On Riemann-Lebesgue theorem for the systems of Chebyshev ridge polynomials

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Let
\[ \mathbb{B}^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \] (1)
denote the unit disc on the plane and
\[ u_m(t) := \frac{1}{\sqrt{\pi}} \sin \left(\frac{(m + 1) \arccos t}{\sqrt{1 - t^2}}\right), \] (2)
m = 0, 1, \ldots, t \in [-1, 1], are the Chebyshev polynomials of the second kind. For an arbitrary sequence of real phases \( \{\varphi_m\}_{m=0}^{\infty} \), we get on \( \mathbb{B}^2 \) the corresponding discrete sequence of Chebyshev ridge polynomials
\[ \left\{ u_m(x \cos \left(\frac{k\pi}{m+1} + \varphi_m\right) + y \sin \left(\frac{k\pi}{m+1} + \varphi_m\right)) \right\}_{k=0}^{m} \] (3)
These systems are very useful tool in the theory of approximation of functions by feed–forward neural networks [1], [2]. It is known [2] that for an arbitrary sequence of real phases \( \{\varphi_m\}_{m=0}^{\infty} \), the system (3) is a complete orthonormal system in \( L^2(\mathbb{B}^2) \). We consider convergence problem to zero for Fourier coefficients \((0 \leq k < m + 1, m = 0, 1, \ldots)\)
\[ a_m(f, k, \varphi_m) := \int_{\mathbb{B}^2} f(x, y) u_m \left( x \cos \left(\frac{k\pi}{m+1} + \varphi_m\right) + y \sin \left(\frac{k\pi}{m+1} + \varphi_m\right) \right) dx dy \] (4)
of a function \( f \in L^p(\mathbb{B}^2) \) with respect to the systems (3). The partial \( L^p \)-integral moduli of continuity of a function \( f \in L^p(\mathbb{B}^2) \) are defined as follows
\[ \omega_1(\delta; f)_p := \sup_{|h| \leq \delta} \left( \int_{\mathbb{B}^2 \cap \mathbb{B}^2(1, h)} |f(x + h, y) - f(x, y)|^p dx dy \right)^{\frac{1}{p}}, \] (5)
and
\[
\omega_2(\delta; f)_p := \sup_{|h| \leq \delta} \left( \int_{B^2(1, h)} \int_{B^2(2, h)} |f(x, y + h) - f(x, y)|^p \, dx \, dy \right)^{\frac{1}{p}}.
\]  
(6)

where

\[
B^2(1, h) := \{(x, y) \in \mathbb{R}^2 : (x + h, y) \in B^2\}, \quad B^2(2, h) := \{(x, y) \in \mathbb{R}^2 : (x, y + h) \in B^2\}.
\]  
(7)

In the present article we shall prove the following theorems.

**Theorem 1** Let \{\varphi_m\}_{m=0}^{\infty} be an arbitrary sequence of real numbers and \(f \in L^p(B^2), p > \frac{3}{2}\). Then the ridge Chebyshev–Fourier coefficients of \(f\) tend to zero:

\[
\lim_{m \to \infty} \max_{0 \leq k \leq m} |a_m(f, k, \varphi_m)| = 0.
\]  
(8)

**Theorem 2** There exists a function \(g \in L^{\frac{3}{2}}(B^2)\) such that

\[
\omega_1(\delta; g)^{\frac{1}{2}} = O \left( \left( \frac{1}{\log \delta} \right)^{\frac{1}{2}} \right), \quad (\delta \to 0^+); \quad \omega_2(\delta; g)^{\frac{1}{2}} = 0, \quad (\delta \in (0, 1))
\]  
(9)

and for each sequence \{\varphi_m\}_{m=0}^{\infty} the following inequality holds true

\[
\limsup_{m \to \infty} \max_{0 \leq k \leq m} |a_m(g, k, \varphi_m)| \geq C_1 > 0,
\]  
(10)

where \(C_1\) is an absolute constant.

The next statement follows from Theorem 2.

**Corollary 1** There exists a function \(g \in L^{\frac{3}{2}}(B^2)\) that satisfies (9) and for each sequence \{\varphi_m\}_{m=0}^{\infty} Fourier series of \(g\) with respect to the system (3) diverges in \(L^{\frac{3}{2}}(B^2)\).

**Proof of the Corollary.** First we prove that for \(m = 0, 1, \ldots, k = 0, 1, \ldots, m\), and for each sequence \{\varphi_m\}_{m=0}^{\infty} we have

\[
\int_{B^2} \left| u_m \left( x \cos \left( \frac{k\pi}{m+1} + \varphi_m \right) + y \sin \left( \frac{k\pi}{m+1} + \varphi_m \right) \right) \right| \, dx \, dy \geq \frac{\sqrt{\pi}}{2}.
\]  
(11)
Indeed, according to (1) and (2)

\[
\int_{\mathbb{B}^2} \left| u_m \left( x \cos \left( \frac{k\pi}{m+1} + \varphi_m \right) + y \sin \left( \frac{k\pi}{m+1} + \varphi_m \right) \right) \right| \, dxdy
\]

\[
= \int_{\mathbb{B}^2} |u_m(x)| \, dxdy = \frac{2}{\sqrt{\pi}} \int_{-1}^{1} |\sin (m+1) \arccos x| \, dx
\]

\[
= \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} |\sin (m+1) \vartheta| \sin \vartheta d\vartheta \geq \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} (\sin (m+1) \vartheta \sin \vartheta)^2 \, d\vartheta
\]

\[
= \frac{1}{2\sqrt{\pi}} \int_{0}^{\pi} (1 - \cos 2(m+1) \vartheta)(1 - \cos 2\vartheta) \, d\vartheta = \frac{\sqrt{\pi}}{2}.
\]

Consequently for the function \( g \) from Theorem 2 we get

\[
\max_{0 \leq k \leq m} \left\| a_m(g, k, \varphi_m) u_m \left( x \cos \left( \frac{k\pi}{m+1} + \varphi_m \right) + y \sin \left( \frac{k\pi}{m+1} + \varphi_m \right) \right) \right\|_{3/2}
\]

\[
\geq C_2 \max_{0 \leq k \leq m} |a_m(g, k, \varphi_m)|
\]

for each sequence \( \{\varphi_m\}_{m=0}^{\infty} \) and \( m = 0, 1, \ldots \), where \( C_2 \) is an absolute positive constant. Now the Corollary follows from (10).

**Proof of Theorem 1.** First we note that for each \( \epsilon \in (0, 1) \) there exists a constant \( B_{\epsilon} \) such that

\[
\int_{\mathbb{B}^2} |u_m(x)|^{3-\epsilon} \, dxdy \leq B_{\epsilon}, \quad m = 0, 1, \ldots
\]  

(12)

Indeed

\[
\int_{\mathbb{B}^2} |u_m(x)|^{3-\epsilon} \, dxdy = 2 \left( \frac{1}{\sqrt{\pi}} \right)^{3-\epsilon} \int_{-1}^{1} |\sin (m+1) \arccos x|^{3-\epsilon} \left( \sqrt{1-x^2} \right)^{\epsilon-2} \, dx
\]

\[
= 4 \left( \frac{1}{\sqrt{\pi}} \right)^{3-\epsilon} \int_{0}^{\pi/2} |\sin (m+1) \vartheta|^{3-\epsilon} (\sin \vartheta)^{\epsilon-1} \, d\vartheta
\]

\[
= 4 \left( \frac{1}{\sqrt{\pi}} \right)^{3-\epsilon} \left( m+1 \right)^{3-\epsilon} \pi^{1-\epsilon} 2^{\epsilon-1} \int_{0}^{\pi} \vartheta^2 \, d\vartheta + 4 \pi^{1-\epsilon} 2^{\epsilon-1} \int_{\pi/2}^{\pi} \vartheta^{\epsilon-1} \, d\vartheta
\]

\[
= 4 \left( \frac{1}{\sqrt{\pi}} \right)^{3-\epsilon} \left( o(1) + \frac{4\pi}{2\epsilon} \right) \quad \text{as} \quad m \to \infty.
\]
Now let \( p > \frac{3}{2} \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then using Helder’s inequality and (12) we obtain that for an arbitrary \( f \) from \( L^p(\mathbb{B}^2) \) the following inequality holds true:

\[
\max_{0 \leq k \leq m} |a_m(f, k, \varphi)| \leq \|f\|_p \|u_m\|_q, \quad m = 0, 1, \ldots.
\]

For a given positive \( \delta \) we can find a function \( h = h_\delta \in L^2(\mathbb{B}^2) \) such that

\[
\|f - h\|_p \leq \delta.
\]

Consequently (cf.(12))

\[
\max_{0 \leq k \leq m} |a_m(f, k, \varphi)| \leq \|f - h\|_p \|u_m\|_q + \max_{0 \leq k \leq m} |a_m(h, k, \varphi)|
\]

\[
\leq O_q(\delta) + o(1) \quad \text{as} \quad m \to \infty
\]

Theorem 1 is proved. The next statement is essential in the proof of Theorem 2.

**Lemma 1** For each \( m, m \geq m_0 \), there exists a function \( q_{m-1}(x) \) of one variable, defined on \([-1,1]\) such that the function \( Q_{m-1}(x, y) \) defined by

\[
Q_{m-1}(x, y) := q_{m-1}(x) \quad \text{for} \quad (x, y) \in \mathbb{B}^2,
\]

satisfies the following conditions:

\[
\max_{0 \leq k \leq m-1} |a_{m-1}(Q_{m-1}, k, \varphi)| \geq C_3 (\log m)^{\frac{1}{3}} \quad \text{for all real} \quad \varphi,
\]

\[
\|Q_{m-1}\|_3 \leq C_4,
\]

\[
\omega_1(\delta; Q_{m-1}) := \begin{cases} 
C_3(m^2 \delta)^{\frac{1}{2}} (\log m)^{-\frac{1}{3}} & \text{for} \quad 0 \leq \delta \leq \frac{2}{m^2} \\
2C_5 & \text{for} \quad \delta > \frac{2}{m^2}
\end{cases}
\]

\[
\omega_2(\delta; Q_{m-1}) = 0 \quad \text{for all} \quad \delta \in (0,1),
\]

where \( C_3, C_4, m_0, C_5 \) are positive absolute constants and

\[
a_m(f, k, \varphi) := \int_{\mathbb{B}^2} f(x, y) u_m \left( x \cos \left( \frac{k\pi}{m+1} + \varphi \right) + y \sin \left( \frac{k\pi}{m+1} + \varphi \right) \right) \, dx \, dy.
\]
Proof of the Lemma. Consider the functions \( f_k^m(x), -1 \leq x \leq 1, k = 1, 2, \ldots, \lfloor \sqrt{m} \rfloor \):

\[
f_k^m(x) = \begin{cases} \frac{1}{k^2} & \text{for } x \in \left[ \cos \frac{2k+1}{m}\pi, \cos \frac{2k}{m}\pi \right], \\ 0 & \text{otherwise on } [-1, 1] \end{cases}
\]

and let

\[
q_{m-1}(x) := \frac{m^2}{(\log m)^{3/2}} \sum_{k=1}^{\lfloor \sqrt{m} \rfloor} f_k^m(x).
\]

Now introduce the function \( Q_{m-1}(x, y) \) defined on the unit disc \( \mathbb{B}^2 \)

\[
Q_{m-1}(x, y) := q_{m-1}(x) \quad \text{for } (x, y) \in \mathbb{B}^2.
\]

First we prove that for \( m \geq m_0^{(1)} \)

\[
\|Q_{m-1}\|_2^2 \leq C_4,
\]

for some absolute constant \( m_0^{(1)} \). Indeed (cf. (19), (20), (21))

\[
\int_{\mathbb{B}^2} |Q_{m-1}(x, y)|^2 \, dxdy = 2 \int_{-1}^{1} |q_{m-1}(x)|^2 \sqrt{1-x^2} \, dx
\]

\[
= 2 \sum_{l=1}^{\lfloor \sqrt{m} \rfloor} \int_{\cos \frac{2l+1}{m}\pi}^{\cos \frac{2l}{m}\pi} |q_{m-1}(x)|^2 \sqrt{1-x^2} \, dx = 2 \frac{m^3}{\log m} \sum_{l=1}^{\lfloor \sqrt{m} \rfloor} \frac{1}{l^3} \int_{\cos \frac{2l+1}{m}\pi}^{\cos \frac{2l}{m}\pi} \sqrt{1-x^2} \, dx
\]

\[
\leq 2 \frac{m^3}{\log m} \sum_{l=1}^{\lfloor \sqrt{m} \rfloor} \frac{1}{l^3} \sin \frac{(2l+1)\pi}{m} \left( \cos \frac{2l\pi}{m} - \cos \frac{(2l+1)\pi}{m} \right)
\]

\[
= 4 \frac{m^3}{\log m} \sum_{l=1}^{\lfloor \sqrt{m} \rfloor} \frac{1}{l^3} \sin \frac{(2l+1)\pi}{m} \sin \frac{\pi}{2m} \sin \frac{(4l+1)\pi}{2m}
\]

\[
\leq \frac{C_5}{\log m} \sum_{l=1}^{\lfloor \sqrt{m} \rfloor} \frac{1}{l} \quad \text{for } m \geq m_0^{(1)},
\]

where \( m_0^{(1)}, C_5, C_6 \) are absolute positive constants. Now we prove that for \( m \geq m_0^{(2)} \) the following inequality is true

\[
\max_{0 \leq k \leq m} |a_{m-1}(Q_{m-1}, k, \varphi)| \geq C_3 (\log m)^{1/2} \quad \text{for all real } \varphi,
\]

(23)
where \( C_3 \) and \( m_0^{(2)} \) are absolute positive constants. Indeed, it is known [2] that if \( F \in L^2_w([-1, 1]), \quad w(t) = 2\sqrt{1-t^2}, \quad t \in [-1, 1], \) then for the function

\[
P(x, y) := F(x) \quad (x, y) \in \mathbb{B}^2
\]

we have

\[
a_m(P, k, \varphi) := \frac{\sqrt{\pi}}{m+1} \hat{F}(m) u_m \left( \cos \left( \frac{k\pi}{m+1} + \varphi \right) \right)
\]

where \( k = 0, 1, \ldots, m \), \( \varphi \in (-\infty, \infty) \) and

\[
\hat{F}(m) := 2 \int_{-1}^{1} F(t) u_m(t) \sqrt{1-t^2} dt.
\]

Further we show that for some absolute positive constant \( C_7 \)

\[
|\hat{q}_{(m-1)}(m-1)| \geq C_7 (\log m) \frac{1}{\sqrt{m}}.
\]

According to (19), (20), (26) we get

\[
\hat{q}_{(m-1)}(m-1) := 2 \int_{-1}^{1} q_{m-1}(t) u_{m-1}(t) \sqrt{1-t^2} dt
\]

\[
= 2 \frac{m^2}{(\log m)^{\frac{1}{2}}} \sum_{k=1}^{\sqrt{m}} \int_{-1}^{1} f_k^{(m)}(t) u_{m-1}(t) \sqrt{1-t^2} dt
\]

\[
= 2 \frac{m^2}{(\log m)^{\frac{1}{2}}} \sum_{k=1}^{\sqrt{m}} \frac{1}{k^2} \int \cos \frac{2k+1}{m} \pi \sin m \arccos t dt
\]

\[
\geq 2 \frac{m^2}{(\log m)^{\frac{1}{2}}} \sum_{k=1}^{\sqrt{m}} \frac{1}{k^2} \int \cos \frac{2k+1}{m} \pi \sin m \vartheta \sin \vartheta d\vartheta
\]

\[
\geq C_8 \frac{1}{(\log m)^{\frac{1}{2}}} \sum_{k=1}^{\sqrt{m}} \frac{1}{k} \geq C_7 (\log m)^{\frac{1}{2}}
\]

where \( C_7 \) and \( C_8 \) are positive absolute constants. Let \( \varphi_0 \) be such that \( \varphi_0 = \varphi \pmod{\pi} \) and \( 0 \leq \varphi_0 < \pi \). Now we prove that there exists an integer \( k_1 = k_1(\varphi_0) \) with the properties

\[
\left| u_{m-1} \left( \cos \left( \frac{k_1\pi}{m} + \varphi_0 \right) \right) \right| \geq \frac{2}{\pi} m \quad \text{and} \quad 0 \leq k_1 \leq m - 1.
\]
Consider the following cases: let first $0 \leq \varphi_0 < \frac{\pi}{2m}$. In this case we take $k_1 := 0$. We see that then (cf. (2))

$$
|u_{m-1}\left(\cos\left(\frac{k_1 \pi}{m} + \varphi_0\right)\right)| = |u_{m-1}(\cos \varphi_0)| \frac{|\sin m \arccos (\cos \varphi_0)|}{|\sin \varphi_0|} \geq \frac{2}{\pi} m.
$$

If now $\frac{\pi}{2m} \leq \varphi_0 < \frac{\pi}{m}$ then we choose $k_1 = m - 1$. It is clear that then

$$
|u_{m-1}\left(\cos\left(\frac{k_1 \pi}{m} + \varphi_0\right)\right)| = |u_{m-1}\left(\cos\left(\frac{(m-1) \pi}{m} + \varphi_0\right)\right)|
$$

$$
= |u_{m-1}\left(-\cos\left(\frac{\pi}{m} - \varphi_0\right)\right)| = |u_{m-1}(\cos (\frac{\pi}{m} - \varphi_0))|
$$

$$
= \frac{|\sin m (\frac{\pi}{m} - \varphi_0)|}{|\sin (\frac{\pi}{m} - \varphi_0)|} \geq \frac{2}{\pi} m.
$$

Now it remains only the case $\frac{\pi}{m} \leq \varphi_0 < \pi$. Let $k_0 = k_0(\varphi_0)$ be the integer such that

$$
\frac{k_0 \pi}{m} + \varphi_0 < \pi \leq \frac{(k_0 + 1) \pi}{m} + \varphi_0.
$$

(29)

It is clear that in this case (cf. (29), (30))

$$
\frac{k_0 \pi}{m} < \pi - \varphi_0 \leq \pi - \frac{\pi}{m} \quad \text{and} \quad \frac{(k_0 + 1) \pi}{m} \geq \pi - \varphi_0 > 0,
$$

and consequently,

$$
0 \leq k_0 < m - 1 \quad \text{and} \quad 0 < \pi - \left(\frac{k_0 \pi}{m} + \varphi_0\right) \leq \frac{\pi}{m}.
$$

(30)

Now we have two subcases:

$$
0 < \pi - \left(\frac{k_0 \pi}{m} + \varphi_0\right) \leq \frac{\pi}{2m}
$$

(31)

and

$$
\frac{\pi}{2m} < \pi - \left(\frac{k_0 \pi}{m} + \varphi_0\right) \leq \frac{\pi}{m}.
$$

(32)
Let
\[ k_1 := k_0 \quad \text{in the first subcase and} \quad k_1 := k_0 + 1 \quad \text{in the second subcase.} \quad (33) \]

It is clear that in both cases (cf. (30))
\[ 0 \leq k_1 \leq m - 1. \]

In the first subcase we have (cf. (2), (33), (31))
\[
\left| u_{m-1} \left( \cos \left( \frac{k_1\pi}{m} + \varphi_0 \right) \right) \right| = \left| u_{m-1} \left( \cos \left( \frac{k_0\pi}{m} + \varphi_0 \right) \right) \right| = \left| \sin m \left( \frac{\pi - \left( \frac{k_0\pi}{m} + \varphi_0 \right)}{m} \right) \right| \geq \frac{2}{\pi^2}.
\]

And at last for the second subcase we get (cf. (13), (33), (32))
\[
\left| u_{m-1} \left( \cos \left( \frac{(k_0+1)\pi}{m} + \varphi_0 - \pi \right) \right) \right| = \left| \sin m \left( \frac{\pi - \left( \frac{(k_0+1)\pi}{m} + \varphi_0 - \pi \right)}{m} \right) \right| \geq \frac{2}{\pi^2}.
\]

The inequality (28) and consequently the inequality (23) are proved. Now we will estimate \( \omega_1(\delta; Q_{m-1}) \). Taking account of the fact that for \( k = 1, 2, \ldots, \lfloor \sqrt{m} \rfloor \),
\[
\frac{\cos (2k+1)\pi}{m} - \frac{\cos (2k+2)\pi}{m} \geq 2 \sin \frac{\pi}{2m} \sin \frac{(4k+3)\pi}{2m} \geq \frac{2}{m^2},
\]
we get for \( |h| \leq \frac{2}{m^2} \) (cf. (19), (20), (21))
\[
\int_{B^2 \cap B^2(1,h)} |Q_{m-1}(x+h, y) - Q_{m-1}(x, y)|^{\frac{3}{2}} \, dx \, dy \leq |h| \frac{m^3}{\log m} \sum_{k=1}^{\lfloor \sqrt{m} \rfloor} \frac{1}{k^3} \sin \frac{(2k+2)\pi}{m} \leq C_9 |h| \frac{m^2}{\log m},
\]
and for \( |h| > \frac{2}{m^2} \) we have (cf. (22))
\[
\int_{B^2 \cap B^2(1,h)} |Q_{m-1}(x+h, y) - Q_{m-1}(x, y)|^{\frac{3}{2}} \, dx \, dy \leq \left( 2 \|Q_{m-1}\|_{\frac{3}{2}} \right)^{\frac{3}{2}} \leq C_{11},
\]
for some absolute $C_9, C_{10}$ and $C_{11}$. From (19), (20), (21) we see that the Lemma is established completely.

**Proof of Theorem 2.** We define an increasing sequence of positive integers \( \{m_l\}_{l=1}^{\infty} \) by induction. Let \( m_1 = m_0 + 1 \) where \( m_0 \) is the number from the Lemma. Now let numbers \( m_1, m_2 \ldots m_{l-1} \) be already defined. Introduce the functions defined on \( \mathbb{B}^2 \) and \([-1,1]\) correspondingly

\[
A_{l-1}(x, y) := \sum_{k=1}^{l-1} \frac{1}{(\log m_k)^\frac{3}{2}} Q_{m_k-1}(x, y),
\]

and

\[
B_{l-1}(x) := \sum_{k=1}^{l-1} \frac{1}{(\log m_k)^\frac{3}{2}} q_{m_k-1}(x),
\]

where \( Q_{m_k-1}(x, y) \) and \( q_{m_k-1}(x) \) are functions from the Lemma corresponding to the number \( m_k \). It is clear that (cf. (24), (19), (20)) \( A_{l-1}(x, y) \in L^2(\mathbb{B}^2), B_{l-1}(x) \in L^2([-1,1]) \) and (cf. (24), (25), (26))

\[
|a_{m-1}(A_{l-1}, k, \varphi)| \leq \pi |\hat{B}_{l-1}(m-1)|, \text{ for all real } \varphi.
\]

It is clear that

\[
\lim_{m \to \infty} |\hat{B}_{l-1}(m-1)| = 0.
\]

From the last equation we conclude that there is the number \( N_{l-1} \) such that for all \( m \geq N_{l-1} \)

\[
|\hat{B}_{l-1}(m-1)| \leq \frac{C_3}{2\pi}
\]

where \( C_3 \) is the constant from the Lemma. Now we define \( m_l \) so that the following relations are satisfied:

\[
m_l > m_{l-1}, \quad m_l \geq N_{l-1},
\]

\[
\frac{m_{l-1}}{(\log m_l)^\frac{3}{2}} \leq \frac{1}{l+1},
\]

\[
2(\log m_l)^{-\frac{1}{3}} \leq (\log m_{l-1})^{-\frac{1}{3}},
\]

and

\[
\frac{m_l^4}{\log m_l} \geq 2 \frac{m_{l-1}^4}{\log m_{l-1}}.
\]
Thus we have constructed the infinite increasing sequence of integers \( \{m_l\}_{l=1}^{\infty} \). Consider the function
\[
g(x, y) := \sum_{k=1}^{\infty} \frac{1}{(\log m_k)^{\frac{1}{3}}} Q_{m_k-1}(x, y)
\] (41)
defined on \( \mathbb{B}^2 \). It is obvious that (cf. (41), (22), (39))
\[
\|g\|_{\frac{3}{2}} \leq \sum_{k=1}^{\infty} \frac{C_4}{(\log m_k)^{\frac{1}{3}}} < \infty.
\] (42)

Let \( \{\varphi_m\}_{m=0}^{\infty} \) be an arbitrary sequence of real numbers. According to (34), (41) we get for each \( k = 0, 1, \ldots, m_l - 1, l = 1, 2, \ldots \),
\[
a_{m_l-1}(g, k, \varphi_{m_l-1}) = a_{m_l-1}(A_{l-1}, k, \varphi_{m_l-1}) + a_{m_l-1}(Q_{m_l-1}(\log m_l)^{-\frac{1}{3}}, k, \varphi_{m_l-1})
\]
\[+ a_{m_l-1}(E_l, k, \varphi_{m_l-1}),
\]
where
\[
E_l(x, y) := \sum_{k=l+1}^{\infty} \frac{1}{(\log m_k)^{\frac{1}{3}}} Q_{m_k-1}(x, y).
\] (43)

According to (25), (34), (35), (36), (26), (37) for each \( k = 0, 1, \ldots, m_l - 1, l = 1, 2, \ldots \) the following inequality holds true
\[
|a_{m_l-1}(A_{l-1}, k, \varphi_{m_l-1})| \leq \frac{C_3}{2}.
\] (44)

On the other hand, it follows from (39), (15), (38) and (43) that for each \( k = 0, 1, \ldots, m_l - 1, l = 1, 2, \ldots \), we have
\[
|a_{m_l-1}(E_l, k, \varphi_{m_l-1})| = O \left( \sum_{k=l+1}^{\infty} \frac{1}{(\log m_k)^{\frac{1}{3}}} \right) = O \left( \frac{1}{l+1} \right) \text{ as } l \to \infty.
\] (45)

Now it is easy to see that (cf. (45), (44)) for each \( k = 0, 1, \ldots, m_l - 1, l = 1, 2, \ldots \), we get
\[
|a_{m_l-1}(Q_{m_l-1}(\log m_l)^{-\frac{1}{3}}, k, \varphi_{m_l-1})| \leq |a_{m_l-1}(g, k, \varphi_{m_l-1})|
\]
\[+ |a_{m_l-1}(A_{l-1}, k, \varphi_{m_l-1})| + |a_{m_l-1}(E_l, k, \varphi_{m_l-1})| \leq |a_{m_l-1}(g, k, \varphi_{m_l-1})|
\]
\[
\frac{C_3}{2} + O \left( \frac{1}{l+1} \right) \leq \max_{0 \leq k \leq m_l-1} |a_{m_l-1}(g, k, \varphi_{m_l-1})|
\]
\[
\frac{C_3}{2} + O \left( \frac{1}{l+1} \right) \text{ as } l \to \infty
\]
and therefore according to the Lemma (cf. (14))

\[
\max_{0 \leq k \leq m-1} |a_{m-1}(g, k, \varphi_{m-1})| \geq \max_{0 \leq k \leq m-1} |a_{m-1}(Q_{m-1}(\log m_l)^{-\frac{1}{3}}, k, \varphi_{m-1})|
\]

\[
\frac{C_3}{2} - O \left( \frac{1}{l+1} \right) \geq C_3 - \frac{C_3}{2} - o(1) \quad \text{as} \quad l \to \infty.
\]

We see now that the relation (10) of theorem 2 is established. It is obvious from (13), (41) and the Lemma that the function \( g(x, y) \) is in fact a function of one variable and consequently the second equation in (17) is true. It remains only to estimate \( \omega_1(\delta; g)^{\frac{3}{2}} \). Let for a given \( \delta > 0 \) the integer \( l_0 = l_0(\delta) \) be such that

\[
\frac{2}{m_{l_0+1}^2} < \delta \leq \frac{2}{m_{l_0}^2}.
\]

From (16), (41), (40), (39) we see that

\[
\omega_1(\delta; g)^{\frac{3}{2}} \leq \sum_{k=1}^{\infty} \frac{1}{(\log m_k)^{\frac{3}{4}}} \omega_{m_k-1}(\delta) \leq C_5 \delta^{\frac{3}{2}} \sum_{k=1}^{l_0} \frac{m_k^{\frac{3}{4}}}{\log m_k}
\]

\[
+ 2C_5 \sum_{k=l_0+1}^{\infty} \frac{1}{(\log m_k)^{\frac{3}{4}}} \leq 2C_5 \frac{m_{l_0}^{\frac{3}{4}}}{\log m_{l_0}} \delta^{\frac{3}{2}} + 4C_5 \frac{1}{(\log m_{l_0+1})^{\frac{3}{4}}} \leq C_{12} \frac{1}{\log \frac{1}{\delta}}
\]

\[
+ C_{13} \frac{1}{(\log \frac{1}{\delta})^{\frac{3}{4}}} = O \left( \frac{1}{(\log \frac{1}{\delta})^{\frac{3}{4}}} \right) \quad \text{as} \quad \delta \to 0^+.
\]

Theorem 2 is now proven.

References


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