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A successful concept for measuring non-planarity of graphs: the crossing number

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Abstract

This paper surveys how the concept of crossing number, which used to be familiar only to a limited group of specialists, emerges as a significant graph parameter. This paper has dual purposes: first, it reviews foundational, historical, and philosophical issues of crossing numbers, second, it shows a new lower bound for crossing numbers. This new lower bound may be helpful in estimating the crossing number of complete graphs.

1 Foundational issues

Pach and Tóth [36] noted that although researchers seem to agree in what they understand under the concept of "crossing number", "drawings" are defined in a variety of ways in the literature, and the possibility is there that some definitions might not be equivalent. Pach and Tóth [36] introduced two new versions of the crossing number problem, and there is a fourth version, implicitly present in Tutte [52]. First I give a careful definition of three classes of drawings, in which all four kind of crossing numbers can be conveniently set.

A *drawing* D of a finite graph G on the plane is an injection ϕ from the vertex set $V(G)$ into the plane, and a mapping of the edge set $E(G)$ into the set of simple plane curves (i.e. homeomorphic images of the interval $[0,1]$), such that the curve corresponding to the edge $e = uv$ has endpoints $\phi(u)$ and $\phi(v)$, and contains no other vertices.

We also speak about the images of vertices as vertices, and about the curves as edges. The *number of crossings* $cr(D)$ in the drawing D is the sum of the number of intersection points of all unordered pairs of interiors of edges (i.e. endpoints are not counted in the intersections).

A *drawing* D is *normal* if it satisfies (i) and (ii):

(i) any two of the curves have finitely many points in common
(ii) no two curves have a point in common in a tangential way, i.e. if we define locally the "left side" and the "right side" of the curves at the common point, both curves are present at both sides of each in every small neighborhood of that point

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Assumption (ii) allows for speaking about points of crossing instead of points of intersection. For normal drawings in most cases we will also assume (iii):
(iii) no three curves cross each other in the same point

Requirement (iii) is convenient, since using it one can simplify the definition of $cr(D)$ to the number of points of crossing in the drawing. Also, some proof techniques about crossing numbers introduce a new planar graph from a drawing D by introducing a new vertices of degree 4 in the points of crossing. However, many drawings in applications, especially straight line drawings, do not satisfy (iii). Notice that if (iii) fails and some k curves cross each other in a normal drawing in a single point, this can easily be transformed locally into a drawing where any two of the k curves cross each other once, and the number of crossings in the drawing does not change. Therefore we assume (iii) for normal drawings and it will not cause any problem that some drawings that we use fail (iii).

A *drawing* D is *nice*, if it is normal, and in addition satisfies

(iv) no two adjacent edges (i.e. edges sharing an endpoint) cross
(v) any two edges cross at most once

The *crossing number* $CR(G)$ of the graph G is the minimum of $cr(D)$ over all drawings of G. We call a drawing D optimal if it realizes $cr(D) = CR(G)$. It is easy to see that an optimal drawing must satisfy (i) and (ii), and a little work shows that it also must satisfy (iv) and (v). Therefore, we have an equivalent definition of $CR(G)$: the minimum of $cr(D)$ over all normal, nice drawings of G.

We show (v) first. Assume that the curves p and q corresponding to edges UZ and XY cross in points R and T. Call p_1, p_2, p_3 and q_1, q_2, q_3 the pieces of p and q determined by R and T, with p_2 and q_2 denoting the RT sections. Redefine the curves as

$$p' = p_1 \cup q_2 \cup p_3 \qquad \text{and} \qquad q' = q_1 \cup p_2 \cup q_3. \tag{1}$$

Now we can eliminate the tangential intersections of p' and q' at R and T. A problem is that p' and q' may not be simple curves (i.e we may have created self-crossings), but we can shortcut them, and this does not increase the number of crossings in the drawing. The proof of (iv) is similar, use the shared endvertex for R, and T for a crossing point. p_1 or p_3 (q_1 or q_3) degenerates for a point. The surgery (1) works again.

Pach and Tóth [36] introduced two new variant of the crossing number problem:
the pairwise crossing number $CR\text{-}PAIR(G)$ is equal to the minimum number of unordered pairs of edges that cross each other at least once (i.e. they are counted once instead of as many times they cross), over all normal drawings of G
the odd crossing number $CR\text{-}ODD(G)$ is equal to the minimum number of unordered pairs of edges that cross each other odd times, over all normal drawings of G

In Tutte's work [52] another kind of crossing number is implicit:
the independent-odd crossing number $CR\text{-}IODD(G)$ is equal to the minimum number of unordered pairs of non-adjacent edges that cross each other odd times, over all normal drawings of G

The following chain of inequalities is obvious from the definitions:

$$CR\text{-}IODD(G) \leq CR\text{-}ODD(G) \leq CR\text{-}PAIR(G) \leq CR(G). \qquad (2)$$

No example of strict inequality is known. Pach [34] considers the problem if all these numbers are always equal as the most important open problem on crossing numbers.

The smallest graphs with $CR(G) = 1$ are K_5 and $K_{3,3}$. For these graphs we have

Theorem 1.1 [Chojnacki [13] 1934]

$$CR\text{-}IODD(K_5) = 1 = CR\text{-}IODD(K_{3,3})$$

For other proofs and generalizations, see [52, 36]).

It is clear that if similar crossing number problems are posed for the sphere instead of the plane, stereographic projection shows that the corresponding planar and spheric crossing numbers are always equal. Crossing number problems can be posed on orientable and non-orientable surfaces of higher genus, and many of the results discussed in this paper generalizes for them, see [44, 48, 39].

It is not the purpose of the present paper to give a comprehensive survey of the literature of crossing numbers. Much of the literature falls into one of two categories: the first investigates exact values of crossing numbers or makes lower bounds on crossing numbers based on information on the crossing number of a certain small graph, the second tries to prove bounds based on structural properties of the graph. We call the first the theory of small graphs, the second the theory of large graphs. During the early history of crossing numbers the theory of small graphs existed only. For more information on the early history and the theory of small graphs, see White and Beineke [55], for the modern history and the theory of large graphs, see Shahrokhi, Sýkora, Székely and Vrťo [48], and for the most recent results see Pach [34]. A bibliography of papers on crossing numbers by I. Vrťo is available online [54].

2 Theory of small graphs

2.1 Turán's Brick Factory Problem

It was Paul Turán who introduced the concept of crossing numbers. Turán [51] tells about how he posed the problem, while in a forced labor camp in World War II: "There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all storage yards. ... the trouble was only at crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time ... the idea occured to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings?"

Put in technical terms, Turán's Brick Factory Problem is: *what is the crossing number $CR(K_{n,m})$ of the complete bipartite graph $K_{n,m}$?*

Place $\lfloor n/2 \rfloor$ vertices to negative positions on the x-axis, $\lceil n/2 \rceil$ vertices to positive positions on the x-axis, $\lfloor m/2 \rfloor$ vertices to negative positions on the y-axis, $\lceil m/2 \rceil$ vertices to positive positions on the y-axis, and draw nm edges by straight line segments to obtain a drawing of $K_{n,m}$. It is not hard to check that the following formula gives the number of crossings in this particular drawing:

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor . \tag{3}$$

Zarankiewicz's Crossing Number Conjecture is that the drawing described above is optimal.

Fáry's theorem [19] telling that planar graphs can be drawn using straight line segments for edges and Zarankiewicz's Crossing Number Conjecture may suggest that optimal drawings can be done using straight line segments for edges. This is not the case. Guy showed that first for K_9 [24], and later Bienstock and Dean [10, 11] constructed graphs with crossing number four for any number k, such that drawings of those graphs using straight line segments for edges have more crossings than k.

The conjectured crossing number of the complete graph K_n is

$$\frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor . \tag{4}$$

We show a drawing with this number of crossings for even n, the construction is due to Guy [25] and Blažek and Koman [5]: take a soup can, which is homeomorphic to a sphere, place $n/2$ vertices equidistantly on the perimeter of the top disk and on the perimeter of the bottom disk, respectively. Draw a $K_{n/2}$ with straight line segments on the top disk and on the bottom disk, respectively. From one point of the bottom disk, draw shortest helical curves to all vertices of the top disk. Repeat this for all $n/2$ vertices on the bottom disk. Although this is not a straight line drawing of K_n, interestingly, the curves that we use are "geodetic" on the soup can.

It is usually not hard to come up with drawings of graph whose optimality is intuitively clear. The difficulty lies in proving matching lower bounds for the crossing numbers. In the old days, every lower bound depended somehow on lower bounds (or exact values) obtained for the crossing number of some small graphs.

2.2 Euler's formula

The simplest lower bound for the crossing number of a simple graph with $n \geq 2$ vertices and m edges is

$$m - 3n + 6. \tag{5}$$

This immediately follows from Euler's polyhedral formula, and already gives $CR(K_5) \geq 1$. A counterpart of this formula for triangle-free graphs $CR(G) \geq m - 2n + 4$, which proves $CR(K_{3,3}) \geq 1$. Formula (5) can give interesting lower bounds for small graphs only, since the magnitude of the crossing number can be as large as m^2. It was Pach and Tóth who observed that (5) sets a lower bound for *all four* crossing numbers in (2), and this extends to all lower bounds which solely depend on (5). We shortly reproduce their proof for the smallest crossing number, $CR\text{-}IODD(G)$. If $m \leq 3n - 6$, then there is nothing to prove. If $m \geq 3n - 5$, then G is non-planar, and hence contains by Kuratowski's

Theorem a subdivision of a K_5 or a $K_{3,3}$ (in fact both). Hence in any normal drawing of G there is a normal subdrawing of a K_5 or a $K_{3,3}$. By Theorem 1.1, there are two vertex disjoint paths of G which cross each other an odd number of times. Hence, there is an edge e from the first path and an edge f from the second path that cross each other odd times. If formula (5) holds for $G - e$, then it holds for G, and the base case for this induction proof is $m = 3n - 5$.

2.3 The standard counting method

A basic technique to obtain a lower bound for the crossing number of a larger graph from that of a sample graph is the *standard counting method*. We take a hypothetic {normal,nice, optimal} drawing of the large graph, find many copies of the sample graph in it, each exhibiting as many crossings as its crossing number, add up those numbers, and divide by the largest multiplicity with which a crossing may have been counted in different copies of the sample graph. Make this more tangible by the following example: $CR\text{-}IODD(K_n) \geq (1 + o(1))n^4/120$. Take a normal drawing of K_n. Any 5 vertices span a normal subdrawing of a K_5, which exhibit at least one pair of non-adjacent edges crossing odd times. We find at least total of $\binom{n}{5}$ such edge pairs, and every such edge pair occurs in exactly in $(n - 4)$ 5-tuples of vertices. The claim follows.

Applying the standard counting argument for K_{n+1} with sample graph=K_n, or for $K_{n+1,n+1}$ with sample graph=$K_{n,n}$, one obtains that

$$\frac{CR(K_n)}{24\binom{n}{4}} \qquad \text{and} \qquad \frac{CR(K_{n,n})}{4\binom{n}{2}^2} \tag{6}$$

are nondecreasing and bounded. Therefore the sequences in (6) have a limit, which provides asymptotic formulae $CR(K_n) \sim c_1 n^4$ and $CR(K_{n,n}) \sim c_2 n^4$ [41, 55]. However, the values of c_1 and c_2 are not known. The drawings shown above imply $c_1 \leq \frac{1}{64}$ and $c_2 \leq \frac{1}{16}$, and if the drawings are optimal, equalities hold.

D. J. Kleitman showed that (3) holds for $m \leq 6$ [27] and also proved that the smallest counterexample to the Zarankiewicz's conjecture must occur for odd n and m. D. R. Woodall used elaborate computer search to show that (3) holds for $K_{7,7}$ and $K_{7,9}$. Thus, the smallest unsettled instances of Zarankiewicz's conjecture are $K_{7,11}$ and $K_{9,9}$. Woodall's result for $K_{7,9}$ implies $\frac{1}{21} \leq c_2$ by a standard counting argument. Kleitman's cited result [27] allows us to use $K_{n-6,6}$ as a sample graph to count crossings in K_n, and one obtains $\frac{1}{80} \leq c_1$. Applying the standard counting argument to K_n with sample graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ [41] shows, that if $c_2 = 1$ then $c_1 = \frac{1}{64}$. The converse of this implication is not known.

2.4 Miscellanea

The graph minor community also has an interest in crossing numbers. Their usual approach is characterization in terms of excluded minors. Robertson and Seymour [43] calls a graph H singly crossing provided H is a minor of a graph that can be drawn on the sphere with at most one crossing. They show that a graph is singly crossing if and only if it does not have one of 41 explicitly given graphs as a minor.

For computing crossing numbers of specific graphs, ad hoc methods are often needed. For example, there is a longstanding conjecture of Harary, Kainen and Schwenk [20], which states that for $n \geq m \geq 3$, the crossing number of the Cartesian product of two cycles, $CR(C_n \times C_m)$ is $n(m-2)$. There is a simple drawing with this number of crossings, the difficulty lies in proving that $n(m-2)$ crossings are, in fact, needed. Proving the conjecture for different small values of n and m took separate, highly technical papers; and the case $n = m = 8$ is still open [9, 42, 28, 40, 3, 4]. Richter and Thomassen [40] introduced here the most general approach so far: consider n red closed curves and m blue closed curves, where each may cover certain points twice, such that every blue curve intersects every red curve, and no point of the plane is covered three times. What is then the minimum number of intersection points of curves? This problem is rather geometric than graph theoretic, and is a better subject to inductive arguments than the Cartesian product of two cycles.

Shahrokhi, Sýkora, Székely and Vrťo [46] showed $CR(C_n \times C_m) = \Omega(nm)$ applying basic results on posets to certain posets arising from drawings of this graph, but this already belongs to the theory of large graphs.

3 Theory of large graphs

The modern history started with Leighton's thesis [30]. Leighton introduced methods to set lower bounds for crossing numbers which instead of crossing numbers of small graphs, depended on certain parameters of the large graphs. He introduced three methods that become classic: lower bounds in terms of number of edges, bisection width, and graph embedding.

3.1 Number of edges

Ajtai *et al.* [2] and Leighton [30] independently discovered that for graphs with $m \geq cn$ edges, the crossing number is at least

$$CR(G) \geq \frac{c-3}{c^3}\frac{m^3}{n^2}. \tag{7}$$

The maximum constant factor in (7) is $\frac{1}{64}$, achieved at selecting $c = 4$. It follows from the argument after (5) that (7) holds for all four crossing numbers in (2). The original proofs of (7) went by induction, a folklor probabilistic proof can be found in [48] and also made it to the Book [1].

For $c = 4$, Pach and Tóth [37] improved $\frac{1}{64}$ to $\frac{1}{33.75}$, but this improved lower bound is not known to extend for all kinds of crossing numbers.

Erdős and Guy [17] conjectured (7) (although those who proved it were not aware of it), and even more. If $\kappa(n,m)$ denotes the minimum crossing number of graphs with n vertices and m edges, they conjectured that $\lim \kappa(n,m)n^2/m^3$ has a limit if $m/n \to \infty$. Recently, Pach, Spencer and Tóth [39] proved this conjecture if, in addition, $n^2/m \to \infty$ is assumed, and observed that some additional restriction is needed. The value of this limit is not known.

3.2 Bisection width

We define here the *bisection width* $b(G)$ of a graph G as the smallest number of edges between two classes of vertices V_1 and V_2, which make a bipartition of

$V(G)$ with $|V_1|, |V_2| \geq |V|/3$. Leighton [30] proved the following theorem:

Theorem 3.1 [Leighton [30] 1983] *For any graph G of bounded degree,*

$$CR(G) + n = \Omega(b(G)^2).$$

Proof. Consider a drawing of G with $CR(G)$ crossings. Introduce a new vertex at each crossing to obtain a graph H with $N = n + CR(G)$ vertices drawn in the plane without crossings. Assign weight 0 to each new vertex and weight of $1/n$ to all other vertices. Since, G has bounded degree, H has also bounded degree. One can apply the Lipton-Tarjan separator theorem [33] to find a separating set S in H, $|S| \leq \sqrt{8(n + CR(G))}$, such that one finds in $V(H) \setminus S$ two sets of vertices, A and B, with no edges between A and B, and the weights of A and B do not exceed 2/3. Since G is degree bounded, the number of edges adjacent to S is $O(\sqrt{n + CR(G)})$. Having deleted those edges, we can add the points of S either to A or B to obtain A' and B' with weight *at least* 1/3, and between A' and B' we have in H at most $O(\sqrt{n + CR(G)})$ edges.

Theorem 3.1 was generalized by Pach, Shahrokhi, and Szegedy [38] the Gazit–Miller separator theorem: Let $d_i, i \in V(G)$ denote the degrees of the vertices in the G, then

$$(1.58)^2 \left(16CR(G) + \sum_{i \in V} d_i^2 \right) \geq b(G)^2.$$

3.3 Graph embedding

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, $|V_1| \leq |V_2|$. An *embedding* ω, of G_1 in G_2 is a pair of injections ϕ, ψ with

$$\phi : V_1 \to V_2$$

$$\psi : E_1 \to \{\text{set of all paths in } G_2\}$$

such that if $uv \in E_1$ then $\psi(uv)$ is a path between $\phi(u)$ and $\phi(v)$. For any $e \in E_2$ and any $u \in V_2$ define

$$\begin{aligned} \mu_\omega(e) &= |\{f \in E_1 \,:\, e \in \psi(f)\}| \text{ and} \\ m_\omega(u) &= |\{f \in E_1 \,:\, u \in \psi(f)\}|. \end{aligned}$$

Moreover, we define,

$$\begin{aligned} \mu_\omega &= \max_{e \in E_2} \mu_\omega(e), \text{ and} \\ m_\omega &= \max_{u \in V_2} m_\omega(u). \end{aligned}$$

We refer to μ_ω and m_ω as the *edge congestion* and *vertex congestion* of ω, respectively. Leighton [30] introduced the concept of graph embedding and used this technique for deriving lower bounds on the crossing number of the perfect shuffle and the mesh of trees and other degree bounded graphs. Shahrokhi and Székely [47] applied Leighton's approach to arbitrary graphs and gave estimates of embedding K_n into a large class of symmetric graphs. Further generalization was obtained by Shahrokhi, Sýkora, Székely and Vrťo [44], [45].

Theorem 3.2 [47] *Let $G = (V, E)$ and $H = (V', E')$ be graphs with $|V'| \leq |V| = n$ and let ω be an embedding of H into G. Then*

$$CR(G) \geq \frac{CR(H)}{{\mu_\omega}^2} - \frac{n}{2}\left(\frac{m_\omega}{\mu_\omega}\right)^2.$$

This theorem is usually applied with $H = K_n$ or $H = K_{n,m}$. The embedding method yields, for example, the crossing number of n-dimensional hypercube the tight lower bound $\Omega(4^n)$ [44].

3.4 Computational complexity

Garey and Johnson [21] proved that testing $CR(G) \leq k$ is NP-complete, Pach and Tóth [36] extended this to $CR\text{-}PAIR$ and $CR\text{-}ODD$. The reduction uses the NP-completeness of Linear Arrangement. Testing planarity, and therefore testing $CR(G) \leq k$ for any fixed k can be done in polynomial time—introduce at most k new vertices for crossing points in all possible ways and test planarity. Leighton and Rao [31] designed the first provably good approximation algorithm for crossing numbers. This algorithm approximates $n + CR(G)$ within a factor of $\log^4 n$ for degree bounded graphs (and therefore provides little information on small crossing numbers). A recent paper Even, Guha, and Schieber [18] reduced the factor to $\log^3 n$. We know nothing that would exclude the possibility of approximation within a constant multiplicative factor.

4 Corroborating Lakatos

K. Zarankiewicz and K. Urbaník independently claimed and published that $CR(K_{n,m})$ was actually equal to (3), their proof was reprinted in a book [6], cited, and used in follow-up papers. Kainen and Ringel discovered a flaw in the proof and the flaw withstood all attempts for correction. Richard Guy deserves much credit for rectifying this confused state of art [25] and also for pointing out "much more sweeping assumptions than the overt hypotheses of the theorem" in some other crossing number papers [26].

Imre Lakatos, who applied the Popperian epistemology to mathematics, carried out his arguments [29] on the paradigmatic example of Euler's polyhedral formula. Actually, crossing numbers, closely connected to Euler's polyhedral formula by (5), could also have served as his paradigmatic example.

In a recent paper Pach and Tóth [36] scrutinize the very definition of crossing numbers! They point out that some authors might have thought of $CR\text{-}PAIR$ instead of CR.

How is it possible that decades in research of crossing numbers passed by and no major confusion resulted from these foundational problems? The answer is the following: the conjectured optimal drawings are usually normal and nice, and the lower bounds—as (5,7)—usually also apply for all kind of crossing numbers.

5 Applications of crossing numbers

Many concepts have been introduced in the literature which measure quantitatively "how far" a non-planar graph is from being a planar graph: genus, crossing number, thickness, splitting number, skewness, vertex deletion number, etc. [55, 48]. Computing these quantities (or their slight variations) is known or conjectured to be NP-hard [21], and apart from this, with the exception of genus and crossing number, there is not much to tell about them.

So far, only familiarity with the genus was a must for every discrete mathematician. Now the crossing number aligns with the genus, since it has applications and is connected to other areas of mathematics.

Ringel discovered that the Turán number $T(n,5,4)$ sets a lower bound for the crossing number of the complete graph on n vertices. Consider an optimal (normal, nice) drawing of the complete graph. Define a 4-uniform hypergraph on the vertex set of the complete graph by the quadruplets of vertices of pairs of crossing edges. Since K_5 is non-planar, any 5 element subset of vertices does contain an edge of the 4-uniform hypergraph.

Leighton's interest in crossing numbers was motivated by VLSI, and he used the crossing number to set lower bound for the VLSI layout area of the graph. In fact, the relevance of crossing number for engineering was well-known already in the pre-VLSI "transistor age" [6].

Székely [50] used the cited theorem of Ajtai *et al.* [2] and Leighton [30] to give a new proof for the Szemerédi–Trotter theorem, which tells how many incidences can be among n points and m lines in the plane. The proof consists of comparing the lower bound (7) to an upper bound, coming from a given drawing, for a certain graph. This crossing number method also yielded simple proofs [50] for the best available results regarding two classic Erdős problems: Given n points in the plane, how many unit distances can be among them? Given n points in the plane, what is the least number of distinct distances among them? Just in a couple of years, the crossing number method gave a number of other applications to discrete geometry [35, 7, 37], etc. Surprisingly, this crossing number method is also cited in number theory, see [14, 15, 16, 32, 23, 8].

Pach, Spencer and Tóth [39] proved a conjecture of Simonovits, improving the bound of (7). If G has girth $> 2r$ and $m \geq 4n$, then

$$CR(G) = \Omega\left(\frac{m^{r+2}}{n^{r+1}}\right), \tag{8}$$

and proved an even more general theorem for graphs G satisfying a monotone graph property. Since $m^2 > CR(G)$, (8) immediately implies that a graph with girth $> 2r$ has at most $O(n^{1+\frac{1}{r}})$ edges, which is the best known result, tight within a constant multiplicative factor for $r = 2, 3, 5$. This may be thought of as an unfair application, since the proof in [39] uses these facts from extremal graph theory, but this is a new genuine connection between crossing numbers and extremal graph theory.

6 Formulae for CR-$IODD$

To have a graph parameter that we cannot even asymptotically evaluate for complete graphs is rather annoying. In addition, knowing the crossing numbers of complete graphs would immediately imply improved lower bounds on the crossing numbers of many other graphs, either by the standard counting argument or by graph embedding.

The present section yields formulae for *CR-IODD*, which are far from obvious how to evaluate, but give a hope to evaluate *CR-IODD* for complete graphs.

6.1 Tutte's theory

Earlier, Tutte [52] introduced an algebraic theory of crossing numbers and proved Chojnacki's Theorem 1.1 from this theory. Tutte's theory is very complicated, since it tries to follow closely not just crossing numbers but drawings. Tutte studies normal drawings. Denoting the vertex set by $V = \{1, 2, ..., n\}$, he defines two orientation for every edge, connecting vertices i and j, ij and ji. The orientation ij defines locally a left side and and a right side of the curve, as if we were facing j on the curve. Tutte denotes by $\lambda(ij, kl)$, for two non-adjacent oriented edges ij and kl, the difference of the following two numbers: number of left-to-right crossings that oriented edge ij does on kl and the number of right-to-left crossings that oriented edge ij does on kl. He observes that $\lambda(ij, kl)$ has the same parity as the number of crossings of ij and kl, and fixing an orientation for every edge, he suggests the lower bound

$$\min_{\text{normal drawings}} \sum |\lambda(ij, kl)|, \tag{9}$$

(where summation goes for unerdered pairs of non-adjacent edges) for the crossing number, and poses the question if equality holds. It is clear that *CR-ODD* $\leq$(9)$\leq$ *CR*. There is an enigmatic sentence of Tutte "We are taking the view that crossings of adjacent edges are trivial, and easily get rid of." We interpret this sentence as a philosophical view and not a mathematical claim.

Pach and Tóth [36] had some formulae, or rather discrete integer programs, for the value of *CR-ODD*, which involved pairs of edges. Tutte, and Pach and Tóth described how their respective formulae transform when an edge is "pulled over" a vertex, a generic step to move from one drawing to another. I am not aware of any paper which draws further conclusions on crossings numbers from Tutte's theory. There seems to be a trade-off between getting tangible results and following faithfully the drawing. Our results, presented next, is a mod 2 version of Tutte's theory. This requires only maintaining information on (edge, vertex) type pairs, which simplifies everything. We show how these results can be used to prove Chojnacki's Theorem 1.1.

6.2 The new results

Assume now, that the vertices $A_1, A_2, ..., A_n$ of a graph G are put in this cyclic order on a circle S. We say that the two non-adjacent edges of G, say XY and UZ are in *acyclic order*, if the cyclic order of S restricted to these 4 vertices is X, U, Y, Z or X, Z, Y, U. Otherwise, two edges are in *cyclic order*. These relations are clearly symmetric. Under bipartition of a set we understand its unordered partition into two subsets, one of which may be empty. We use the notation $||$ for a bipartition, and write $U||V$ to express that U and V belong to different classes, and $-||UV$ to express that U and V belong to the same class. For every edge $XY \in E(G)$, consider an arbitrary bipartition $||_{XY}$ of $V(G) \setminus \{X, Y\}$, and define now the relations O and P as follows:

$$O(XY, UZ) = \begin{cases} 1 & \text{if } XY \text{ and } UZ \text{ are in cyclic order,} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

$$P(XY,UZ) \quad = \quad \begin{cases} 1 & \text{if } U||_{XY}Z, \\ 0 & \text{otherwise, i.e. if } -||_{XY}UZ \text{ holds.} \end{cases} \tag{11}$$

Note that O does not change if we interchange the values of X and Y and/or U and Z, or the edge XY with the edge UZ. P does not change if we interchange the values of X and Y and/or U and Z, however, may change if we interchange the pair XY with the pair UZ. Let

$$\mathcal{B} = \{||_{XY} : \ XY \in E(G)\} \tag{12}$$

denote a set of bipartitions. Define

$$\begin{aligned} cr(XY,UZ) \quad &= \quad [1 - O(XY,UZ)][1 - P(XY,UZ)][1 - P(UZ,XY)] \\ &+ \quad [1 - O(XY,UZ)]P(XY,UZ)P(UZ,XY) \\ &+ \quad O(XY,UZ)[1 - P(XY,UZ)]P(UZ,XY) \\ &+ \quad O(XY,UZ)P(XY,UZ)][1 - P(UZ,XY)] \end{aligned} \tag{13}$$

Note that cr does not change if we interchange X and Y, U and Z, or XY and UZ.

Theorem 6.1 *We have*

$$CR\text{-}IODD(G) = \min_{\mathcal{B}} \sum_{XY,UZ \in E(G)} cr(XY,UZ), \tag{14}$$

where the summation goes for unordered pairs of non-adjacent edges, and the minimization goes for all possible sets of bipartitions.

Here we show a reformulation of Theorem 6.1 that we hope will be evaluated for complete graphs. This is just a quartic expression evaluated on ± 1 values, which is highly symmetric in the case of complete graphs.

Theorem 6.2 *Let N denote the number of unordered pairs of non-adjacent edges in G. For every $AB \in E(G)$ let $Q_{AB} : \ V(G) \setminus \{A,B\} \to \{-1,+1\}$ be a function , such that $Q_{AB} = Q_{BA}$, and $\mathcal{Q} = \{Q_{AB} : \ AB \in E(G)\}$. Now we have $CR\text{-}IODD(G) =$*

$$\frac{N}{2} - \max_{\mathcal{Q}} \sum_{XY,UZ \in E(G)} O(XY,UZ)Q_{XY}(U)Q_{XY}(Z)Q_{UZ}(X)Q_{UZ}(Y), \tag{15}$$

where the summation goes for unordered pairs of non-adjacent edges.

6.3 Proof to Chojnacki's Theorem 1.1

In order to show that Theorem 6.1 is a promising combinatorial approach to crossing numbers, we give an unexpectedly purely combinatorial proof to Chojnacki's Theorem 1.1.

Proof. We need to prove $CR\text{-}IODD \geq 1$. Consider the vertices of K_5 in the cyclic order $1,2,3,4,5$. Assume that (14) equals zero. Then, the summation has to be termwise zero, for all pairs of non-adjacent edges all four terms in every cr formula. (All summations in this proof, even if not spelled out, go for non-adjacent edges.) Observe that the summation (14) goes for 15 unordered pairs of non-adjacent edges. Note that for 10 of them $O = 1$, and and for 5 of them $O = 0$. Hence we have in (14) for a certain P, resulting from bipartitions,

$$\begin{aligned} 0 &= \sum_{\substack{\{XY,UZ\}\\ O(XY,UZ)=0}} \Big([1 - P(XY,UZ)][1 - P(UZ,XY)] \\ &\qquad + P(XY,UZ)P(UZ,XY) \Big) \\ &+ \sum_{\substack{\{XY,UZ\}\\ O(XY,UZ)=1}} \Big([1 - P(XY,UZ)P(UZ,XY) \\ &\qquad + P(XY,UZ)][1 - P(UZ,XY)] \Big), \end{aligned} \tag{16}$$

where the summations still go for unordered pairs of unordered pairs. Using the abbreviation $P_1 = P(XY,UZ)$ and $P_2 = P(UZ,XY)$, a generic term in the first sum is $1 + 2P_1P_2 - P_1 - P_2$, while a generic term in the second sum is $-2P_1P_2 + P_1 + P_2$. Therefore, taking (16) mod 2 yields

$$0 \equiv 5 + \sum_{XY}\sum_{UZ} P(XY,UZ) \qquad \text{mod } 2. \tag{17}$$

We are going to evaluate the summation in the right-hand side of (17) in a different way. Observe that any bipartition $||_{XY}$ of 3 elements is either 3:0 or 2:1, and in both of them the number of separated pairs is even. Recall that $P(XY,UZ) = 1$ iff $||_{XY}$ has U and Z in different classes. Therefore for an arbitrary XY, $\sum_{UZ} P(XY,UZ) = 2k_{XY}$. ¿From here,

$$\sum_{XY}\sum_{UZ} P(XY,UZ) = 2k. \tag{18}$$

Substituting (18) into the right-hand side of (17), we obtain $0 \equiv 1$ mod 2 by (18), a contradiction.

In order to prove the theorem for $K_{3,3}$, start with a copy of $K_{3,3}$ in which the colorclasses are $\{1,3,5\}$ and $\{2,4,6\}$, respectively. Put the vertices into cyclic order 1,2,3,4,5,6. $K_{3,3}$ has 9 edges, and 18 unordered pairs of non-adjacent edges. It is easy to see that for 3 of them $O = 0$ and for 15 of them $O = 1$. We call the colorclasses of $K_{3,3}$ Red and White vertices. We repeat a slight variation of the counting argument above. Formula (14) turns into an 18 term summation, and we end up with an analogue of (16), where the first summation goes for $\{XY,UZ\}: \ XY \in E(K_{3,3}), UZ \in E(K_{3,3}), O(XY,UZ) = 0$, and the second summation goes for $\{XY,UZ\}: \ XY \in E(K_{3,3}), UZ \in E(K_{3,3}), O(XY,UZ) = 1$. Again, we are going to do calculations mod 2. First we need that for an arbitrary $XY \in E(K_{3,3})$,

$$\sum_{UZ \in E(K_{3,3})} P(XY,UZ) = 2k_{XY}. \tag{19}$$

To prove (19), we study how many Red-White vertex pairs a bipartition P can separate. The possibilities are $RR||WW$, $RW||RW$, $R||RWW$, $W||WRR$,

$-||RRWW$; in each case the number of separated Red-White vertex pairs is even. Summing up (19), we obtain

$$\sum_{XY \in E(K_{3,3})} \sum_{UZ \in E(K_{3,3})} P(XY, UZ) = 2k. \tag{20}$$

Evaluating the analogue of (16) mod 2, we obtain the analogue of (17):

$$0 \equiv 3 + \sum_{XY \in E(K_{3,3})} \sum_{UZ \in E(K_{3,3})} P(XY, UZ) \mod 2 \tag{21}$$

by and (20) we have a contradiction again.

7 Proofs

Let us be given four points A, B, C, D in this cyclic order on a circle S in the plane $\mathbb{R}^2$. Call a, b, c, d the four rays, perpendicular to S, which connect the points A, B, C, D to ∞ and stay outside the circle. Then, a, d, c, b is the cyclic order of the rays at ∞. Assume now that $X, Y \in \{A, B, C, D\}$, $X \neq Y$ are connected by a simple curve q. This simple curve may intersect the rays x and y several times, however, we assume that at intersection points different from X and Y "crossing" happens, i.e. "tangential situation" is not allowed. The following Lemma is easy to see:

Lemma 7.1 *Consider $U, Z \notin q \cup x \cup y$, $U \neq Z$, and a simple curve r connecting U and Z, such that it does not pass through any of the points X, Y, ∞, and intersection points of x, y, and q, and not "tangential" to q. Then, the parity of the number of crossings of r and q is independent of the selection of r.*

We say that a simple curve q (as above) *separates* U and Z, if q and r crosses odd number of times, when r is selected according to the conditions of Lemma 7.1. Another easy Lemma claims that

Lemma 7.2 *"Not being separated by q" is an equivalence relation on $\mathbb{R}^2 \setminus \{q \cup x \cup y\}$ with two classes.*

We use the notation $U|_q Z$ if U and Z are in different equivalence classes, and $-|_q UZ$, if U and Z are in the same equivalence class. ¿From now on we use this notation only for $\{U, Z\} = \{A, B, C, D\} \setminus \{X, Y\}$.

Assume now that another simple curve, p, connects U and Z, with similar conditions that q satisfied for X and Y. Assume that p and q have finitely many points in common, and at any points in common the curves cross, i.e. no touching situation is allowed. Now we claim the following simple Lemma.

Lemma 7.3 *If the cyclic order induced by S on the four points is X, U, Y, Z or X, Z, Y, U, and either*

$$-|_q UZ \quad \textit{and} \quad -|_p XY \qquad \textit{or}$$
$$U|_q Z \quad \textit{and} \quad X|_p Y,$$

then p and q crosses odd many times. If the cyclic order induced by S on the four points is X, Y, U, Z, X, Y, Z, U, X, Z, U, Y or X, U, Z, Y, and either

$$-|_q UZ \quad and \quad X|_p Y \qquad or$$
$$U|_q Z \quad and \quad -|_p XY,$$

then p and q crosses, again, odd many times.

Note that the conditions in the two parts of Lemma 7.3 read in terms of Section 6 *as if* we had XY and UZ edges in a graph, and they were in acyclic order and cyclic order, respectively. We freely extend the O relation in (10) for unordered pairs of unordered pairs of points on S, all four points distinct. We introduce

$$P^*(XY, UZ) = \begin{cases} 1 & \text{if } U|_q Z, \\ 0 & \text{otherwise, i.e. if } -|_q UZ \text{ holds.} \end{cases} \tag{22}$$

Lemma 7.3 immediately implies the next Lemma:

Lemma 7.4 *The value of the quantity*

$$\begin{aligned} CR(p,q) &= [1 - O(XY,UZ)][1 - P^*(XY,UZ)][1 - P^*(UZ,XY)] \\ &+ [1 - O(XY,UZ)]P^*(XY,UZ)P^*(UZ,XY) \\ &+ O(XY,UZ)[1 - P^*(XY,UZ)]P^*(UZ,XY) \\ &+ O(XY,UZ)P^*(XY,UZ)][1 - P^*(UZ,XY)] \end{aligned} \tag{23}$$

is 1, if p and q crosses odd times, and 0 otherwise.

Proof to Theorem 6.1: Any drawing of graph G can be transformed, without changing which edges cross how many times, into a new drawing, such that the vertices $A_1, A_2, ..., A_n$ are in this cyclic order on a circle S. ¿From now on we study such a drawing D. Lemma 7.4 above applies to any pair of non-adjacent edges of G, where edge $XY \in E(G)$ represented by curve q, and edge UZ is represented by curve p in the drawing. Then the actual number of unordered pairs of non-adjacent pairs of edges, which cross odd times, is

$$\sum CR(p,q), \tag{24}$$

where the summation goes for non-adjacent pairs of edges, p, q are the curves realizing two generic non-adjacent edges in the drawing. Observe that every drawing induces a set of bipartitions as required in (12), and every set of bipartitions in (12) easily can be induced by a graph drawing under the correspondence which defines $\mathcal{B}$ by $||_{XY} = |_q$. Under this equivalence formula (24) coincides with the minimum of (14) over all drawings, and we finished the proof.
Proof to Theorem 6.2: Write $P_{XY}(UZ) = 1$, if $U||_{XY}Z$, and $P_{XY}(UZ) = -1$ otherwise. It is easy to see from (13) that

$$\begin{aligned} cr(XY,UZ) = [1 \quad &- \quad O(XY,UZ)]\frac{1 + P_{XY}(UZ)P_{UZ}(XY)}{2} \\ &+ \quad O(XY,UZ)\frac{1 - P_{XY}(UZ)P_{UZ}(XY)}{2}. \end{aligned} \tag{25}$$

Observe that P_{XY}, which is defined on pairs of vertices, can be written in terms of Q_{XY}, which is defined on vertices, such that $Q_{XY} = 1$ on one class of the bipartition $||_{XY}$, and $Q_{XY} = -1$ on the other class, since then $P_{XY}(UZ) = Q_{XY}(U)Q_{XY}(Z)$. There is a bijective correspondence between $\mathcal{B}$'s and equivalence classes of $\mathcal{Q}$'s, where $\mathcal{Q}\sim \mathcal{Q}'$ if and only for all edges AB, $Q_{AB} = \pm Q'_{AB}$. Rewriting (25) in terms of $\mathcal{Q}$, we obtain (15).

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