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On Lebesgue functions of uniformly bounded orthonormal systems

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Abstract

A comlement to A.M.Olevskii's fundamental inequality on logaritmic growth of Lebesgue functions of an arbitrary uniformly bounded orthonormal system on a set of positive measure is made. Namely, the index where the Lebesgue functions have growth slightly weaker than logarithm can be chosen independent of the variable. The theorem proved in this paper improves one result established earlier by the author.

1 Introduction

The role of Lebesgue functions in divergence phenomena is crucial. The fundamental inequality of A.M.Olevskii on growth of Lebesgue functions on sets of positive measure for general uniformly bounded orthonormal systems is well known [1-3]:

Theorem 1 *(A. M. Olevskii). Let* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *be an arbitrary ONS on* $[0,1]$ *that satisfy the conditions:*

$$|\varphi_n(x)| \le M, \; n = 1, 2, 3, \dots, \; x \in [0,1]. \tag{1}$$

Then for each $n > 1$ *the following inequality holds*

$$\mu \left\{ x \in [0,1] : \max_{1 \le m \le n} L_m(x) \ge C \log_2 n \right\} \ge \gamma > 0, \tag{2}$$

where μ *is the Lebesgue measure on* $[0,1]$, C *and* γ *are positive constants that depend only on* M *and*

$$L_m(x) = \int_0^1 \left| \sum_{k=1}^{m} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta, \; m = 1, 2, \dots, \; x \in [0,1], \tag{3}$$

denotes the m-th Lebesgue function of the system.

In this paper we prove a theorem that complements to some extent inequality (2). The method of proof used by us is originated with the method of A.M.Olevskii. Namely, the following is true:

Theorem 2 . *Let $\{\varphi_n(x)\}_{n=1}^{\infty}$ be an orthonormal system on $[0,1]$ that satisfies (1). Then for each $n > p_0$ there exists an index $m_n < n$ such that*

$$\mu\left\{x \in [0,1] :\ L_{m_n}(x) \geq C_1 \frac{\log_2 n}{\log_2 \log_2 \log_2 n}\right\} \geq \gamma_1 > 0, \tag{4}$$

where p_0, C_1 and γ_1 are positive constants that depend only on M in (1).

The novelty of the inequality (4) is that the index m_n does not depend on x. The weaker inequality than (4) was proved by us in [4]. In connection with the inequality (2) we should also note [5] and [6].

We will prove in fact the " 2^n- version" of Theorem 2 because it is convenient for notations. So we will establish the following

Theorem 3 . *Let $\{\varphi_n(x)\}_{n=1}^{\infty}$ be an ortonormal system on $[0,1]$ that satisfies (1). Then for each $n > n_0$ there exists an index N_n such that $N_n < 2^{n+4}$ and*

$$\mu\left\{x \in [0,1] :\ L_{N_n}(x) \geq C_2 \frac{n}{\log_2 \log_2 n}\right\} \geq \gamma_2 > 0, \tag{5}$$

where n_0, C_2, γ_2 are positive constants that depend only on M in (1).

Consider the sets

$$F_N = \left\{x \in [0,1] : L_N(x) \geq \frac{n}{324}\right\}, \quad N = 1, 2, \ldots. \tag{6}$$

We may assume that

$$\mu F_N \leq \frac{1}{256 M^2} \quad \text{for all } N = 1, 2, \ldots, 2^{n+4}, \tag{7}$$

otherwise Theorem 3 is already established.

It is obvious that there exists a finite sequence of pairwise disjoint measurable sets $e_j^{(n)}$, $j = 1, 2, \ldots, q(n)$, such that

$$\bigcup_{j=1}^{q(n)} e_j^{(n)} = [0, 1] \tag{8}$$

and for all $j = 1, 2, \ldots, q(n)$ and $k = 1, 2, \ldots, 2^{n+4}$ we have the inequalities

$$|\varphi_k(x_1) - \varphi_k(x_2)| \leq \frac{1}{2^{3n}} \tag{9}$$

for $x_1 \in e_j^{(n)}$, $x_2 \in e_j^{(n)}$.

Let also

$$k(n) = \left[\frac{n}{3 \log_2 n} \right], \tag{10}$$

where by $[a]$ is denoted the integer part of the real number a.

2 The basic Lemma

Now we formulate the basic

Lemma 1 . *Suppose* (7) *holds. Then for each $n > n_0$ there exist finite sequences:*

of positive integers

$$m_0 = 2^n \leq m_1 \leq m_2 \leq \ldots \leq m_{k(n)} < 2^{n+4}, \tag{11}$$

of the sets

$$H_n^{(p)} \subset [0, 1], \quad p = 1, 2, \ldots k(n), \tag{12}$$

of the sets $\Omega_n^{(p)}(x) \subset [0, 1]$,that are defined for all $x \in [0, 1]$ and $p = 1, 2, \ldots k(n)$, such that

$$m_p - m_{p-1} \leq \frac{2^{n+2}}{n^{3(p-1)}}, \ \textit{for all } p = 1, 2, \ldots k(n) \tag{13}$$

and

$$\mu H_n^{(p)} \geq \frac{1}{512 M^2}, \ \textit{for all } p = 1, 2, \ldots \ k(n). \tag{14}$$

More than that for all $p = 1, 2, \ldots k(n)$

$$\Omega_n^{(p)}(x_1) = \Omega_n^{(p)}(x_2), \quad x_1 \in e_j^{(n)}, \ \ x_2 \in e_j^{(n)}, \quad j = 1, 2. \ldots, q(n), \tag{15}$$

and for all $x \in [0,1]$

$$\Omega_n^{(i)}(x) \bigcap \Omega_n^{(j)}(x) = \phi \quad , 1 \le i < j \le k(n). \tag{16}$$

Also the following inequalities hold:

$$\mu \Omega_n^{(p)}(x) \le \frac{n^{3(p-1)} n^2 \log_2 n}{2^n} \quad \text{for all} \quad p = 1, 2 \dots \quad k(n), \quad x \in [0,1] \tag{17}$$

and for each $x \in H_n^{(p)}$ *we have*

$$\int_{\Omega_n^{(p)}(x)} \left| \sum_{k=1}^{m_p} \varphi_k(x) \varphi_k(\vartheta) \right| d\vartheta \ge \frac{1}{386 M^4} \left[\frac{\log_2 n}{3 \log_2 \log_2 n} \right], \tag{18}$$

where n_0 *depends only on* M *from* (1)*.*

3 Proof of Lemma 1

The proof of the first step in the Lemma is completely similar to the construction of the $p+1$-th step. So we will show only the latter.

Now we suppose that al the steps in Lemma 1 have been carried out from the first up to the step $p, p < k(n)$.

3.1 The Assumption After the p-th step and its Analysis

Let $F_{p,m}$, where $m > m_p$, be the set of all points that posses the following property:

for x there exists a measurable set $G_0(x) \subset [0,1]$ such that

$$\int_{G_0(x)} \left| \sum_{k=m_p+1}^{m} \varphi_k(x) \varphi_k(\vartheta) \right| d\vartheta \ge \frac{\log_2 n}{48} \tag{19}$$

and

$$\mu G_0(x) \le \frac{n^{3p} n^2 \log_2 n}{2^n}. \tag{20}$$

We will need the following

Lemma 2 . *Suppose that some integer m', satisfies simultaneously the following conditions*

$$m_p \le m' \le m_p + \frac{2^{n+2}}{n^{3p}} \tag{21}$$

and

$$\mu F_{p,m'} \ge \frac{1}{256M^2}. \tag{22}$$

Then the $p+1$-th step of Lemma 1 can be constructed.

Proof of Lemma 2. Indeed let j be an index such that

$$e_j^{(n)} \cap F_{p,m'} \ne \phi, \quad 1 \le j \le q(n).$$

Then for each such j we choose an arbitrary point $x_j^{(1)}$ from $e_j^{(n)} \cap F_{p,m'}$ and keep it fixed. It is clear that for each such point there exists a set $G_0(x_j^{(1)}) \subset [0,1]$ that satisfy the inequality

$$\int_{G_0(x_j^{(1)})} \left| \sum_{k=m_p+1}^{m'} \varphi(x_j^{(1)})\varphi(\vartheta) \right| d\vartheta \ge \frac{\log_2 n}{48} \tag{23}$$

and at the same time

$$\mu G_0(x_j^{(1)}) \le \frac{n^{3p} n^2 \log_2 n}{2^n}. \tag{24}$$

We introduce the set

$$\Omega_n^{(p+1)}(x_j^{(1)}) = G_0(x_j^{(1)}) \setminus \bigcup_{i=1}^{p} \Omega_n^{(i)}(x_j^{(1)}). \tag{25}$$

Then by the assumption up to the step p in Lemma 1 we get (cf.(17),(21),(1)))

$$\begin{aligned} &\int_{\cup_{i=1}^{p} \Omega_n^{(i)}(x_j^{(1)})} \left| \sum_{k=m_p+1}^{m'} \varphi_k(x_j^{(1)})\varphi_k(\vartheta) \right| d\vartheta \\ &\le \frac{4M^2 2^n}{n^{3p}} \sum_{i=1}^{p} \frac{n^{3(i-1)} n^2 \log_2 n}{2^n} \le \frac{8M^2 n^2 n^{3(p-1)} \log_2 n}{n^{3p}} \\ &= \frac{8M^2 \log_2 n}{n} \end{aligned}$$

Consequently, we get for $n > n_0$ (cf.(23)-(25))

$$\int_{\Omega_n^{(p+1)}(x_j^{(1)})} \left| \sum_{k=m_p+1}^{m'} \varphi_k(x_j^{(1)})\varphi_k(\vartheta) \right| d\vartheta \geq \frac{\log_2 n}{49}. \tag{26}$$

and

$$\mu\Omega_n^{(p+1)}(x_j^{(1)}) \leq \frac{n^{3p} n^2 \log_2 n}{2^n}. \tag{27}$$

We now define the sets $\Omega_n^{(p+1)}(x)$ for all $x \in [0,1]$ on each $e_j^{(n)}$, $j = 1.2.\dots q(n)$, by the following equalities

$$\Omega_n^{(p+1)}(x) = \Omega_n^{(p+1)}(x_j^{(1)}) \text{ for all } x \in e_j^{(n)} \text{ if } e_j^{(n)} \cap F_{p,m'} \neq \phi \tag{28}$$

and

$$\Omega_n^{(p+1)}(x) = \phi \quad \text{for all} \quad x \in e_j^{(n)} \text{ if } e_j^{(n)} \cap F_{p,m'} = \phi. \tag{29}$$

Let $x \in F_{p,m'}$ be arbitrary. Then (cf.(8))there exists an integer j_0 that depend on x such that $x \in e_{j_0}^{(n)} \cap F_{p,m'}$. We have already chosen the point $x_{j_0}^{(1)}$ from $e_{j_0}^{(n)} \cap F_{p,m'}$ such that (cf. (27), (26))

$$\int_{\Omega_n^{(p+1)}(x_{j_0}^{(1)})} \left| \sum_{k=m_p+1}^{m'} \varphi_k(x_{j_0}^{(1)})\varphi_k(\vartheta) \right| d\vartheta \geq \frac{\log_2 n}{49}, \tag{30}$$

where

$$\mu\Omega_n^{(p+1)}(x_{j_0}^{(1)}) \leq \frac{n^{3p} n^2 \log_2 n}{2^n}. \tag{31}$$

But then by the definition of $e_j^{(n)}$ (cf.(9),(30),(31)and (28)) we conclude

$$\int_{\Omega_n^{(p+1)}(x)} \left| \sum_{k=m_p+1}^{m'} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \geq \frac{\log_2 n}{50}, \tag{32}$$

where (cf.(31))

$$\mu\Omega_n^{(p+1)}(x) \leq \frac{n^{3p} n^2 \log_2 n}{2^n}. \tag{33}$$

¿From the assumption of induction up to the step p (cf.(29),(15)) and (25) we have

$$\Omega_n^{(p+1)}(x) \cap \Omega_n^{(j)}(x) = \phi \quad , 1 \leq j \leq p \quad , x \in [0,1]. \tag{34}$$

Now we define the set $H_n^{(p+1)}$ and index m_{p+1} from Lemma 1. Introducing the set

$$T_n^{(p+1)} = \left\{ x \in F_{p,m'} \quad : \int_{\Omega_n^{(p+1)}(x)} \left| \sum_{k=1}^{m_p} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \geq \frac{\log_2 n}{100} \right\},$$

we let

$$H_n^{(p+1)} = T_n^{(p+1)} \text{ and } m_{p+1} = m_p \text{ if } \mu T_n^{(p+1))} \geq \frac{1}{512 M_2}.$$

and

$$H_n^{(p+1)} = F_{p,m'} \setminus T_n^{(p+1)} \text{ and } m_{p+1} = m' \text{ if } \mu T_n^{(p+1))} < \frac{1}{512 M_2}.$$

Lemma 2 (cf.(32)-(34))is proved.

The next is

Lemma 3 . *Let l be an integer such that*

$$1 \leq l \leq \left[\frac{\log_2 n}{3 \log_2 \log_2 n} \right], \tag{35}$$

integer m_l' — any index exceeding m_p from Lemma 1 and r_l' defined as

$$r_l' = m_l' + \left[\frac{2^n}{n^{3p} (\log_2 n)^{3l}} \right]. \tag{36}$$

Suppose that for some $x \in [0,1]$ we have

$$\int_{A_{p,l}} \left| \sum_{k=m_l'+1}^{r_l'} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \geq \frac{\log_2 n}{24}, \tag{37}$$

where

$$A_{p,l} = \left\{ \vartheta \in [0,1] : \frac{2^n}{n^{3p} (\log_2 n)^{3l} n} \leq \left| \sum_{k=m_l'+1}^{r_l'} \varphi_k(x)\varphi_k(\vartheta) \right| \leq \frac{2^n}{n^{3p} (\log_2 n)^{3l} \log_2 n} \right\} \tag{38}$$

Then

$$x \in F_{p,m_l'} \cup F_{p,r_l'}. \tag{39}$$

Proof of lemma 3. Indeed from (37) and (38) it follows that there exists a measurable set $G_1(x)$ with the following properties:

$$\int_{G_1(x)}\left|\sum_{k=m_l'+1}^{r_l'}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta=\frac{\log_2 n}{24},$$

and

$$G_1(x)\subset A_{p,l}.$$

According to (38) we have

$$\frac{2^n}{n^{3p}(\log_2 n)^{3l}n}\mu G_1\le\frac{\log_2 n}{24}$$

and ,consequently,

$$\mu G_1(x)\le\frac{n^{3p}n^2\log_2 n}{2^n} \tag{40}$$

So we get that either

$$\int_{G_1(x)}\left|\sum_{k=m_p+1}^{r_l'}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta\ge\frac{\log_2 n}{48},$$

and then $x\in F_{p,r_l'}$, or

$$\int_{G_1(x)}\left|\sum_{k=m_p+1}^{m_l'}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta\ge\frac{\log_2 n}{48},$$

and then $x\in F_{p,m_l'}$.

Lemma 3 is established.

3.2 Construction of the p+1-th step of Lemma 1

We may assume that

$$\mu F_{p,m}\le\frac{1}{256M^2}\quad\text{for}\quad m_p+1\le m\le m_p+\left[\frac{2^{n+2}}{n^{3p}}\right], \tag{41}$$

otherwise, by Lemma 2, it would remain nothing to construct.

Let

$$r(n) = \left[\frac{\log_2 n}{3\log_2 \log_2 n}\right]. \tag{42}$$

We will show first the following

Lemma 4 . *Suppose* (41) *holds. Then for each* $n > n_0$ *there exist finite sequences: of positive integers*

$$m'_0 = m_p + 1 \le m'_1 \le m'_2 \le \ldots \le m'_{r(n)} \le m_p + \left[\frac{2^{n+4}}{n^{3p}}\right], \tag{43}$$

of the measurable sets

$$h^{(l)} \subset [0,1], \quad l = 1, 2, \ldots r(n), \tag{44}$$

of the measurable sets $\omega^{(l)}(x) \subset [0,1]$ *,that are defined for all* $x \in [0,1]$*, and* $l = 1, 2, \ldots r(n)$*, such that*

$$m'_l - m'_{l-1} \le \frac{2^n}{(\log_2 n)^{3(l-1)} n^{3p}}, \ \textit{for all } l = 1, 2, \ldots r(n) \tag{45}$$

and

$$\mu h^{(l)} \ge \frac{7}{32M^2}, \ \textit{for all } l = 1, 2, \ldots \ r(n). \tag{46}$$

More than that for all $l = 1, 2, \ldots r(n)$ *we have*

$$\omega^{(l)}(x_1) = \omega^{(l)}(x_2), \quad x_1 \in e_j^{(n)}, \ x_2 \in e_j^{(n)}, \quad j = 1, 2. \ldots, r(n), \tag{47}$$

and for all $x \in [0,1]$

$$\omega^{(i)}(x) \cap \omega^{(j)}(x) = \phi \quad , 1 \le i < j \le r(n). \tag{48}$$

Also the following inequalities hold:

$$\mu\omega^{(l)}(x) \le \frac{(\log_2 n)^{3(l-1)} n^{3p} \log_2 n}{4M^2 2^n} \quad \textit{for all} \quad l = 1, 2 \ldots \quad r(n), \ \ x \in [0,1] \tag{49}$$

and for each $x \in h^{(l)}$ *we have*

$$\int_{\omega^{(l)}(x)} \left| \sum_{k=m_p+1}^{m'_l} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \ge \frac{1}{20M^2}, \tag{50}$$

where n_0 *depends only on* M *from* (1).

Proof of Lemma 4. We will prove this lemma by induction. We note that the construction of the 1-st step is absolutely similar to the construction of the l+1-th step. So we will give only the construction of the latter one.

Now suppose that all the steps up to the l-th step have been carried out and $l < r(n)$.

We introduce the integer

$$r'_l = m'_l + \left[\frac{2^n}{n^{3p}(\log_2 n)^{3l}}\right]. \tag{51}$$

It is clear that (cf.(11),(45),(43),(51))

$$2^n \leq m_p + 1 \leq m'_l \leq r'_l \leq m_p + \frac{2^{n+2}}{n^{3p}} \leq 2^{n+4}. \tag{52}$$

Let

$$k'_{l+1} = \left\{ x \in [0,1] : \sum_{k=m'_l+1}^{r'_l} \varphi_k^2(x) \geq \frac{1}{2}\left[\frac{2^n}{n^{3p}(\log_2 n)^{3l}}\right] \right\}. \tag{53}$$

Then (cf.(53),(51))

$$\begin{aligned} \left[\frac{2^n}{n^{3p}(\log_2 n)^{3l}}\right] &\leq \int_0^1 \sum_{k=m'_l+1}^{r'_l} \varphi_k^2(x)dx \\ = \int_{k'_{l+1}} + \int_{[0,1]\setminus k'_{l+1}} &\leq \left(M^2 \mu k'_{l+1} + \frac{1}{2}\right)\left[\frac{2^n}{n^{3p}(\log_2 n)^{3l}}\right] \end{aligned}$$

and, consequently,

$$\mu k'_{l+1} \geq \frac{1}{2M^2}. \tag{54}$$

Next we introduce the set

$$p'_{l+1} = k'_{l+1} \setminus \left(F_{m'_l} \bigcup F_{r'_l} \bigcup F_{p,m'_l} \bigcup F_{p,r'_l}\right). \tag{55}$$

By the assumptions (7) and (41) and also (52),(54) we have

$$\mu p'_{l+1} \geq \frac{1}{2M^2} - \frac{4}{256M^2} \geq \frac{7}{16M^2}. \tag{56}$$

Let now i be an index such that

$$e_i^{(n)} \cap p'_{l+1} \neq \phi, \quad 1 \leq i \leq q(n).$$

Then for each such i we choose an arbitrary point $y_i^{(l+1)}$ from $e_i^{(n)} \cap p'_{l+1}$ and keep it fixed. So we have(cf.(51),(55))

$$\frac{1}{2}\left[\frac{2^n}{n^{3p}(\log_2 n)^{3l}}\right] \leq \int_0^1 \left(\sum_{k=m'_l+1}^{r'_l} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta)\right)^2 d\vartheta$$

$$\leq M^2\left[\frac{2^n}{n^{3p}(\log_2 n)^{3l}}\right] \int_{B(p,l)} \left|\sum_{k=m'_l+1}^{r'_l} \varphi_k(y_i^{l+1})\varphi_k(\vartheta)\right| d\vartheta$$

$$+\frac{2^n}{n^{3p}(\log_2 n)^{3l}\log_2 n} \int_{A(p,l)} \left|\sum_{k=m'_l+1}^{r'_l} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta)\right| d\vartheta$$

$$+\frac{2^n}{n^{3p}(\log_2 n)^{3l}\ n} \int_{C(p,l)} \left|\sum_{k=m'_l+1}^{r'_l} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta)\right| d\vartheta = I_1 + I_2 + I_3,$$

where

$$A_{p,l} = \left\{\vartheta \in [0,1] : \frac{2^n}{n^{3p}(\log_2 n)^{3l} n} \leq \left|\sum_{k=m'_l+1}^{r'_l} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta)\right| \leq \frac{2^n}{n^{3p}(\log_2 n)^{3l}\log_2 n}\right\}$$

and

$$B_{p,l} = \left\{\vartheta \in [0,1] : \left|\sum_{k=m'_l+1}^{r'_l} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta)\right| \geq \frac{2^n}{n^{3p}(\log_2 n)^{3l}\log_2 n}\right\} \tag{57}$$

and

$$C_{p,l} = \left\{\vartheta \in [0,1] : \left|\sum_{k=m'_l+1}^{r'_l} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta)\right| \leq \frac{2^n}{n^{3p}(\log_2 n)^{3l} n}.\right\}$$

According to the definition of the sets F_N (cf.(6),(3),(52)) we have

$$I_3 \leq \frac{2^n}{n^{3p}(\log_2 n)^{3l} n}\frac{n}{162} \leq \frac{2^n}{162 n^{3p}(\log_2 n)^{3l}}.$$

¿From the definition of the sets $F_{p,m}$, Lemma 3 and (19),(35),(36),(39),(20),(37),(55),(52)we get

$$I_2 \leq \frac{2^n}{24n^{3p}(\log_2 n)^{3l}}.$$

Consequently, we conclude for $n > n_0$ that

$$\int_{B(p,l)} \left| \sum_{k=m_l'+1}^{r_l'} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta) \right| d\vartheta \geq \frac{1}{4M^2}.$$

It is obvious that one can find a set

$$g_{l+1}'(y_i^{(l+1))}) \subset B(p,l)$$

such that

$$\int_{g_{l+1}'(y_i^{l+1})} \left| \sum_{k=m_l'+1}^{r_l'} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta) \right| d\vartheta = \frac{1}{4M^2}.$$

¿From (57) we have

$$\frac{2^n}{n^{3p}(\log_2 n)^{3l}\log_2 n} \mu g_{l+1}'(y_i^{l+1}) \leq \frac{1}{4M^2}$$

and, consequently,

$$\mu g_{(l+1)}'(y_i^{(l+1)}) \leq \frac{1}{4M^2} \frac{n^{3p}(\log_2 n)^{3l}\log_2 n}{2^n}. \tag{58}$$

Let

$$\omega^{(l+1)}(y_i^{(l+1)}) = g_{l+1}'(y_i^{(l+1)}) \setminus \bigcup_{j=1}^{l} \omega^{(j)}(y_i^{(l+1)}). \tag{59}$$

then we have from the assumption of the induction (cf.(51),(49))

$$\int_{\cup_{j=1}^{l}\omega^{(j)}(y_i^{(l+1)})} \left| \sum_{k=m_l'+1}^{r_l'} \varphi_k(y_{l+1}^{(i)})\varphi_k(\vartheta) \right| d\vartheta$$
$$\leq \frac{M^2 2^n}{4M^2 n^{3p}(\log_2 n)^{3l}} \sum_{j=1}^{l} \frac{(\log_2 n)^{3(j-1)} n^{3p}\log_2 n}{2^n} \leq \frac{2(\log_2 n)^{3(l-1)}\log_2 n}{4(\log_2 n)^{3l}}$$
$$= \frac{1}{2(\log_2 n)^2}.$$

So we get for $n > n_0$ (cf.(59))

$$\int_{\omega^{(l+1)}(y_i^{(l+1)})} \left| \sum_{k=m'_l+1}^{r'_l} \varphi_k(y_i^{(l+1)})\varphi_k(\vartheta) \right| d\vartheta \geq \frac{3}{20M^2}. \tag{60}$$

We now define the sets $\omega^{(l+1)}(x)$ for all $x \in [0,1]$ on each $e_i^{(n)}$, $i = 1, 2, \ldots q(n)$, by the following equalities

$$\omega^{(l+1)}(x) = \omega^{(l+1)}(y_i^{(l+1)}) \text{ for all } x \in e_i^{(n)} \text{ if } e_i^{(n)} \cap p'_{l+1} \neq \phi \tag{61}$$

and

$$\omega^{(l+1)}(x) = \phi \quad \text{for all} \quad x \in e_i^{(n)} \text{ if } e_i^{(n)} \cap p'_{l+1} = \phi. \tag{62}$$

According to (8), (9), (61), we get for all $x \in p'_{l+1}$ and $n > n_0$

$$\int_{\omega^{(l+1)}(x)} \left| \sum_{k=m'_l+1}^{r'_l} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \geq \frac{1}{10M^2}. \tag{63}$$

¿From (59),(47) it is obvious that for all $x \in [0,1]$

$$\omega^{(l+1)}(x) \cap \omega^{(i)}(x) = \phi \quad, 1 \leq i < l+1 \leq r(n). \tag{64}$$

and from (61),(62)

$$\omega^{(l+1)}(x_1) = \omega^{(l+1)}(x_2), \quad x_1 \in e_i^{(n)}, \ x_2 \in e_i^{(n)}, \quad i = 1, 2, \ldots, r(n). \tag{65}$$

Now we Introduce the set

$$t^{(l+1)} = \left\{ x \in p'_{l+1} \quad : \int_{\omega^{(l+1)}(x)} \left| \sum_{k=m_p+1}^{m'_l} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \geq \frac{1}{20M^2} \right\}.$$

We define

$$h^{(l+1)} = t^{(l+1)} \text{ and } m'_{l+1} = m'_l \text{ if } \mu t^{(l+1)} \geq \frac{7}{32M^2}.$$

and

$$h^{(l+1)} = p'_{l+1} \setminus t^{(l+1)} \text{ and } m'_{l+1} = r'_l \text{ if } \mu t^{(l+1))} < \frac{7}{32M^2}.$$

It is clear that (cf.(41)-(53),(56),(58), (63)-(64)) the $l+1$-th step of the lemma is now constructed and so Lemma 4 is proved.

Now we start the construction of the p+1-th step Lemma 1 using Lemma 4.

Let

$$e_n^{(p+1)} = \left\{ x \in [0,1] : \sum_{l=1}^{r(n)} \chi_{l,p}(x) \geq \frac{7}{64M^2} r(n) \right\} \tag{66}$$

where each $\chi_{l,p}$ is the characteristic function of the set $h^{(l)}$ from Lemma 4.

It is obvious that (cf.(14),(12),(66),(46))

$$\frac{7}{32M^2} r(n) \leq \int_0^1 \sum_{l=1}^{r(n)} \chi_{l,p}(x) dx = \int_{e_n^{(p+1)}} + \int_{[0,1]\setminus e_n^{(p+1)}}$$
$$\leq r(n)\mu e_n^{(p+1)} + \frac{7}{64M^2} r(n),$$

and, consequently,

$$\mu e_n^{(p+1)} \geq \frac{7}{64M^2} r(n). \tag{67}$$

Now we take an arbitrary point x from $e_n^{(p+1)}$ and keep it fixed. Then according to (66) there exists a finite sequense of integers depending on x, $1 \leq l_1 < \ldots < l_{j'(n)} \leq r(n)$ such that

$$x \in h^{(l_i)}, \quad i = 1, 2, \ldots, j'(n) \tag{68}$$

and

$$j'(n) \geq \frac{7}{64M^2} r(n). \tag{69}$$

Taking account of (45) we conclude for all $i = 1, 2, 3, \ldots j'(n)$ that

$$m'_{r(n)} - m'_{l_i} = \sum_{s=l_i+1}^{r(n)} (m'_s - m'_{s-1}) \leq \sum_{s=l_i+1}^{r(n)} \frac{2^n}{(\log_2 n)^{3(s-1)} n^{3p}} \leq \frac{2^{n+1}}{(\log_2 n)^{3l_i} n^{3p}}. \tag{70}$$

And for all $i = 1, 2, 3, \ldots j'(n)$

$$\int_{\omega^{(l_i)}(x)} \left| \sum_{k=m_p+1}^{m'_{l_i}} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \geq \frac{1}{20M^2}. \tag{71}$$

According to (1),(68)-(71),(49),(48) it is clear that

$$\int_{\cup_{l=1}^{r(n)}\omega^{(l)}(x)}\left|\sum_{k=m_p+1}^{m'_{r(n)}}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta \geq \sum_{i=1}^{j'(n)}\int_{\omega^{(l_i)}(x)}\left|\sum_{k=m_p+1}^{m'_{r(n)}}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta$$

$$\geq \sum_{i=1}^{j'(n)}\int_{\omega^{(l_i)}(x)}\left|\sum_{k=m_p+1}^{m'_{l_i}}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta - \sum_{i=1}^{j'(n)}\int_{\omega^{(l_i)}(x)}\left|\sum_{k=m'_{l_i}+1}^{m'_{r(n)}}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta$$

$$\geq \frac{j'(n)}{20M^2} - M^2\sum_{i=1}^{j'(n)}\mu\omega^{(l_i)}(x)(m'_{r(n)} - m'_{l_i})$$

$$\geq \frac{j'(n)}{20M^2} - 2M^2\sum_{i=1}^{j'(n)}\frac{\log_2 n(\log_2 n)^{3(l_i-1)}}{4M^2(\log_2 n)^{3l_i}}$$

$$\geq \frac{j'(n)}{20M^2} - \frac{j'(n)}{2(\log_2 n)^2} \geq \frac{1}{192M^4}\left[\frac{\log_2 n}{3\log_2\log_2 n}\right].$$

Consequently we have constructed the set $e_n^{(p+1)}$ such that (67) holds and for all $x \in e_n^{(p+1)}$ we have

$$\int_{G_n^{(p+1)}(x)}\left|\sum_{k=m_p+1}^{m'_{r(n)}}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta \geq \frac{1}{192M^4}\left[\frac{\log_2 n}{3\log_2\log_2 n}\right]. \tag{72}$$

where

$$G_n^{(p+1)}(x) = \bigcup_{l=1}^{r(n)}\omega^{(l)}(x). \tag{73}$$

We note that (cf.(49),(42),(73)) for all $x \in [0,1]$

$$\mu G_n^{(p+1)}(x) \leq \sum_{l=1}^{r(n)}\frac{(\log_2 n)^{3(l-1)}n^{3p}\log_2 n}{4M^2 2^n} \leq \frac{2(\log_2 n)^{3(r(n)-1)}n^{3p}\log_2 n}{4M^2 2^n}$$

$$\leq \frac{n^{3p}n\log_2 n}{4M^2 2^n}.$$

Now we define the set $\Omega_n^{(p+1)}(x)$ in Lemma 1 as follows

$$\Omega_n^{(p+1)}(x) = G_n^{(p+1)}(x) \setminus \bigcup_{i=1}^{p}\Omega_n^{(i)}(x). \tag{74}$$

We have from the assumption of the induction in Lemma 1 up to the p-th step (cf.(17))

$$\int_{\cup_{i=1}^{p}\Omega_n^{(i)}(x)}\left|\sum_{k=m_p+1}^{m'_{r(n)}}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta \le M^2\left(m'_{r(n)}-m_p\right)\sum_{i=1}^{p}\mu\Omega_n^{(i)}(x)$$
$$\le \frac{M^2 2^n}{n^{3p}}\sum_{i=1}^{p}\frac{n^{3(i-1)}n^2\log_2 n}{2^n} \le \frac{2M^2\log_2 n}{n}.$$

We conclude from (74),(72) that for $n > n_0$ and for all $x \in e_n^{(p+1)}$

$$\int_{\Omega_n^{(p+1)}(x)}\left|\sum_{k=m_p+1}^{m'_{r(n)}}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta \ge \frac{1}{193M^4}\left[\frac{\log_2 n}{3\log_2\log_2 n}\right].$$

We introduce the set

$$Q_n^{(p+1)} = \left\{x \in e_n^{(p+1)} : \int_{\Omega_n^{(p+1)}(x)}\left|\sum_{k=1}^{m_{(p)}}\varphi_k(x)\varphi_k(\vartheta)\right|d\vartheta \ge \frac{1}{386M^4}\left[\frac{\log_2 n}{3\log_2\log_2 n}\right]\right\}.$$

Now we define the set $H_n^{(p+1)}$ and index m_{p+1} from Lemma 1. Let

$$H_n^{(p+1)} = Q_n^{(p+1)} \text{ and } m_{p+1} = m_p \text{ if } \mu Q_n^{(p+1))} \ge \frac{7}{128M^2}.$$

and

$$H_n^{(p+1)} = e_n^{(p+1)} \setminus Q_n^{(p+1)} \text{ and } m_{p+1} = m'_{r(n)} \text{ if } \mu Q_n^{(p+1))} < \frac{7}{128M^2}.$$

Lemma 1 (cf.(14),(74),(73)) is proved.

4 Proof of Theorem 3.

We introduce the set

$$E_n = \left\{x \in [0,1] : \sum_{p=1}^{k(n)}\chi_{l,p}^{(1)}(x) \ge \frac{1}{1024M^2}k(n)\right\} \tag{75}$$

where $\chi_{l,p}^{(1)}$ is the characteristic function of the set $H_n^{(p)}$ from Lemma 1

Analogiously to (67) we get

$$\mu E_n \geq \frac{1}{1024M^2}. \tag{76}$$

Now we take an arbitrary point x from E_n and keep it fixed. Then (cf.(75)) there exists a finite sequense of indices depending on x, $1 \leq p_1 < \ldots < p_{j(n)} \leq k(n)$ such that

$$x \in H_n^{(p_i)}, \quad i = 1, 2, \ldots, j(n) \tag{77}$$

and

$$j(n) \geq \frac{1}{1024M^2} k(n). \tag{78}$$

Now let

$$N_n = m_{k(n)}.$$

Taking account of (13) we have for all $i = 1,\ 2\ \ldots\ k(n)$

$$N_n - m_{p_i} = \sum_{p=p_i}^{k(n)-1} (m_{p+1} - m_p) \leq \sum_{p=p_i}^{k(n)-1} \frac{2^{n+2}}{n^{3p}} \leq \frac{2^{n+3}}{n^{3p_i}}.$$

as well as

$$N_n = m_0 + \sum_{p=0}^{k(n)-1} (m_{p+1} - m_p) \leq 2^n + 2^{n+2} \sum_{p=0}^{k(n)-1} \frac{1}{n^{3p}} \leq 2^{n+4}.$$

According to (77),(78),(10),(16)-(18) it is obvious that

$$\begin{aligned}
&\int_0^1 \left| \sum_{k=1}^{N_n} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \geq \sum_{i=1}^{j(n)} \int_{\Omega_n^{(p_i)}(x)} \left| \sum_{k=1}^{N_n} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \\
&\geq \sum_{i=1}^{j(n)} \int_{\Omega_n^{(p_i)}(x)} \left| \sum_{k=1}^{m_{p_i}} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta - \sum_{i=1}^{j(n)} \int_{\Omega_n^{(p_i)}(x)} \left| \sum_{k=m_{p_i}+1}^{N_n} \varphi_k(x)\varphi_k(\vartheta) \right| d\vartheta \\
&\geq \frac{C_2 j(n) \log_2 n}{\log_2 \log_2 n} - \sum_{i=1}^{j(n)} M^2 (N_n - m_{p_i}) \mu \Omega_n^{(p_i)}(x) \\
&\geq \frac{C_2 j(n) \log_2 n}{\log_2 \log_2 n} - 8n^2 \log_2 n \sum_{i=1}^{j(n)} \frac{n^{3(p_i-1)}}{n^{3p_i}} \\
&\geq \frac{C_3 n}{\log_2 \log_2 n},
\end{aligned}$$

for some constants C_2 and C_3 that depend only on M from (1).

Theorem 3 (cf.(3),(5),(76)) and, consequently, Theorem 2 are proved completely.

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