Constructing complete projectively flat connections

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CONSTRUCTING COMPLETE PROJECTIVELY FLAT CONNECTIONS

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ABSTRACT. On any open subset $U$ of the Euclidean space $\mathbb{R}^n$ there is complete torsion free connection whose geodesics are reparameterizations of the intersections of the straight lines of $\mathbb{R}^n$ with $U$. For any positive integer $m$ there is a complete projectively flat torsion free connection on the two dimensional torus such that for any point $p$ there is another point $q$ so that any broken geodesic from $p$ to $q$ has at least $m$ breaks. This example is also homogeneous with respect to a transitive Lie group action.

1. Introduction.

The propose of this note is to tie up a couple of loose ends in the classical theory of linear connections. First, in [6, p. 395], Spivak rises the question of if, on a compact manifold with complete connection, any two points can be joined by a geodesic. The answer is "no" even when the connection is projectively flat and homogeneous:

Theorem 1. Let $T^2$ be the two dimensional torus. Then for any positive integer $m$ there is a complete torsion free projectively flat connection, $\nabla$, on $T^2$ such that for any point $p \in T^2$ there is a point $q \in T^2$ with the property that any broken $\nabla$-geodesic between $p$ and $q$ has at least $m$ breaks. Moreover if $T^2$ is viewed as a Lie group in the usual manner, this connection is invariant under translations by elements of $T^2$.

Another natural question is: For a connected open subset, $U$, of the Euclidean space, $\mathbb{R}^n$, is the usual flat connection restricted to $U$ projectively equivalent to complete torsion free connection on $U$? This is true and is a special case of a more general result about connections on incomplete Riemannian manifolds.

Theorem 2. Let $(M, g)$ be a not necessarily complete Riemannian manifold. Then there is a complete torsion free connection on $M$ that is projective with the metric connection on $M$. In particular any connected open subset $M$ of the Euclidean space, $\mathbb{R}^n$, has a complete torsion free connection $\nabla$ such that the geodesics of $\nabla$ are reparameterizations of straight line segments of $M \subseteq \mathbb{R}^n$.

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The main tool is Proposition 2.2 which gives an elementary method of constructing complete torsion free connections that are projective with a given torsion free connection.

1.1. Definitions, notation and preliminaries. All of our manifolds are smooth (i.e. $C^\infty$), Hausdorff, paracompact, and connected. The tangent bundle of $M$ is denoted by $T(M)$. If $f: M \to N$ is a smooth map between manifolds, then the derivative map is $f_*: T(M)_x \to T(M)_{f(x)}$.

We will use the term connection to stand for a linear connection on the tangent bundle (also called a Koszul connection) as defined in [4, Prop. 2.8 p. 123 and Prop. 7.5 p. 143] or [6, p. 241]. Let $c: (a, b) \to M$ be a smooth immersed curve. Then $c$ is a $\nabla$-geodesic iff $\nabla_{c'(t)} c'(t) = 0$. The curve is a $\nabla$-pregeodesic iff there is a reparameterization of $c$ that is a geodesic. This is equivalent to $\nabla_{\omega'(t)} c'(t) = \alpha(t) c'(t)$ for some smooth function $\alpha: (a, b) \to \mathbb{R}$. Given a pregeodesic $c: (a, b) \to M$ then an affine parameterization of $c$ is a reparameterization $\sigma: (a_1, b_1) \to (a, b)$ so that $c \circ \sigma$ is a geodesic.

If $f: M \to N$ is a local diffeomorphism and $\nabla$ is a connection on $N$ then the pull back connection is the connection $f^*\nabla$ defined on $M$ by $f_*((f^*\nabla)_X Y) = \nabla_{f_*X} f_*Y$. The connection $\nabla$ on $M$ is homogeneous on $M$ iff there is a transitive action on $M$ by a Lie group, $G$, so that $\varphi^*\nabla = \nabla$ for all $\varphi \in G$.

Two connections $\nabla$ and $\tilde{\nabla}$ on $M$ are projective iff all geodesics of $\nabla$ are pregeodesics of $\tilde{\nabla}$. This is an equivalence relation on the set of connections on $M$. If $\nabla_i$ is a connection on $M_i$ for $i = 1, 2$ then a map $f: M_1 \to M_2$ is a projective map iff it is a local diffeomorphism and maps $\nabla_1$-geodesics to $\nabla_2$-pregeodesics. This is equivalent to the connections $\nabla_1$ and $f^*\nabla_2$ on $M_1$ being projective. The connection $\nabla$ is projectively flat iff every point $p \in M$ has an open neighborhood $U$ and projective map $f: U \to \mathbb{R}^n$ where $\mathbb{R}^n$ has its standard flat connection. Or what, is the same thing for every geodesic $c$ of $M$ the image $f \circ c$ is a reparameterization of interval in a line of $\mathbb{R}^n$. There is a well known criterion, due to Hermann Weyl, for two connections to be projective. A proof can be found in [6, Cor 19 p. 277].

1.1. Proposition (H. Weyl). Two connections $\nabla$ and $\tilde{\nabla}$ on a manifold are projective and have the same torsion tensor if and only if there is a smooth one form $\omega$ so that the connections are related by

$$\nabla_X Y = \tilde{\nabla}_X Y + \omega(X) Y + \omega(Y) X.$$  

Therefore if this relation holds and $\nabla$ is torsion free, then so is $\tilde{\nabla}$.  

Only the easy direction of this result will be used. That is if $\nabla$ is torsion free and $\nabla$ is given by (1.1) then $\tilde{\nabla}$ is torsion free and projective with $\tilde{\nabla}$. Note in this case if $c: (a, b) \to M$ is a $\nabla$-geodesic then (1.1) implies $\nabla_{c'(t)} c'(t) = 2\omega(c'(t)) c'(t)$ and therefore $c$ is a $\tilde{\nabla}$-pregeodesic. That $\tilde{\nabla}$ is torsion free is equally as elementary.

The connection $\nabla$ is complete iff every $\nabla$-geodesic defined on a subinterval of $\mathbb{R}$ extends to a $\nabla$-geodesic defined on all of $\mathbb{R}$. Letting $\exp^\nabla$ be the
exponential of $\nabla$ (cf. [4, p. 140]), then $\nabla$ is easily seen to be complete if and only if the domain of $\exp^\nabla$ is all of $T(M)$. A curve $c: [0, b) \to M$ is an \textit{inextendible} $\nabla$-geodesic ray iff $c$ is a $\nabla$-geodesic and has no extension to $[0, b + \varepsilon)$ for any $\varepsilon > 0$. Therefore when $b = \infty$, so that $[0, \infty)$ is the domain of $c$, $c$ is always inextendible.

1.2. Proposition. Let $\nabla$ be a torsion free connection on the manifold $M$ and let $\nabla$ be torsion free and projective with $\nabla$. Then $\nabla$ is complete if and only if every inextendible $\nabla$-geodesic ray $c: [0, b) \to M$ has an orientation preserving reparameterization $\sigma: [0, \infty) \to [0, b)$ such that $\sigma \circ \sigma$ is a $\nabla$-geodesic.

Proof. First assume that the reparameterization condition holds and we will show that $\nabla$ is complete by showing the domain of the exponential map of $\nabla$ is all of $T(M)$. Let $v \in T(M)$. As $0$ is in the domain of $\exp^\nabla$, assume $v \neq 0$. Let $c: [0, b) \to M$ be the inextendible $\nabla$-geodesic ray with $c'(0) = v$. By assumption there is an orientation preserving reparameterization $\sigma: [0, \infty) \to [0, b)$ such that $\hat{c} := c \circ \sigma$ is a $\nabla$-geodesic. As the reparameterization is orientation preserving $\hat{c}'(0) = \lambda c'(0) = \lambda v$ for some positive constant $\lambda$. Then $\hat{c}: [0, \infty) \to M$ given by $\hat{c}(t) := \hat{c}(t/\lambda)$ is also a $\nabla$-geodesic and $\hat{c}'(0) = v$. From the definition of $\exp^\nabla$ we have for all $t \geq 0$ that $tv$ is in the domain of $\exp^\nabla$ and $\exp^\nabla(tv) = \hat{c}(t)$. In particular letting $t = 1$ shows that $v$ is in the domain of $\exp^\nabla$ and completes the proof that $\nabla$ is complete.

Conversely assume $\nabla$ is complete and let $c: [0, b) \to M$ be an inextendible $\nabla$-geodesic ray. Assume, toward a contradiction, there is an orientation preserving reparameterization $\sigma: [0, b_1) \to [0, b)$ with $b_1 < \infty$ and so that $\hat{c} = c \circ \sigma$ is a $\nabla$ geodesic. Then, as $\nabla$ is complete, the curve $\hat{c}$ extends to a $\nabla$-geodesic $\hat{c}: [0, \infty) \to M$ and therefore is a proper extension of $\hat{c}$. But then $\hat{c}$ can be reparameterized as a $\nabla$-geodesic that extends $c$, contradicting that $c$ was an inextendible $\nabla$-geodesic ray and completing the proof.

2. Constructing complete projectively equivalent connections on incomplete Riemannian manifolds.

We first observe that for some choices of the one form $\omega$ in Weyl’s result 1.1 there is an explicit formula for reparameterizing a $\nabla$-geodesic as a $\nabla$-geodesic.

2.1. Lemma. Let $\nabla$ be a smooth manifold and let $\nabla$ be a connection on $M$ and let $\nu: M \to (0, \infty)$ be a smooth positive function. Define a new connection by

$$(2.1) \quad \nabla_X Y = \nabla_X Y + \frac{1}{2\nu}d\nu(X)Y + \frac{1}{2\nu}d\nu(Y)X$$

Let $c: (a, b) \to M$ be a $\nabla$-geodesic and $\sigma: (\alpha, \beta) \to (a, b)$ an orientation preserving reparameterization of $c$ so that $\tilde{c} = c \circ \sigma$ is a $\nabla$-geodesic. Then
the inverse of \( \sigma, \sigma^{-1} : (a, b) \rightarrow (\alpha, \beta) \), is given by

\[
(2.2) \quad \sigma^{-1}(t) = C_0 + C_1 \int_{t_0}^t \nu(c(\tau)) \, d\tau
\]

where \( t_0 \in (a, b) \), \( C_0, C_1 \in \mathbb{R} \) and \( C_1 > 0 \).

**Proof.** Let \( t \) be the natural coordinate on \((a, b)\) and \( s \) the coordinate on \((\alpha, \beta)\) related to \( t \) by \( t = \sigma(s) \). Our goal is to find \( s = s(t) = \sigma^{-1}(t) \). Note \( dt = \sigma'(s) \, ds \) so that \( \sigma'(s) = \frac{ds}{dt} \). Therefore

\[
c'(s) = (c \circ \sigma)'(s) = \sigma'(s) c'(\sigma(s)) = \frac{dt}{ds} \frac{dc}{ds} \bigg|_{s=\sigma^{-1}(t)}.
\]

Because of this, and because it makes applications of the chain rule easier to follow, we will denote \( c'(s) \) as \( \frac{dc}{ds} \) and think of \( s \) as “the affine parameter for \( \nabla \) along \( c \).” We will abuse notation a bit and write \( v(t) = \nu(c(t)) \). As \( \nabla \frac{dc}{dt} = - \nabla c'(t)c'(t) = 0 \), we have using (2.1) that \( \nabla \frac{dc}{dt} = \frac{dc}{ds} \nabla \frac{ds}{dt} = 0 \), and

\[
0 = \frac{dc}{dt} = \frac{dc}{ds} \frac{ds}{dt} = \frac{1}{v} \frac{dv}{ds} \frac{dc}{ds} + \frac{d}{ds} \left( \frac{dt}{ds} \right) = \frac{d}{ds} \left( \frac{dt}{ds} \right) = \frac{d}{ds} \left( \frac{dt}{ds} \right) = \frac{d}{ds} \left( \frac{dt}{ds} \right) = \frac{d}{ds} \left( \frac{dt}{ds} \right) = \frac{d}{ds} \left( \frac{dt}{ds} \right) = \frac{d}{ds} \left( \frac{dt}{ds} \right)
\]

This shows that \( \ln \left( \frac{dt}{ds} \right) \), and therefore also \( \frac{dt}{ds} \), is constant. As \( v, \frac{dv}{ds} > 0 \) (the reparametrization is orientation preserving implies \( \frac{dv}{ds} = \sigma'(s) > 0 \)) there is a constant \( C_1 > 0 \) such that

\[
v(t) \frac{dt}{ds} = \frac{1}{C_1}.
\]

This differential equation can be integrated to give \( s(t) = \sigma^{-1}(t) \) as a function of \( t \) and the result is the required formula (2.2). \( \quad \Box \)

**2.2. Proposition.** Let \( M \) be a smooth manifold with smooth torsion free connection \( \nabla \) and let \( \nu : M \rightarrow (0, \infty) \) be a smooth positive function. Then the connection \( \nabla \) defined by (2.1) is a torsion free connection projective with \( \nabla \) and \( \nabla \) is complete if and only if for each inextendible \( \nabla \)-geodesic ray \( c : [0, b) \rightarrow M \) the growth condition

\[
(2.3) \quad \int_0^b \nu(c(t)) \, dt = \infty.
\]

holds.

**Proof.** That \( \nabla \) is projective to \( \nabla \) and torsion free follows from Proposition 1.1 using \( \omega = (2\nu)^{-1} dv \). So all that is left to check is that \( \nabla \) is complete if and only if (2.3) holds along inextendible \( \nabla \)-geodesic rays.
First assume that the growth condition \((2.3)\) holds along inextendible \(\nabla\)-geodesic rays. Let \(c: [0, b) \to M\) be be such a ray and let \(\sigma: [0, \beta) \to [0, b)\) be an orientation preserving reparameterization of \(c\) so that \(\tilde{c} = c \circ \sigma\) is a \(\nabla\)-geodesic. We claim that \(\beta = \infty\). By Lemma 2.1 \(\sigma^{-1}(t)\) is given by

\[
(2.4) \quad \sigma^{-1}(t) = C_1 \int_0^t v(c(\tau)) \, d\tau
\]

with \(C_1 > 0\). But then the growth condition \((2.3)\) implies \(\beta = C_1 \int_0^b v(c(\tau)) \, d\tau = \infty\). As \(c\) was any inextendible \(\nabla\)-geodesic ray, the completeness of \(\nabla\) follows from Proposition 1.2.

Conversely assume \(\nabla\) is complete and let \(c: [0, b) \to M\) be an inextendible \(\nabla\)-geodesic ray. Then by Proposition 1.2 there is an orientation preserving reparameterization \(\sigma: [0, \infty) \to [0, b)\) so that \(\tilde{c} = c \circ \sigma\) is a \(\nabla\)-geodesic. Again Lemma 2.1 implies that \(\sigma^{-1}\) is given by \((2.4)\). Therefore \(C_1 \int_0^b v(c(\tau)) \, d\tau = \lim_{t \to \infty} \sigma^{-1}(t) = \infty\) which shows that the condition \((2.3)\) holds along all inextendible \(\nabla\)-geodesic rays. \(\square\)

For a general connection, \(\nabla\), it is not clear how to choose a positive smooth function \(v\) so that the growth condition \((2.3)\) holds along all inextendible \(\nabla\)-geodesics rays. However when \(\nabla\) is the metric connection of a Riemannian metric the behavior of geodesics is closely related to the properties of the distance function of the metric and this can be exploited to find an appropriate \(v\).

**Proof of Theorem 2.** If \((M, g)\) is complete as a metric space, then the metric connection \(\nabla\) is complete (cf. [7, p. 462]) and taking \(\nabla = \nabla\) completes the proof. Therefore assume that \(M\) is incomplete. Let \(\overline{M}\) be the completion of \(M\) as a metric space and let \(\partial M = \overline{M} \setminus M\) be the boundary of \(M\) in \(\overline{M}\). For \(x \in M\) let \(\delta(x)\) be the distance of \(x\) from \(\partial M\). A standard partition of unity argument shows that there is a smooth function \(v\) on \(M\) so that

\[
v(x) \geq \max\{1, 1/\delta(x)\}
\]

for all \(x \in M\). Let \(c: [0, b) \to M\) be an inextendible \(\nabla\)-geodesic ray. There are two cases: \(b = \infty\) and \(b < \infty\). In the case \(b = \infty\), then from the definition of \(v\) we have \(v(c(t)) \geq 1\) and so \(\int_0^b v(c(t)) \, dt \geq \int_0^\infty 1 \, dt = \infty\) and the condition \((2.3)\) holds in this case.

In the second case, where \(b < \infty\), the length of the velocity vector \(c'(t)\) is constant and thus there is a constant \(C > 0\) so that for all \(t_1, t_2 \in [0, b)\) the distance \(d(c(t_1), c(t_2))\) between \(c(t_1)\) and \(c(t_2)\) satisfies

\[
d(c(t_1), c(t_2)) \leq C|t_2 - t_1|.
\]

Therefore in the completion \(\overline{M}\) the limit \(p = \lim_{t \to b} c(t)\) will exist and from the definition of \(\delta\) as the distance from the boundary \(\partial M\) the estimate
\[ \delta(c(t)) \leq d(c(t), p) \leq C|b - t| \text{ holds. This yields} \]
\[ \int_0^b v(c(t)) \, dt \geq \int_0^b \frac{dt}{\delta(c(t))} \geq \int_0^b \frac{dt}{C|b - t|} = \infty. \]

Thus (2.3) holds in all cases and therefore \( \nabla \) is complete by Proposition 2.2

2.3. Remark. In a complete Riemannian manifold any two points can be joined by a geodesic. For complete connections this is no longer true and Hicks [3] has constructed an example of a complete connection on a manifold, \( \tilde{M} \), so that for any positive integer \( m \) there are two points of \( \tilde{M} \) that not only can not be connected by a geodesic, but any broken geodesic between the points must have at least \( m \) breaks. For open sets \( U \) in \( \mathbb{R}^2 \) the behavior of geodesics is easy to visualize and, using Theorem 2, it is trivial to generate such examples that are also projectively flat. For example, set

\[ K := \bigcup_{k=-\infty}^{\infty} \{2k\} \times [-1, \infty) \cup \bigcup_{k=-\infty}^{\infty} \{2k + 1\} \times (-\infty, 1], \]
which is a union of rays parallel to the \( y \)-axis, and let \( U = \mathbb{R}^2 \setminus K \) (See Figure 1). Use Theorem 1 to put a complete projectively flat connection on \( U \) that has line segments as its geodesics and polygonal paths as its broken geodesics. With this connection \( U \) has the property that any broken geodesic between the points \((1/2, 0)\) and \((m + 1/2, 0)\) must have at least \( m + 1 \) corners.

3. Homogeneous examples

Before specializing to two dimensions for the proof of Theorem 1 we do the preliminary calculations in arbitrary dimensions. This leads to higher dimensional examples.

Let \( \nabla \) be the standard flat connection on \( \mathbb{R}^n \) and let \( U := \mathbb{R}^n \setminus \{0\} \) be \( \mathbb{R}^n \) with the origin deleted. Then any nonsingular linear map \( A: \mathbb{R}^n \rightarrow \mathbb{R}^n \) preserves the connection \( \nabla \) and therefore the general linear group \( \text{GL}(n, \mathbb{R}) \) has a transitive action on \( U \) that preserves \( \nabla \). Let \( O(n) \) be the orthogonal group of the standard inner product, \( \langle , \rangle \), on \( \mathbb{R}^n \) and let \( \mathbb{R}^+ \) be the multiplicative group of positive real numbers. Let \( G \) be the product group \( G = O(n) \times \mathbb{R}^+ \). View \( G \) as a subgroup of \( \text{GL}(n, \mathbb{R}) \) by letting it act on \( \mathbb{R}^n \) by \( (P, c)x = cx \). This action of \( G \) is transitive on \( U \) and preserves the connection \( \nabla \). Let \( v: U \rightarrow (0, \infty) \) be the function \( v(x) = 1/\|x\| \). Then, if \( g = (P, c) \in G \), the
pull back of \( v \) by \( g \) is 
\[
(g^*v)(x) = v(gx) = \|cPx\|^{-1} = c^{-1}\|x\|^{-1} = c^{-1}v(x)
\]
as \( P \in O(n) \) so that \( \|Px\| = \|x\| \). The pull back of the one form \( dv/v \) is 
\[
g^* \left( \frac{dv}{v} \right) = g^*\frac{dv}{g^*v} = \frac{d(g^*v)}{g^*v} = \frac{d(c^{-1}v)}{c^{-1}v} = \frac{dv}{v}
\]
and so \( dv/v \) is invariant under the action of \( G \). Therefore if we define a 
connection \( \nabla \) on \( U \) by 
\[
(3.1) \quad \nabla_X Y = \nabla_X Y + \nabla_X Y + \frac{1}{2v}\left( dv(X)Y + dv(Y)X \right) \quad \text{with} \quad v(x) = \frac{1}{\|x\|}
\]
then \( \nabla \) will be invariant under the action of the group \( G \). The inextendible 
\( \nabla \)-geodesic rays in \( U \) are the curves \( c: [0, b) \to U \) given by \( c(t) = x_0 + tx_1 \)
where \( x_1 \neq 0 \) and either \( b = \infty \) or \( c(b) := \lim_{t \to b} c(t) = 0 \). In either case it 
is easy to check that \( \int_0^b v(c(t)) \, dt = \infty \) and therefore by Proposition 2.2 the 
connection \( \nabla \) is complete and projectively flat on \( U \).

To get compact examples let \( \lambda > 1 \) and let \( \Gamma \) be the cyclic subgroup of 
\( G \) given by \( \Gamma := \{(I, \lambda^k) : k \in \mathbb{Z}\} \) where \( \mathbb{Z} \) is the integers. The action of 
\( \Gamma \) on \( U \) is fixed point free and properly discontinuous and therefore if \( M \) is 
defined to be the quotient space \( M := \Gamma \backslash U \) then \( M \) is a smooth manifold 
(cf. [1, Thm 8.3 p. 97]) and it is not hard to see that \( M \) is diffeomorphic to 
\( S^{n-1} \times S^1 \). Let \( \pi: U \to M \) be the natural projection. Then \( \pi \) is a projective 
map and \( \Gamma \) is the group of deck transformations. As the connection \( \nabla \) is 
invariant under these transformations it follows there is a unique connection 
\( \nabla^M \) on \( M \) so that \( \pi^*\nabla^M = \nabla \). The \( \nabla^M \)-geodesics on \( M \) are \( \pi \circ c \) where \( c \) 
is a \( \nabla \)-geodesic on \( U \). As the \( \nabla \)-geodesics in \( U \) are complete, it follows that 
the \( \nabla^M \) geodesics in \( M \) are complete. Also this implies that \( \pi \) is a projective 
map and therefore \( \nabla^M \) is projectively flat on \( M \).

For any \( g = (P, c) \in G \) and \( a = (I, \lambda^k) \in \Gamma \) we have \( ag = ga \). As 
for \( x \in U \) the image \( \pi(x) \) is the orbit \( \pi(x) = \Gamma x \) we see for \( g \in \Gamma \) that 
\( \pi(gx) = \Gamma gx = g\Gamma x = g\pi(x) \). Therefore there is a well defined action of 
\( G \) on \( M \) given by \( g\pi(x) = \pi(gx) \). This action is transitive on \( M \) as \( G \) is 
transitive on \( U \).

We now claim that if \( x \in U \) and \( y = -\alpha x \) for \( \alpha > 0 \), then there is no 
geodesic from \( \pi(x) \) to \( \pi(y) \) in \( M \). Assume, toward a contradiction, that 
there is a geodesic \( c: [a, b] \to M \) with \( c(a) = \pi(x) \) and \( c(b) = \pi(y) \). Then 
there is is a unique geodesic \( \tilde{c}: [a, b] \to U \) with \( \tilde{c}(a) = x \) and \( \pi \circ \tilde{c} = c \). 
Therefore \( \pi(c(b)) = c(b) = \pi(y) \) which implies that \( \tilde{c}(b) = ay \) for some 
\( a \in \Gamma \). From the definition of \( \Gamma \) this implies that for some \( k \in \mathbb{Z} \) that 
\( \tilde{c}(b) = \lambda^k y = -\lambda^k x \). But as \( \nabla \) is projective with the flat metric \( \nabla \) the 
geodesics segments of \( \nabla \) are reparameterizations of straight line segments in \( U \). But then \( \tilde{c} \) is a reparameterization of a straight line segment of \( U \) 
form \( \tilde{c}(a) = x \) to \( \tilde{c}(b) = -\lambda^k x \), which is impossible as \( \lambda^k \alpha > 0 \) so that any 
line segment connecting these points must pass through the origin, which is 
not in \( U \). This contradiction verifies our claim that there is no geodesic
of $M$ from $\pi(x)$ to $\pi(y)$. Letting $\alpha$ vary over the positive real numbers we get uncountably many points $\pi(y)$ that cannot be connected to $\pi(x)$ by a geodesic. As every point $p \in M$ is of the form $p = \pi(x)$ this can be summarized as:

3.1. Proposition. Let $M = \Gamma \backslash U$ and $\nabla^M$ be the manifold and connection just constructed. Then $M$ is diffeomorphic to $S^{n-1} \times S^1$ and the connection $\nabla^M$ on $M$ is complete, projectively flat and with homogeneous with respect to the group action of $G$ on $M$. For any $p \in M$ there are uncountable many points $q$ that cannot be connected to $p$ by a $\nabla^M$-geodesic. □

3.1. Proof of Theorem 1. In the case that $n = 2$ it is possible to be more explicit. On $U = \mathbb{R}^2 \setminus \{0\}$ there are several sets of coordinates that will be convenient to use. First the standard Euclidean coordinates $x$ and $y$. With respect to these coordinates the standard flat connection $\nabla$ is given by

$$\nabla \frac{\partial}{\partial x} = \nabla \frac{\partial}{\partial y} = \nabla \frac{\partial}{\partial x} = \nabla \frac{\partial}{\partial y} = 0.$$

The simply connected covering space, $\hat{U}$, of $U$ is diffeomorphic to $\mathbb{R}^2$. Using polar coordinates $r, \theta$ on $\hat{U}$ (with $(r, \theta) \in (0, \infty) \times \mathbb{R}$) we have the usual formula for the covering map: $x = r \cos \theta$ and $y = r \sin \theta$. In polar coordinates the connection is given by

$$\nabla \frac{\partial}{\partial r} = 0, \quad \nabla \frac{\partial}{\partial \theta} = \nabla \frac{\partial}{\partial \theta} = 1 \frac{\partial}{\partial r}, \quad \nabla \frac{\partial}{\partial \theta} = -r \frac{\partial}{\partial r}.$$

(More precisely this is the pull back of the connection $\nabla$ to $\hat{U}$ by the covering map. We will still denote this connection by $\nabla$.) The function $v = \|(x, y)\|^{-1}$ used in the definition (3.1) of the connection $\nabla$ is given in polar coordinates a $v = r^{-1}$. Then $dv = -r^{-2}dr$. Using this in (3.1) gives

$$\nabla_X Y = \nabla_X Y - \frac{1}{r} (dr(X) Y + dr(Y) X)$$

and therefore $\nabla$ is given explicitly in polar coordinates as

$$\nabla \frac{\partial}{\partial r} = -\frac{1}{r} \frac{\partial}{\partial r}, \quad \nabla \frac{\partial}{\partial \theta} = \nabla \frac{\partial}{\partial \theta} = \frac{1}{2r} \frac{\partial}{\partial \theta}, \quad \nabla \frac{\partial}{\partial \theta} = -r \frac{\partial}{\partial r}.$$

The formulas for $\nabla$ simplify even farther if we replace the coordinate $r$ on $\hat{U}$ by $\rho$ related to $r$ by $r = e^\rho$. The vector field $\frac{\partial}{\partial r}$ is related to the vector field $\frac{\partial}{\partial \rho}$ by $\frac{\partial}{\partial r} = r \frac{\partial}{\partial \rho}$ and $\frac{\partial}{\partial r} = e^{-\rho} \frac{\partial}{\partial \rho}$. Therefore in the coordinates $\rho, \theta$ the connection $\nabla$ is given by

$$\nabla \frac{\partial}{\partial \rho} = 0, \quad \nabla \frac{\partial}{\partial \theta} = \nabla \frac{\partial}{\partial \theta} = \frac{1}{\rho} \frac{\partial}{\partial \theta}, \quad \nabla \frac{\partial}{\partial \theta} = -\frac{\partial}{\partial \rho}.$$

This explicit form of the connection $\nabla$ makes it clear that it is invariant under translations $\rho \mapsto \rho + a$ and $\theta \mapsto \theta + b$. From the construction $\nabla$ is complete and projectively flat.

Using the coordinates $\rho$ and $\theta$ and letting $\mathbb{Z}$ be the integers, then the original open set $U$ is naturally identified with the quotient group $\mathbb{R}^2/(\{0\} \times 2\pi \mathbb{Z})$ (that is identify $(\rho, \theta)$ with $(\rho, \theta + 2k\pi)$ for $k \in \mathbb{Z}$). As in the original
set $U$ the $\nabla$-geodesics are reparameterized line segments it is not hard to see that a point $z \in U$ can be connected to a point $z_0$ on the positive real axis by a $\nabla$-geodesic if and only if $z$ is not on the negative real axis. That is $z$ can be connected to $z_0$ by a $\nabla$-geodesic if and only if $|\theta(z)| < \pi$. (See Figure 2.) But because of the homogeneity of the connection with respect to translations $\theta \mapsto \theta + b$ this implies:

3.2. Lemma. Two points $z_1, z_2 \in \bar{U}$ can be connected by a $\nabla$-geodesic if and only if $|\theta(z_1) - \theta(z_2)| < \pi$. Therefore if $z_1, z_2$ satisfy $|\theta(z_1) - \theta(z_2)| \geq m\pi$ for some positive integer $m$ any piecewise broken geodesic from $z_1$ to $z_2$ must have at least $m$ breaks.

3.3. Remark. There is a less geometric, but possibly more informative, proof of this lemma. Using the coordinates $\rho, \theta$ on $\bar{U}$ and the coordinates $x, y$ on $U$, the covering map from $\bar{U}$ to $U$ is given by $x = e^\rho \cos \theta$ and $y = e^\rho \sin \theta$. In $U$ the $\nabla$-geodesics are reparameterization of straight lines and thus along a $\nabla$-geodesic the coordinates $x$ and $y$ are related by $ax + by = 0$ (if geodesic goes through the origin) or $ax + by = 1$ (if it does not pass through the origin).

The first case leads to a relation between $\rho$ and $\theta$ of the form $e^\rho (a \cos \theta + b \sin \theta) = 0$ along the geodesic which implies $\theta = \theta_0$ on the geodesic, for some constant $\theta_0$. In the second case we get $e^\rho (a \cos \theta + b \sin \theta) = 1$ along the geodesic. Let $A = \sqrt{a^2 + b^2}$ and let $\alpha$ be so that $A \cos \alpha = a$ and $A \sin \alpha = b$. Then the equation between $\rho$ and $\theta$ becomes $e^\rho A \cos (\theta - \alpha) = 1$.

From this it follows that given a point in $\bar{U}$ with coordinates $(\rho_0, \theta_0)$ the $\nabla$-geodesics of $\bar{U}$ through this point are the line $\theta = \theta_0$ and the curves defined for $|\theta - \alpha| < \pi/2$ by the equation

$$e^\rho \cos(\theta - \alpha) = e^{\rho_0} \cos(\theta_0 - \alpha) \tag{3.2}$$

where $\alpha$ varies over real numbers with $|\alpha - \theta_0| < \pi/2$. This makes it clear a point $(\rho_1, \theta_1)$ with $|\theta_1 - \theta_0| \geq \pi$ cannot be on a geodesic through $(\rho_0, \theta_0)$. And conversely if $|\theta_1 - \theta_0| < \pi$ then either $\theta_1 \neq \theta_0$, and the points are both on the geodesic $\theta = \theta_0$, or $\theta_1 = \theta_0$ and straightforward calculus argument shows that there is a unique $\alpha \in (\theta_0 - \pi/2, \theta_0 + \pi/2) \cap (\theta_1 - \pi/2, \theta_1 + \pi/2)$ so that $e^{\rho_1} \cos(\theta_1 - \alpha) = e^{\rho_1} \cos(\theta_0 - \alpha)$. For this choice of $\alpha$ both of the points $(\rho_0, \theta_0)$ and $(\rho_1, \theta_1)$ will be on the $\nabla$-geodesic defined by (3.2).
We now complete the proof of Theorem 1. Given the positive integer \( m \) let \( k \) be an integer with \( k \geq m \). Let \( T^2 \) be the torus

\[
T^2 = \hat{U}/(\mathbb{Z} \times 2\pi k \mathbb{Z})
\]

(that is identify \((\rho, \theta)\) with \((\rho + j, \theta + 2\pi k\ell)\) for \( j, \ell \in \mathbb{Z} \)). As the connection \( \nabla \) is translation invariant it well defined as a connection on \( T^2 \) and will be invariant under translations of \( T^2 \) when \( T^2 \) is viewed as a Lie group. We have already seen that \( \nabla \) is complete and projectively flat. Let \( \varpi: \hat{U} \to T^2 \) be the covering map. We now claim that any broken \( \nabla \)-geodesic in \( T^2 \) from \( \varpi(\rho_0, \theta_0) \) to \( \varpi(\rho_0, \theta_0 + m\pi) \) must have at least \( m \) breaks. For let \( c: [a, b] \to T^2 \) be such a broken geodesic. By the Path Lifting Theorem ([2, p. 22] or [5, p. 67]) there is a unique curve \( \hat{c}: [a, b] \to M \) with \( \hat{c}(a) = (\rho_0, \theta_0) \) and \( \varpi \circ \hat{c} = c \). This curve will also be a broken geodesic. Also \( \varpi(\hat{c}(b)) = c(b) = \varpi(\rho_0, \theta_0 + m\pi) \), and therefore \( \hat{c}(b) = (\rho_0 + j, \theta_0 + m\pi + 2\pi k\ell) \) for some \( j, \ell \in \mathbb{Z} \). The difference in the \( \theta \) coordinates of the ends of \( \hat{c} \) is

\[
|\theta_0 + m\pi + 2\pi k\ell - \theta_0| = |m + 2k\ell|\pi \geq m\pi
\]
as \( k \geq m \). By Lemma 3.2 this implies that \( \hat{c} \) has at least \( m \) breaks. But then \( c = \varpi \circ \hat{c} \) also has at least \( m \) breaks. As \( \varpi(\rho_0, \theta_0) \) was an arbitrary point of \( T^2 \) this completes the proof of Theorem 1. \( \square \)

3.4. Remark. The connection \( \nabla \) has another property worth noting. If \( c(t) = (\rho(t), \theta(t)) \) is a smooth curve in \( \hat{U} \) then the equations for \( c \) to be a \( \nabla \)-geodesic are

\[
\ddot{\rho} = \dot{\theta}^2, \quad \ddot{\theta} = -\dot{\rho} \dot{\theta}.
\]

These imply

\[
\frac{1}{2} \frac{d}{dt}(\dot{\rho}^2 + \dot{\theta}^2) = \dot{\rho}\ddot{\rho} + \dot{\theta}\ddot{\theta} = \ddot{\rho}\dot{\rho}^2 - \dot{\rho}\dot{\theta} \dot{\theta} = 0.
\]

Therefore \( \dot{\rho}^2 + \dot{\theta}^2 \) is constant along \( \nabla \)-geodesics. Thus all \( \nabla \)-geodesics have constant speed with respect to the flat Riemannian metric \( ds^2 = d\rho^2 + d\theta^2 \) on \( \hat{U} \). As this metric is translation invariant it is also well defined on the torus \( T^2 = \hat{U}/(\mathbb{Z} \times 2\pi k \mathbb{Z}) \) and the \( \nabla \)-geodesics on \( T^2 \) will also have constant speed with respect to this metric. This can be used to give another proof that \( \nabla \) is complete. \( \square \)

References


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