A criterion for convergence of weak greedy algorithms

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1. INTRODUCTION

This paper completes the investigation of necessary and sufficient conditions on the "weakness" sequence \( \tau := \{t_k\}_{k=1}^{\infty} \) for convergence of Weak Greedy Algorithm for all dictionaries \( \mathcal{D} \) and each function (vector) \( f \) in Hilbert space \( H \). This paper is a follow up to the papers [T] and [LT]. The Weak Greedy Algorithms (WGA) were introduced in [T]. The paper [T] contains also historical remarks and some motivation of studying greedy and weak greedy algorithms. We will not repeat historical remarks from [T] here and refer the reader to [T] for prehistory of WGA. We discuss here results on WGA in detail.

We remind first some notations and definitions from the theory of greedy algorithms. Let \( H \) be a real Hilbert space with an inner product \( \langle \cdot, \cdot \rangle \) and the norm \( \|x\| := \langle x, x \rangle^{1/2} \). We say a set \( \mathcal{D} \) of functions (elements) from \( H \) is a dictionary if each \( g \in \mathcal{D} \) has norm one (\( \|g\| = 1 \)) and \( \overline{\text{span}} \mathcal{D} = H \). We give now the definition of WGA (see [T]). Let a weakness sequence \( \tau = \{t_k\}_{k=1}^{\infty}, 0 \leq t_k \leq 1, \) be given.

**Weak Greedy Algorithm.** We define \( f^\tau_0 := f \). Then for each \( m \geq 1 \), we inductively define:

1). \( \varphi^\tau_m \in \mathcal{D} \) is any satisfying

\[
|\langle f^\tau_{m-1}, \varphi^\tau_m \rangle| \geq t_m \sup_{g \in \mathcal{D}} |\langle f^\tau_{m-1}, g \rangle|;
\]

2). \( f^\tau_m := f^\tau_{m-1} - \langle f^\tau_{m-1}, \varphi^\tau_m \rangle \varphi^\tau_m; \)

3). \( G^\tau_m(f, \mathcal{D}) := \sum_{j=1}^{m} \langle f^\tau_{j-1}, \varphi^\tau_j \rangle \varphi^\tau_j. \)

In the case \( t_k = 1, k = 1, 2, \ldots, \) we call WGA by Pure Greedy Algorithm (PGA). The convergence of PGA and WGA with \( t_k = t, 0 < t < 1, \) was established in [J] and [RW]. The first sufficient condition on \( \tau \) which includes sequences with \( \lim_{k \to \infty} t_k = 0 \) was obtained in [T].

**Theorem A.** Assume

\[
(1.1) \quad \sum_{k=1}^{\infty} \frac{t_k}{k} = \infty.
\]

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Then for any dictionary \( \mathcal{D} \) and any \( f \in H \) we have

\[
\lim_{m \to \infty} \|f - G^\tau_m(f, \mathcal{D})\| = 0.
\]

In [T] we reduced the proof of convergence of WGA with weakness sequence \( \tau \)

to some properties of \( l_2 \)-sequences with regard to \( \tau \). Theorem A was derived from

the following two statements proved in [T].

**Proposition 1.1.** Let \( \tau \) be such that for any \( \{a_j\}_{j=1}^\infty \in l_2, \ a_j \geq 0, \ j = 1, 2, \ldots \) we have

\[
\lim \inf_{n \to \infty} a_n \sum_{j=1}^n a_j/t_n = 0.
\]

Then for any \( H, \mathcal{D} \), and \( f \in H \) we have

\[
\lim_{m \to \infty} \|f^\tau_m\| = 0.
\]

**Proposition 1.2 (Lemma 2.3,[T]).** If \( \tau \) satisfies the condition (1.1) then \( \tau \) satis-

fies the assumption of Proposition 1.1.

The following simple necessary condition

\[
\sum_{k=1}^\infty t_k^2 = \infty
\]

was mentioned in [T]. The first nontrivial necessary conditions were obtained in
[LT]. We proved in [LT] the following theorem.

**Theorem B.** In the class of monotone sequences \( \tau = \{t_k\}_{k=1}^\infty, \ 1 \geq t_1 \geq t_2 \geq \cdots \geq 0 \)

the condition (1.1) is necessary and sufficient for convergence of Weak Greedy
Algorithm for each \( f \) and all Hilbert spaces \( H \) and dictionaries \( \mathcal{D} \).

The proof of this theorem is based on a special procedure which we called Equalizer.
The generalization of that procedure plays an important role in this paper
also (see S.3). In [LT] we gave an example of a class of sequences \( \tau \) for which the
condition (1.1) is not a necessary condition for convergence. We also proved in [LT]
a theorem which covers Theorem A.

**Theorem C.** Assume

\[
\sum_{s=0}^\infty \left(2^{-s} \sum_{k=2^s}^{2^{s+1}-1} t_k^2\right)^{1/2} = \infty.
\]

Then for any dictionary \( \mathcal{D} \) and any \( f \in H \) we have

\[
\lim_{m \to \infty} \|f - G^\tau_m(f, \mathcal{D})\| = 0.
\]

We prove in this paper a criterion on \( \tau \) for convergence of WGA. Let us introduce
some notation.

We define by \( \mathcal{V} \) the class of sequences \( x = \{x_k\}_{k=1}^\infty, \ x_k \geq 0, \ k = 1, 2, \ldots \) with

the following property: there exists a sequence \( 0 = q_0 < q_1 < \ldots \) such that

\[
\sum_{s=1}^\infty \frac{2^s}{\Delta q_s} < \infty;
\]
and

\[
(1.3) \quad \sum_{s=1}^{\infty} 2^{-s} \sum_{k=1}^{q_s} x_k^2 < \infty,
\]

where \( \Delta q_s := q_s - q_{s-1} \).

**Remark 1.1.** It is clear from this definition that if \( x \in \mathcal{V} \) and for some \( N \) and \( c \) we have \( 0 \leq y_k \leq cx_k, \, k \geq N \), then \( y \in \mathcal{V} \).

**Theorem 1.1.** The condition \( \tau \notin \mathcal{V} \) is necessary and sufficient for convergence of the Weak Greedy Algorithm with weakly sequence \( \tau \) for each \( f \) and all Hilbert spaces \( H \) and dictionaries \( D \).

Sufficient part is proved in Section 2 and necessary part is proved in Section 3.

2. **Proof of Convergence**

We begin this section with the following lemma.

**Lemma 2.1.** Let \( \{a_j\}_{j=1}^{\infty} \in l_2, \, a_j \geq 0, \, j = 1, 2, \ldots \). Then \( \{a_n \sum_{j=1}^{n} a_j\}_{n=1}^{\infty} \in \mathcal{V} \).

**Proof.** Assume \( \{a_j\}_{j=1}^{\infty} \) contains infinitely many nonzero terms (if not the statement is trivial). Denote \( y_n := a_n \sum_{j=1}^{n} a_j \) and define \( q_s := q_s(y) \) inductively: \( q_0 := 0 \) and for \( q_0, \ldots, q_{s-1} \) defined we choose \( q_s \) as the smallest \( q \) such that

\[
(2.1) \quad (q - q_{s-1}) \sum_{n=q_{s-1}+1}^{q} y_n^2 \geq 2^{2s}.
\]

Denote \( Q_s := (q_{s-1}, q_s] \). Then (2.1) implies

\[
\frac{2^s}{\Delta q_s} \leq 2^{-s} \sum_{n \in Q_s} y_n^2 \leq 2^{-s} \sum_{n=1}^{q_s} y_n^2.
\]

Thus it is sufficient to check only (1.3)

\[
\sum_{s} 2^{-s} \sum_{n=1}^{q_s} y_n^2 < \infty,
\]

From the definition of \( q_s \) we have

\[
(2.2) \quad \sum_{n=q_{s-1}+1}^{q_s-1} y_n \leq (\Delta q_s - 1)^{1/2} \left( \sum_{n=q_{s-1}+1}^{q_s-1} y_n^2 \right)^{1/2} < 2^s.
\]

Next for any \( N \leq M \) we have

\[
\sum_{n=N}^{M} a_n \sum_{j=1}^{n} a_j \geq \sum_{N \leq j \leq n \leq M} a_n a_j =
\]
\begin{equation}
(2.3) \quad \frac{1}{2} \left\{ \sum_{j=N}^{M} a_j^2 + \left( \sum_{j=N}^{M} a_j \right)^2 \right\} \geq \left( \sum_{j=N}^{M} a_j \right)^2 / 2.
\end{equation}

Combining (2.2) and (2.3) we get

\begin{equation}
\sum_{j \in Q_s} a_j = \sum_{j=q_{s-1}+1}^{q_s} a_j + a_q \leq 2^{(s+1)/2} + \|a\|_\infty.
\end{equation}

This implies

\begin{equation}
(2.4) \quad \sum_{j=1}^{q_s} a_j \leq C(a) 2^{s/2}.
\end{equation}

We have now

\begin{equation}
\sum_{s} 2^{-s} \sum_{n=1}^{q_s} y_n^2 = \sum_{s} 2^{-s} \sum_{v=1}^{s} \sum_{n \in Q_v} y_n^2 \leq 2 \sum_{s} 2^{-s} \sum_{n \in Q_s} y_n^2 \leq
\end{equation}

\begin{equation}
2 \sum_{s} 2^{-s} \left( \sum_{j=1}^{q_s} a_j \right)^2 \sum_{n \in Q_s} a_n^2 \leq C(a) \sum_{n} a_n^2 < \infty.
\end{equation}

Lemma 2.1 is proved now.

**Theorem 2.1.** The following two conditions are equivalent

\begin{equation}
(C.1) \quad \tau \notin \mathcal{V},
\end{equation}

\begin{equation}
(C.2) \quad \forall \{a_j\}_{j=1}^{\infty} \in l_2, \quad a_j \geq 0, \quad \liminf_{n \to \infty} a_n \sum_{j=1}^{n} a_j / t_n = 0.
\end{equation}

**Proof.** We prove first that (C.1) \(\Rightarrow\) (C.2). Assume (C.2) is not satisfied: \(\exists \{a_j\}_{j=1}^{\infty} \in l_2, \quad a_j \geq 0, \) such that

\begin{equation}
(2.5) \quad \liminf_{n \to \infty} a_n \sum_{j=1}^{n} a_j / t_n > 0.
\end{equation}

Relation (2.5) implies that for some \(N\) and \(c > 0\) we have for \(n \geq N\) that

\begin{equation}
a_n \sum_{j=1}^{n} a_j / t_n \geq c
\end{equation}

or

\begin{equation}
t_n \leq C a_n \sum_{j=1}^{n} a_j.
\end{equation}
This inequality, Lemma 2.1, and Remark 1.1 imply that \( \tau \in \mathcal{V} \). The first implication is proved now.

We proceed to the second implication \((C.2) \Rightarrow (C.1)\). Let \( \tau \in \mathcal{V} \). We construct a sequence \( \{a_j\}_{j=1}^{\infty} \in l_2 \) such that for all \( n \)

\[
    t_n \leq C a_n \sum_{j=1}^{n} a_j
\]

with some \( C \). This will imply that \((C.2)\) is not satisfied. Let \( \{q_s\} := \{q_s(\tau)\} \) be a sequence from the definition of \( \mathcal{V} \). We define a sequence \( \{a_j\}_{j=1}^{\infty} \) as follows. For \( n \in Q_s \) we set

\[
    a_n := t_n 2^{-s/2} + 2^{s/2}/\Delta q_s.
\]

Then

\[
    a_n^2 \leq 2(t_n^2 2^{-s} + 2^s(\Delta q_s)^{-2})
\]

and

\[
    \sum_n a_n^2 \leq 2 \sum_s 2^{-s} \sum_{n \in Q_s} t_n^2 + 2 \sum_s \frac{2^s}{\Delta q_s} < \infty.
\]

Next,

\[
    \sum_{n \in Q_s} a_n \geq 2^s/2.
\]

Thus for \( n \in Q_s \) we have

\[
    a_n \sum_{j=1}^{n} a_j \geq a_n \sum_{j \in Q_{s-1}} a_j \geq t_n 2^{-1/2}
\]

and

\[
    t_n \leq \sqrt{2} a_n \sum_{j=1}^{n} a_j
\]

for all \( n \).

Theorem 2.1 is proved now.

The sufficient part of Theorem 1.1 follows from Theorem 2.1 and Proposition 1.1.

3. **Construction of a counterexample**

The following procedure which is the generalization of Equalizer from [LT] plays an important role in the construction. Let \( H \) be a Hilbert space with an orthonormal basis \( \{e_j\}_{j=1}^{\infty} \). We take two elements \( e_i, e_j, i \neq j \), and define the following procedure.

**Equalizer with schedule** \( \gamma := \{\gamma_k\} \). Let \( \gamma_k \leq 1/5, f_0 := e_i \). Define:

(3.1) \quad g_1 := \alpha_1 e_i - (1 - \alpha_1^2)^{1/2} e_j; \quad \alpha_1 = \gamma_1; \quad \langle f_0, g_1 \rangle = \gamma_1;

(3.2) \quad f_n := f_{n-1} - \langle f_{n-1}, g_n \rangle g_n; \quad g_n := \alpha_n e_i - (1 - \alpha_n^2)^{1/2} e_j;
\[ \langle f_n, g_{n+1} \rangle = \gamma_{n+1}; \quad f_n = a_ne_i + b_ne_j. \]

We check
\[ a_{n-1} - b_{n-1} \geq 3\sqrt{2}\gamma_n \]
to continue. If
\[ a_{n-1} - b_{n-1} < 3\sqrt{2}\gamma_n \]
then we take \( g_n := 2^{-1/2}(e_i - e_j) \) and
\[ f_n := f_{n-1} - \langle f_{n-1}, g_n \rangle g_n, \]
and stop after this step. We call this step "the final step" and all other steps "regular steps". At each regular step \( l \) we have
\[ a_l - b_l = a_{l-1} - b_{l-1} - \gamma_l(a_l + (1 - \alpha_l^2)^{1/2}) \geq a_{l-1} - b_{l-1} - 2^{1/2}\gamma_l > 0. \]
After the final step we have
\[ a_n = b_n. \]
At each regular step we have by definition that
\[ \langle f_{l-1}, g_l \rangle = \gamma_l. \]
At the final step we have
\[ \langle f_{n-1}, g_n \rangle = 2^{-1/2}(a_{n-1} - b_{n-1}) \geq 2^{-1/2}(a_{n-2} - b_{n-2} - 2^{1/2}\gamma_{n-1}) \geq 2^{-1/2}(2\sqrt{2}\gamma_{n-1}) = 2\gamma_{n-1}. \]
Thus, if \( 2\gamma_{n-1} \geq \gamma_n \) then the above described Equalizer is a WGA with weakness sequence \( \gamma_1, \ldots, \gamma_n \).

At regular step \( l \) we reduce the \( \| \cdot \| \) by \( \gamma_l^2 \). At the final step we reduce the \( \| \cdot \| \) by
\[ \frac{1}{2}(a_{n-1} - b_{n-1})^2 < 9\gamma_n^2. \]
We also have
\[ a_{n-1} - b_{n-1} < 3\sqrt{2}\gamma_n \]
and
\[ a_{n-1} - b_{n-1} \geq 1 - \sqrt{2}\sum_{j=1}^{n-1} \gamma_j. \]
Thus,
\[ \sqrt{2}\sum_{j=1}^{n-1} \gamma_j + 3\sqrt{2}\gamma_n > 1. \]
On the other hand
\[ a_l - b_l \leq a_{l-1} - b_{l-1} - \gamma_l. \]
Therefore,
\[ 0 \leq a_{n-1} - b_{n-1} \leq 1 - \sum_{l=1}^{n-1} \gamma_l \]
and
\[ (3.3) \sum_{l=1}^{n-1} \gamma_l \leq 1. \]

In order to apply the above Equalizer we need to have the inequality \( 2\gamma_{n-1} \geq \gamma_n \) satisfied. Let us use the following regularization procedure.
Regularization. For a given \( \tau = \{t_k\}_{k=1}^{\infty}, \tau \in l_\infty \), we define \( \tau^R := \{t^R_k\}_{k=1}^{\infty} \) with
\[
t^R_k = \sum_{m=0}^{\infty} 2^{-m} t_{n+m}.
\]

**Lemma 3.1.** If \( \tau \in \mathcal{V} \cap l_\infty \) then \( \tau^R \in \mathcal{V} \cap l_\infty \).

**Proof.** Assumption \( \tau \in \mathcal{V} \) implies
\[
(3.4) \quad \sum_s 2^{-s} \sum_{k=1}^{q_s} t^2_k < \infty.
\]

We will prove that
\[
(3.5) \quad \sum_s 2^{-s} \sum_{k=1}^{q_s} (t^R_k)^2 < \infty
\]
with the same \( q_s = q_s(\tau) \) as above. Thus (3.5) will imply \( \tau^R \in \mathcal{V} \). Let us prove (3.5). We have for any \( N \)
\[
\sum_{k=1}^{N} (t^R_k)^2 = \sum_{k=1}^{N} \left( \sum_{m=0}^{\infty} 2^{-m} t_{n+m} \right)^2 = \sum_{k=1}^{N} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} t_{k+m} t_{k+n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \sum_{k=1}^{N} t_{k+m} t_{k+n} \leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \left( \sum_{k=1}^{N} t^2_{k+m} \right)^{1/2} \left( \sum_{k=1}^{N} t^2_{k+n} \right)^{1/2} = \left( \sum_{m=0}^{\infty} 2^{-m} \left( \sum_{k=1}^{N} t^2_{k+m} \right)^{1/2} \right)^2 \leq \left( \sum_{m=0}^{\infty} 2^{-m} \right)^2 \left( \sum_{m=0}^{\infty} 2^{-m} \sum_{k=1}^{N} t^2_{k+m} \right).
\]

Next,
\[
\sum_{k=1}^{N} t^2_{k+m} \leq \sum_{k=1}^{N} t^2_k + m\|\tau\|_\infty^2
\]
and
\[
\sum_{m=0}^{\infty} 2^{-m} \left( \sum_{j=1}^{N} t^2_j + m\|\tau\|_\infty^2 \right) \leq 2 \sum_{j=1}^{N} t^2_j + C(\tau).
\]

Therefore we got
\[
\sum_{j=1}^{N} (t^R_j)^2 \leq 2 \sum_{j=1}^{N} t^2_j + C(\tau)
\]
and
\[
\sum_s 2^{-s} \sum_{k=1}^{q_s} (t^R_k)^2 \leq 2 \sum_s 2^{-s} \sum_{k=1}^{q_s} t^2_k + C(\tau) < \infty.
\]

It is easy to see that \( \|\tau^R\|_\infty \leq 2\|\tau\|_\infty \).

Lemma 3.1 is proved now.
Thus for any $\tau \in \mathcal{V} \cap l_\infty$ we have $\tau^R \in \mathcal{V}$, $\|\tau^R\|_\infty \leq 2\|\tau\|_\infty$, and
\[
2t^R_{n-1} \geq t^R_n, \quad n = 2, 3, \ldots
\]
Clearly, we also have for all $n$
\[
t_n \leq t^R_n.
\]
One more restriction in the Equalizer is $\gamma_n \leq 1/5$. Define a new sequence $\tau'$ by
\[
t'_n := \min\{t^R_n, 1/5\}.
\]
It is clear that $\tau' \in \mathcal{V}$ and also satisfies
\[
2t'_{n-1} \geq t'_n.
\]
Let $\{q_s\} := \{q_s(\tau')\}$ be the sequence for $\tau'$ from the definition of $\mathcal{V}$:
\[
\sum_s \frac{2^s}{\Delta q_s} < \infty, \quad \sum_s 2^{-s} \sum_{n=1}^{q_s} (t'_n)^2 < \infty.
\]
Let $\epsilon$ be a small number which we will specify later and $s_0$ be such that
\[
\sum_{s \geq s_0} \frac{2^s}{\Delta q_s} < \epsilon, \quad \sum_{s \geq s_0} 2^{-s} \sum_{n=1}^{q_s} (t'_n)^2 < \epsilon.
\]
Consider the function
\[
f_{s_0} := 2^{-s_0/2}(e_1 + \cdots + e_{2^{s_0}}).
\]
We have $\|f_{s_0}\| = 1$. Define
\[
t''_k := \max\{t''_k, 2^{s_0+2}(\Delta q_{s_0})^{-1}\}.
\]
We apply a mixture of Equalizer with schedule $\{t''_k\}$ to vectors $e_i$, $i \leq 2^{s_0}$, and the PGA to the corresponding residual of $f_{s_0}$. We do this in the following way. If $t''_1 \geq 1/5$ we use PGA and throw away, say, $2^{-s_0/2}e_{2^{s_0}}$. If $t''_1 < 1/5$ we start using the Equalizer with schedule $\{t''_k\}$ to vectors $e_1$ and $e_{2^{s_0}+1}$. If at some step $t''_k \geq 1/5$ then we use PGA what means throwing away one term of the form $2^{-s_0/2}e_j$, $j \in [1, 2^{s_0}]$. Applying the Equalizer to the very last term of the form $2^{-s_0/2}e_m$ we may encounter with $t''_k \geq 1/5$. In such a case we apply PGA and stop. As a result we get
\[
f_{s_0+1} := \sum_{k \in F_{s_0+1}} c_k^{s_0+1} e_k.
\]
It is clear that for all $k \in F_{s_0+1}$ we have
\[
(c_k^{s_0+1})^2 \leq 2^{-s_0-1}
\]
and also
\[
|F_{s_0+1}| \leq 2^{s_0+1}.
\]
Assume that $\epsilon < 1/20$. Then $2^{s_o+2}(\Delta q_{s_0})^{-1} < 1/5$ and $t''_k \geq 1/5$ is equivalent to $t''_k = t'_k = 1/5$. If $t''_k < 1/5$ then $t'_k < 1/5$ and $t_k \leq t'_k$. Therefore, at all Equalizer steps we have a WGA with weakness parameters $\{t_k\}$. If $t''_k = 1/5$ we apply PGA what is a WGA with any $t_k$ at this step. During this procedure which we call "working on $s_0$-level" we perform $M'_{s_0}$ steps of Equalizer and $M^G_{s_0}$ steps of PGA. Let us estimate $M'_{s_0}$ and $M^G_{s_0}$. It is clear that $M^G_{s_0} \leq 2^{s_o}$. We have applied the Equalizer to terms of the form $2^{-s_o/2}e_j$ at most $2^{s_o}$ times. For each Equalizer application we have $\sum \gamma_j \leq 2$ (see (3.3)). Thus denoting $E(s_0) := \{ k : t'_k < 1/5 \}$ we get

$$
\sum_{k \in E(s_0)} t'_k \leq 2^{s_o+1}.
$$

On the other hand we have

$$
\sum_{k \in E(s_0)} t''_k \geq M'_{s_0}2^{s_o+2}(\Delta q_{s_0})^{-1}
$$

and

$$M'_{s_0} \leq \Delta q_{s_0}/2.
$$

Therefore,

$$N_{s_0} := M'_{s_0} + M^G_{s_0} \leq \Delta q_{s_0}/2 + 2^{s_o} \leq \Delta q_{s_0}.
$$

At each Equalizer step we reduced the $\| \cdot \|^2$ by at most $9(t'_k)^22^{-s_0}$ and at each PGA by at most $25(t'_k)^22^{-s_0}$. Thus the total reduction $\delta_{s_0}$ for the $s_0$-level does not exceed

$$25(2^{-s_0})\sum_{k=1}^{q_{s_0}} (t'_k)^2 + 9(2^{s_0+4})(\Delta q_{s_0})^{-1}.
$$

We are on the $(s_0+1)$-level now and perform the similar procedure. We describe it for the general case of an $s$-level. Assume we have after $N_{s-1} \leq q_{s-1}$ steps of our WGA the function

$$f_s = \sum_{k \in F_s} c^s_k e_k
$$

with

$$(c^s_k)^2 \leq 2^{-s}, \quad |F_s| \leq 2^s.
$$

Define now

$$t''_k := \max\{t'_k, 2^{s+2}(\Delta q_s)^{-1}\}, \quad k > N_{s-1}.
$$

We pick $c^s_k e_k$ with the biggest $c^s_k$ out of $\{c^s_k, k \in F_s\}$ and throw it away if $t''_k = 1/5$ (we remind that assumption $\epsilon < 1/20$ implies $2^{s+2}(\Delta q_s)^{-1} < 1/5$) and apply the Equalizer with schedule $\{t''_k\}$ otherwise. We continue to perform the above described procedure (the mixture of Equalizer and PGA steps) until we get

$$f_{s+1} = \sum_{k \in F_{s+1}} c^{s+1}_k e_k
$$

with

$$(c^{s+1}_k)^2 \leq 2^{-s-1}.
$$
It is clear that then \(|F_{s+1}| \leq 2^{s+1}\). Similarly to the above estimates of \(M^w_{s_0}\) and \(M^G_{s_0}\) we get

\[
M^G_s \leq 2^s
\]

and

\[
M^w_s 2^{s+2}(\Delta q_s)^{-1} \leq \sum_{k \in E(s)} t''_k \leq 2^{s+1}.
\]

Thus

\[
M^w_s + M^G_s \leq \Delta q_s / 2 + 2^s \leq \Delta q_s
\]

and

\[
N_s := N_{s-1} + M^w_s + M^G_s \leq q_s.
\]

The total reduction \(\delta_s\) of the \(\| \cdot \|^2\) from working on the \(s\)-level does not exceed

\[
25(2^{-s}) \sum_{k=1}^{q_s} (t'_k)^2 + 9(2^{s+4})(\Delta q_s)^{-1}.
\]

We continue this process and get that the \(\| \cdot \|^2\) will be reduced by at most

\[
\sum_{s=s_0}^{\infty} \delta_s \leq 25 \sum_{s=s_0}^{\infty} 2^{-s} \sum_{k=1}^{q_s} (t'_k)^2 + 144 \sum_{s=s_0}^{\infty} 2^s (\Delta q_s)^{-1} \leq 169 \epsilon.
\]

Choosing \(\epsilon\) small enough, say, \(\epsilon = 0.005\) we get divergent WGA with the weakness sequence \(\tau\). This completes the construction of the counterexample.

References


