{R}^s)$ of specific subsequences (branches) of the subdivision tree. For example, if we consider the one dimensional construction of orthogonal/biorthogonal wavelet packets, one branch of the wavelet packet tree we can estimate using the methods presented below is the the branch consisting of the wavelet packets $\{w_{2^n-1}\}_{n=1}^{\infty}$. The wavelet packets with index set $\{2^n-1\}_{n=1}^{\infty}$ play a special role when it comes to frequency localization of wavelet packets. It was proved by Séré [20] that with respect to a certain reasonable measure of the frequency localization, the wavelet packet w_{2^n-1} is, in general, the one on scale n with the worst localization. This can be explained by the fact that the subsequence w_{2^n-1} is obtained by iterating only the high-pass filter in (1.3), and the high-pass filter does not generate a convergent subdivision scheme due to the fact that the symbol of the filter has value zero at the origin. The frequency measure Séré used was essentially the L^1 norm of the Fourier transform of the wavelet packets. However, such Fourier transforms turn out not to be nonnegative functions for the wavelet packets associated with finite quadrature filters, so we cannot get any information on the growth in the L^∞ or L^p norms from such estimates.

The p-norm joint spectral radius can be used to obtain very precise information about the asymptotic growth of certain subsequence of the subdivision tree. For simplicity we will assume in this section that the root of the subdivision tree ψ has stable integer shifts in the L^p-space we are looking at, i.e. that (2.7) is satisfied for this p.

Given a finite set $\mathcal{M} = \{a_\epsilon(\alpha),\, \epsilon \in \mathcal{E}\}$ of sequences of $r \times r$ matrices supported on $[0, N]^s$, we consider the following family of linear operators

$$\mathcal{C}^\epsilon = \left\{ \left(a_\epsilon(\eta - 2\alpha - \beta)\right)_{\alpha,\beta\in\{0,\ldots,N-1\}^s} : \eta \in \{0,1\}^s \right\}, \tag{3.1}$$

and we have the following Corollary to Theorem 2.

Corollary 3.1. *Suppose that the shifts of ψ are stable in $L^p(\mathbb{R}^s)$, $1 \le p \le \infty$, i.e. that (2.7) holds and let $\epsilon \in \mathcal{E}$. Then*

$$M_p^\epsilon(\mathcal{M}, \psi) := \lim_{\substack{n\to\infty \\ \epsilon_1=\cdots=\epsilon_n=\epsilon}} \|\psi_{\epsilon_1,\ldots,\epsilon_n}\|_p^{1/n} = 2^{-1/p}\rho_p\left(\mathcal{C}^\epsilon\right).$$

Proof. We put $a_\eta(\alpha) = 0$, $\forall\alpha$ and $\eta \in \mathcal{E}\backslash\epsilon$ and use the same proof as in Theorem 2. $\square$

Remark 3.1. For a given $\epsilon \in \mathcal{E}$ we use the following notation to denote the branch considered above:

$$T^\epsilon(\mathcal{M}, \psi) = \{\psi_{\epsilon_1,\ldots,\epsilon_n} \in T(\mathcal{M}, \psi) | \epsilon_1 = \cdots = \epsilon_n = \epsilon, n \ge 0\}$$

Let us consider one example in one dimension that can be consider the model on which every wavelet packet system is based.

Example 1. A special role is played by the Haar filter given by $a_0(0) = a_0(1) = 1$ since it generates the well known Walsh system, see [21]. The matrices given by (2.4) are $A_0 = A_1 = B_0 = [1]$, $B_1 = [-1]$ and it is clear that

$$\rho_p(A_0, A_1, B_0, B_1) = 4^{1/p}.$$

The Walsh wavelet packets are thus uniformly bounded functions, which is a well known property [21] of the Walsh system. We also have $\rho_p(B_0, B_1) = 2^{1/p}$, which agrees with the result from Corollary 1. $\square$

A more interesting example is the following.

Example 2. We consider the family of quadrature mirror filters with real valued coefficients of length 4. The family of low-pass filters can be parameterized by one parameter θ by considering the equations

$$\sum_{k=0}^{3} a_0(k) = 2 \quad \text{and} \quad \sum_{k=0}^{4} a_0(2j+k)a_0(k) = 2\delta_{0,j}, \quad j = 0, 1.$$

Solving this system of equations gives us

$$a_0(0) = \theta, \quad a_0(1) = \frac{1}{2} + \frac{1}{2}\sqrt{1+4\theta-4\theta^2}, \quad a_0(2) = 1 - a_0(0), \quad \text{and} \quad a_0(3) = 1 - a_0(1),$$

for $0 \le \theta \le 1/2 + 1/2\sqrt{2}$. We are interested in the behavior of the associated high-pass filter. We are not concerned with the interaction of the high-pass filter with the low-pass filter so we can shift the filter so the high-pass filter is given by $a_1(k) = (-1)^k a_0(k)$. One of the two matrices of $\mathcal{C}^1$, see (3.1), is

$$B_0 := \begin{pmatrix} a_1(0) & 0 & 0 \\ a_1(2) & -a_1(1) & a_1(0) \\ 0 & -a_1(3) & a_1(2) \end{pmatrix}$$

and one can check directly that its spectral radius is given by

$$\rho(B_0) = \max\left\{\theta, \frac{1}{2} + \frac{1}{2}\sqrt{1+4\theta-4\theta^2}\right\}.$$

Clearly, $M^1_\infty(\mathcal{M}, \psi) = \rho_\infty(\mathcal{C}^1) \ge \rho(B_0)$ so the estimate shows that the only values of θ that give rise to (potentially) uniformly bounded wavelet packets are $\theta \in \{0, 1\}$. One can check that in those cases the filter of length 4 degenerates to (a shift of) the Haar filter. Figure 1 shows a plot of the lower bounds as a function of θ.

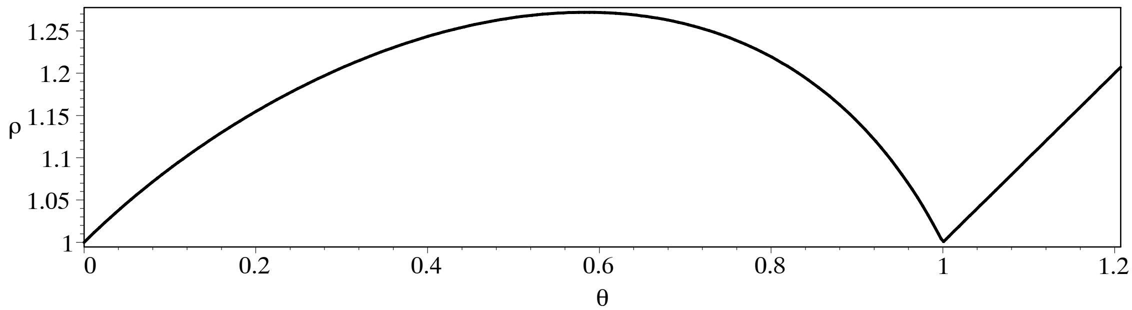

Figure 1. *Lower bound for the uniform joint spectral radius $M_\infty^1(\mathcal{M},\psi) = \rho_\infty(\mathcal{C}^1)$ as a function of θ.*

A "popular" filter of this type is the Daubechies low-filter of length 4 which is given by the choice $\theta = (1+\sqrt{3})/4$ [8], i.e.

$$a_0(0) = \frac{1+\sqrt{3}}{4},\ a_0(1) = \frac{3+\sqrt{3}}{4},\ a_0(2) = \frac{3-\sqrt{3}}{4}, \quad \text{and} \quad a_0(3) = \frac{1-\sqrt{3}}{4},$$

with $a_1(k) = (-1)^k a_0(k)$. By the estimates above we get

$$M_\infty^1(\mathcal{M},\psi^{\mathrm{Daub}}) \geq \frac{\sqrt{11}+\sqrt{3}}{4} \approx 1.26.$$

The associated wavelet packets are therefore not uniformly bounded and they have a growth given by (asymptotically) $\|w_{2^n-1}\|_\infty \geq C_r r^n$, for any $r < \frac{\sqrt{11}+\sqrt{3}}{4}$. More specific examples can be found in [17]. $\square$

3.2. Schauder basis properties of wavelet packets

Let us recall that a Schauder basis for a separable Banach space $\mathbb{B}$ is a collection $\{e_k\}_{k=1}^\infty \subset \mathbb{B}$ with the property that every element $f \in \mathbb{B}$ has a unique norm convergent expansion of the form $f = \sum_{k=1}^\infty \alpha_k e_k$, see e.g. [25]. The ordering of the Schauder basis elements is crucial for convergence, unless the basis also happens to be unconditional.

Using the estimates in the previous section we will give necessary conditions for certain collection of functions extracted from a subdivision tree to be a Schauder basis for $L^p(\mathbb{R}^s)$, and give examples of wavelet packet systems that fail to satisfy the conditions.

Given $\epsilon \in \mathcal{E}$, we let $\mathcal{D} \subset L^2(\mathbb{R}^s) \cap L^\infty(\mathbb{R}^s)$ be a collection of function obtained by selecting one coordinate (function) from each vector in $T^\epsilon(\mathcal{M},\psi)$ and we will assume that there is a uniform constant $c_p > 0$ such that whenever f_i is the coordinate selected from $f \in T^\epsilon(\mathcal{M}, \psi)$, we have

$$\|f_i\|_{L^p(\mathbb{R}^s)} \geq c_p \|f\|_p. \tag{3.2}$$

Then we have the following result

Lemma 3.1. *Given $\epsilon \in \mathcal{E}$, and let $\mathcal{D}$ be obtained from $T^\epsilon(\mathcal{M}, \psi)$. Suppose $\mathcal{D}$ is a subset of an orthonormal basis $\mathbb{B}$ of $L^2(\mathbb{R}^s)$ which has dense span in $L^p(\mathbb{R}^s)$, $1 \leq p < \infty$, and that (3.2) holds. Then a necessary condition for $\mathbb{B}$ to be a Schauder basis of $L^p(\mathbb{R}^s)$, $1 \leq p < \infty$, is*

$$M_p^\epsilon(\mathcal{M}, \mathcal{E}) M_q^\epsilon(\mathcal{M}, \mathcal{E}) = 1,$$

where $p^{-1} + q^{-1} = 1$, or equivalently

$$\rho_p\big(\mathcal{C}^\epsilon\big)\rho_q\big(\mathcal{C}^\epsilon\big) = 2.$$

Proof. It is a well known result [25] that given a Schauder basis $\{e_k\}$ for $L^p(\mathbb{R}^s)$, $1 \leq p < \infty$, the set of coefficient functionals $\{f_k\}$ satisfy $\sup_k \|e_k\|_{L^p(\mathbb{R}^s)} \|f_k\|_{L^q(\mathbb{R}^s)} < \infty$. Since $\mathcal{D}$ is a subset of an orthonormal basis in $L^2(\mathbb{R}^s)$ which has dense span in $L^p(\mathbb{R}^s)$, we see that the coefficient functional of $w_n \in \mathcal{D}$ is just the function itself, $w_n \in L^q(\mathbb{R}^s)$. Thus, the condition becomes

$$\sup_{w \in \mathcal{D}} \|w\|_p \|w\|_q < \infty, \quad p \in [1, \infty) \quad \text{and} \quad p^{-1} + q^{-1} = 1.$$

Hence, using assumption (3.2),

$$M_p^\epsilon(\mathcal{M}, \mathcal{E}) M_q^\epsilon(\mathcal{M}, \mathcal{E}) = \lim_{\substack{n \to \infty \\ \epsilon_1 = \cdots = \epsilon_n = \epsilon}} \left(\|\psi_{\epsilon_1, \ldots, \epsilon_n}\|_p \|\psi_{\epsilon_1, \ldots, \epsilon_n}\|_q\right)^{1/n} \leq 1.$$

The bound $M_p^\epsilon(\mathcal{M}, \mathcal{E}) M_q^\epsilon(\mathcal{M}, \mathcal{E}) \geq 1$ follows easily from Hölder's inequality and the fact that the elements of $\mathcal{D}$ are normalized in $L^2(\mathbb{R}^s)$. $\square$

Remark 3.2. Lemma 3.1 shows that if $\rho_p\big(\mathcal{C}^\epsilon\big)\rho_q\big(\mathcal{C}^\epsilon\big) = \alpha > 2$ for some $p \in (2, \infty)$ then the associated wavelet packet system fails to be a Schauder basis for $L^p(\mathbb{R}^s)$ regardless of the ordering of the elements in the system, i.e. the failure to be a basis is not due to the fact that we have chosen the wrong ordering of the functions. Also, by the log-convexity of the p-norm joint spectral radius [23], we have for $1 \leq \tilde{p} \leq p$, $\tilde{p}^{-1} + \tilde{q}^{-1} = 1$,

$$\rho_{\tilde{p}}(\mathcal{C}^\epsilon)\rho_{\tilde{q}}(\mathcal{C}^\epsilon) \geq 2^{s(1/p-1/\tilde{p})}\rho_p(\mathcal{C}^\epsilon)\rho_q(\mathcal{C}^\epsilon).$$

Thus, $\rho_{\tilde{p}}(\mathcal{C}^\epsilon)\rho_{\tilde{q}}(\mathcal{C}^\epsilon) > 2$ for $\tilde{p} > [1/p + s^{-1}(\log_2 \alpha - 1)]^{-1}$.

Remark 3.3. Consider the orthonormal wavelet packets $\{w_n\}_{n=0}^\infty$ in one dimension constructed using the low-pass filter $\{h(k)\}_k$ and the high-pass filter $\{g(k)\}_k$. We will assume that the underlying multiresolution is such that the wavelet packets have dense span in $L^p(\mathbb{R})$. For $\epsilon = 1$, Lemma 3.1 shows that a necessary condition for $\{w_n\}_{n=0}^\infty$ to be a Schauder basis for $L^p(\mathbb{R})$ is that the high pass filter satisfies $\rho_p(\mathcal{C}^1)\rho_q(\mathcal{C}^1) = 2$.

Example 3. We consider the one dimensional wavelet packets generated by a quadrature mirror filter of length four. By Remark 3.3, if it turns out that $\rho_1(\mathcal{C}^1)\rho_\infty(\mathcal{C}^1) > 2$, then we know that the associated wavelet packets will fail to be a basis for $L^p(\mathbb{R})$ for p large. We already have a lower bound on $\rho_\infty(\mathcal{C}^1)$. To get a lover bound on $\rho_1(\mathcal{C}^1)$ we form the symbol $m_1(\xi) = \sum_k a_1(k)e^{ik\xi}$. Notice that $\{-2\pi/3, 2\pi/3\}$ is an invariant cycle under $\xi \to 2\xi \bmod [-\pi, \pi)$, and that $|m_1(-2\pi/3)| = |m_1(2\pi/3)|$ since the filter-coefficients are real valued. Hence,

$$\rho_1(\mathcal{C}^1) = \lim_{n\to\infty} \|m_1(\xi)m_1(2\xi)\cdots m_1(2^{n-1}\xi)\|_{\ell^1}^{1/n} \geq |m_1(2\pi/3)|,$$

where $\|f\|_{\ell^1}$ denotes the ℓ^1-norm of the Fourier coefficients of f. Figure 2 shows a plot of the ratio to two of the lower bound for $\rho_1(\mathcal{C}^1)\rho_\infty(\mathcal{C}^1)$ obtained this way as a function of the parameter θ introduced in Example 2. We notice that the estimate shows that for parameter $\theta \in (0.2, 0.9)$ the associated wavelet packets fail to be a basis for $L^p(\mathbb{R})$ for p large. This interval clearly includes the Daubechies filter. Outside the interval the test is inconclusive, but it is the belief of the authors that such wavelet packets will also fail to be a basis for at least some values of $p \neq 2$.

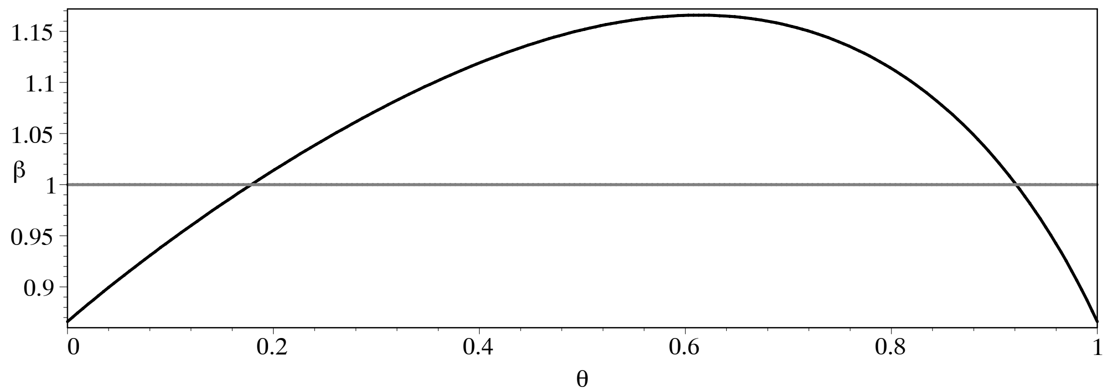

Figure 2. *Lower bound for the product $\beta = M_1(\mathcal{M},\psi)M_\infty(\mathcal{M},\psi)/2$ as a function of θ.* $\square$

Remark 3.4. The estimates giving the negative result on stability in L^p presented here only works for large values of p (or equivalently, for values of p near 1). So, for example, we can only conclude that the Daubechies wavelet packets associated with the filter of length four will fail to be a Schauder basis for L^p for p large. It is an open problem what happens with p near 2, we believe that such wavelet packets will fail to be a basis for L^p for $p \neq 2$. In fact, we conjecture the following:

Conjecture. *The basic wavelet packets associated with a Daubechies filter of length at least 4 will fail to be a Schauder basis for L^p when $p \neq 2$, and the functions will not be uniformly bounded in p-mean across scales for any $p > 2$.*

§4. Instability of Biorthogonal Wavelet Packets

In this section we apply our analysis on the mean size of wavelet packets to give a quantitative estimate for the instability of biorthogonal wavelet packets in L_2.

The biortogonal wavelet packets are defined by means of a pair of biorthogonal refinable functions ϕ and $\tilde{\phi}$ associated with masks a and $\tilde{a}$, respectively. Here the biorthogonality means

$$< \phi, \tilde{\phi}(\cdot - k) >= \delta_{0,k}, \qquad k \in \mathbb{Z}.$$

Set $\mathcal{E} = \{0,1\}$ and $\mathcal{M} = \{a_0 = a, a_1\}$ with a_1 given by $a_1(k) = (-1)^k\overline{\tilde{a}(1-k)}$; $\tilde{\mathcal{M}} = \{\tilde{a}_0, \tilde{a}_1\}$ with $\tilde{a}_0 = \tilde{a}, \tilde{a}_1$ given by $\tilde{a}_1(k) = (-1)^k\overline{a(1-k)}$. Then the biorthogonal wavelet packets are defined by

$$\psi_{\varepsilon_1,\cdots,\varepsilon_n}(x) = \sum_{k\in\mathbb{Z}} a_{\varepsilon_1,\cdots,\varepsilon_n}(k)\phi(2^n x - k), \qquad \varepsilon_1,\cdots,\varepsilon_n \in \{0,1\}, n \in \mathbb{Z}_+$$

and

$$\tilde{\psi}_{\varepsilon_1,\cdots,\varepsilon_n}(x) = \sum_{k\in\mathbb{Z}} \tilde{a}_{\varepsilon_1,\cdots,\varepsilon_n}(k)\tilde{\phi}(2^n x - k), \qquad \varepsilon_1,\cdots,\varepsilon_n \in \{0,1\}, n \in \mathbb{Z}_+$$

Cohen and Daubechies [4] proved that the biorthogonal wavelet packets are instable even in L_2. In particular, if $\phi \neq \tilde{\phi}$ up to an integer shift, then there exist $C > 0$ and $\lambda > 1$ such that

$$\sum_{\varepsilon_1,\cdots,\varepsilon_n\in\{0,1\}} \|\psi_{\varepsilon_1,\cdots,\varepsilon_n}\|_2^2 \geq C 2^n \lambda^n, \qquad n \in \mathbb{N}.$$

Hence the biorthogonal wavelet packets are even not uniformly bounded in L_2. By the analysis in Section, we can find this number λ measuring the instability.

Using the notion of subdivision tree, we know that the wavelet packet $\{\psi_{\varepsilon_1,\cdots,\varepsilon_n} : \varepsilon_1,\cdots,\varepsilon_n \in \{0,1\}, n \in \mathbb{Z}_+\}$ is the subdivision tree $T(\mathcal{M},\phi)$. Hence

$$\lim_{n\to\infty}\left\{\sum_{\varepsilon_1,\cdots,\varepsilon_n\in\{0,1\}} \|\psi_{\varepsilon_1,\cdots,\varepsilon_n}\|_2^2\right\}^{1/2n} = M_2(\mathcal{M},\phi).$$

Thus the following estimate follows from Theorem 2.

Corollary 4.1. *Let $\{\psi_{\varepsilon_1,\cdots,\varepsilon_n}\}$ be defined as above. Assume that a and a_1 are supported on $[0, N]$. Then*

$$\lim_{n\to\infty}\left\{\sum_{\varepsilon_1,\cdots,\varepsilon_n\in\{0,1\}} \|\psi_{\varepsilon_1,\cdots,\varepsilon_n}\|_2^2\right\}^{1/2n} = 2^{-1/2}\rho_2\left(\left\{\left[a_\varepsilon(\eta + 2j - k)\right]_{j,k=0}^{N-1} : \varepsilon,\eta \in \{0,1\}\right\}\right).$$

The 2-norm joint spectral radius of the above four matrices can be computed by means of the spectral radius of a linear operator on a finite dimensional space (a finite matrix). This linear operator is often called transfer operator or transition operator. We denote it as F_{a_0,a_1}. This is a linear operator on the space Π_N^* of all trigonometric polynomials of degree N (with basis $\{e^{ik\xi}\}_{k=-N}^{N}$) defined by

$$\left(F_{a_0,a_1}W\right)(\xi) = \sum_{\eta=0}^{1}\sum_{\varepsilon=0}^{1} |m_\varepsilon(\xi + \eta\pi)|^2 W(\xi + \eta\pi), \qquad W \in \Pi_N^*.$$

Here m_ε is the symbol of a_ε:

$$m_\varepsilon(\xi) = \sum_{k=0}^{N} a_\varepsilon(k) e^{-ik\xi}/2.$$

Then $|m_\varepsilon(\xi)|^2 \in \Pi_N^*$ is a trigonometric polynomial of degree N. It follows that the linear operator F_{a_0,a_1} on Π_N^* has the matrix representation under the basis $\{e^{ik\xi}\}_{k=-N}^{N}$:

$$F_{a_0,a_1} = \big(b(2j-k)\big)_{j,k=-N}^{N}$$

where b is a sequence supported on $[-N, N]$ given by

$$b(k) = \sum_{\varepsilon=0}^{1} \sum_{l=0}^{N} a_\varepsilon(k+l)\overline{a_\varepsilon(l)}/2.$$

By the same procedure as in [14, 11, 22], we can prove the following result concerning the mean size of wavelet packets by means of the spectral radius of a finite matrix.

Corollary 4.2. *Let $\{\psi_{\varepsilon_1,\cdots,\varepsilon_n}\}$ be defined as above. Assume that a and a_1 are supported on $[0, N]$. Then*

$$\lim_{n\to\infty} \left\{ \sum_{\varepsilon_1,\cdots,\varepsilon_n \in \{0,1\}} \|\psi_{\varepsilon_1,\cdots,\varepsilon_n}\|_2^2 \right\}^{1/2n} = \sqrt{\rho(F_{a_0,a_1})}.$$

By the analysis in [4], we know that $\rho(F_{a_0,a_1}) > 2$.

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