On the half-half case of the Zarankiewicz problem

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Abstract

Consider the minimum number $f(m, n)$ of zeroes in a $2m \times 2n$ $(0, 1)$-matrix $M$ that contains no $m \times n$ submatrix of ones. This special case of the well-known Zarankiewicz problem was studied by Griggs and Ouyang, who showed, for $m \leq n$, that $2n + m + 1 \leq f(m, n) \leq 2n + 2m - \gcd(m, n) + 1$. The lower bound is sharp when $m$ is fixed for all large $n$. They proposed determining $\lim_{m \to \infty} \{f(m, m + 1)/m\}$. In this paper, we show that this limit is 3. Indeed, we determine the actual value of $f(m, km + 1)$ for all $k, m$. For general $m, n$, we derive a new upper bound on $f(m, n)$. We also give the actual value of $f(m, n)$ for all $m \leq 7$ and $n \leq 20$. 

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Section 1. Introduction

The terminology and notation in this paper are the same as in the paper [4] by Griggs and Ouyang. We consider rectangular matrices $M$ with entries that are 0 or 1. The intersection of $a$ rows and $b$ columns of a matrix is called an $a \times b$ submatrix. We say that a $2m \times 2n$ matrix $M$ has Property Z if every $m \times n$ submatrix has at least one zero, i.e., $M$ has no half-half all ones submatrix. An equivalent formulation of Property Z, that is typically more useful in our study, is to require that for every $m$ rows of $M$ at least $n+1$ columns contain a zero somewhere in those rows. We denote by $f(m, n)$ the minimum number of zeroes in such a matrix $M$ with Property Z. For simplicity, we often assume that $m \leq n$, since we may switch to the transpose when $m > n$.

In general, we may ask the maximum number $Z = Z_{m,n}(k,l)$ of ones in a $k \times l$ matrix $M$ avoiding $m \times n$ all ones submatrix. (Note that $f(m, n) = 4mn - Z_{m,n}(2m, 2n)$.) In 1951 Zarankiewicz [5] posed the problem of determining $Z_{m,m}(k,k)$ for $k \geq 4$, and the general problem concerning $Z_{m,n}(k,l)$ has also become known as the problem of Zarankiewicz.

By viewing $M$ as the incidence matrix for a bipartite graph, we can obtain the graph-theoretic formulation of Zarankiewicz problem that asks for the maximum number of edges in a bipartite graph $(K,L)$ with part sizes $|K| = k$, $|L| = l$ such that there is no complete bipartite subgraph $K_{m,n}$ with $m$ vertices in $K$ and $n$ vertices in $L$.

A survey of work on the Zarankiewicz problem appears in [1, Sec. VI.2]. Some of the more recent work includes the papers [2, 3, 4].

For the half-half case of the Zarankiewicz problem, Griggs and Ouyang obtained the following results on $f(m, n)$:

**Theorem 1.1.** [4] Assume $m \leq n$. Then

$$f(m, n) \geq 2n + m + 1,$$

where the equality holds precisely when

1. $n$ is a multiple of $m$, or
2. $k + r \geq m$, where $n = km + r$, and $0 < r < m$. 

**Theorem 1.2.** [4] Assume $m < n$. Then

1. $f(m, n) \leq 2km + f(r, m)$, where $n = km + r$, and $0 < r \leq m$,
2. $f(m, n) \leq 2n + 2m - \gcd(m, n) + 1$, where $\gcd(m, n)$ is the greatest common divisor of $m$ and $n$.

By Theorem 1.1 and Theorem 1.2(2), they observed that $3m + 4 \leq f(m, m+1) \leq 4m + 2$ and proposed determining $\lim_{m \to \infty} \{f(m, m+1)/m\}$. In this paper we show that this limit is 3. Indeed, we prove that for all $k$, $m$, $f(m, km + 1) = 2(km + 1) + m + i$, where $i$ is the largest integer such that $\left\lfloor i^2/4 \right\rfloor + 1 < m$. For general $m$, $n$, we also derive a new upper bound on $f(m, n)$.
In Section 2 we consider \( n = km + 1 \) and construct \( 2m \times 2n \) matrices \( M_t \) for \( 1 \leq t \leq m \) such that each matrix \( M_t \) has Property Z. Denoting the number of zeroes in \( M_t \) by \( g(t) \), we prove \( f(m, n) = \min\{g(t) : 1 \leq t \leq m\} \) and derive the formula for \( f(m, n) \).

In Section 3 we consider an extension of matrices \( M_t \) for general \( m, n \), and derive a new upper bound on \( f(m, n) \). In Section 4 we give the actual value of \( f(m, n) \) for small \( m, n \). Some of these values are obtained by tedious analysis of several cases. Finally, in Section 5 we summarize what we now know.

Section 2. The Actual Value of \( f(m, km + 1) \)

When \( n = km + r \) with \( 0 < r < m \) and \( k + r \geq m \), Griggs and Ouyang [4] presented a matrix achieving \( f(m, n) = 2n + m + 1 \). By permuting columns and rearranging the entries in the last row of this matrix, we obtain the matrix shown in Figure 1. (All the blank entries in this figure are ones.)

![Figure 1](image-url)

This matrix inspires us to consider the following construction: Assume \( 2 \leq m < n \) and \( n = km + 1 \). For \( 1 \leq t \leq m \), we construct a \( 2m \times 2n \) \((0, 1)\)-matrix \( M_t \) illustrated in Figure 2. In this construction, \( q, \alpha, \) and \( \beta \) are the integers satisfying \( 2n = km + kt + 2 = (kt - k + 1)q + k\alpha + \beta \), i.e.,

\[
km + kt + 2 = (kt - k + 1)q + k\alpha + \beta,
\]

where \( 0 < k\alpha + \beta \leq kt - k + 1 \) and \( 0 < \beta \leq k \). For example, when \( m = 3 \) and \( n = 4 \), Figure 3 displays the matrices \( M_1, M_2, \) and \( M_3 \).

Denote the number of zeroes in \( M_t \) by \( g(t) \). Then we have
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Figure 2. The matrix $M_t$ for $n = km + 1$.

\[
M_1 = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
M_2 = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix},
\]

\[
M_3 = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]

Figure 3. The matrices $M_t$ for $(m, n) = (3, 4)$.
**Proposition 2.1.** Assume $2 \leq m < n$, $n = km + 1$, and $1 \leq t \leq m$. Then

1. The matrix $M_t$ has Property Z;
2. $g(t) = (2k + 1 + \frac{1}{kt-k+1})m + t + \frac{t+2}{kt-k+1} + \frac{\alpha-(t-1)\beta}{kt-k+1}$;
3. $g(t) = \left[(2k + 1 + \frac{1}{kt-k+1})m + t + \frac{t+2}{kt-k+1}\right]$.

**Proof.** (1) We consider a two-coloring on all zeroes in $M_t$: We assign blue to the first $k$ zeroes in each row, and assign red to all the rest. Then in any $m$ rows, we can find exactly $km$ blue zeroes and at least 2 red zeroes such that all these zeroes are in different columns. Therefore, any $m$ rows have zeroes in at least $km + 2 = n + 1$ columns, and Property Z holds for the matrix $M_t$.

(2) Note that the number of zeroes in $M_t$ is $k(m - t) + (k + 1)((t - 1)q + \alpha) + (k + 2)\frac{q + \beta - 2}{k}$. Since $km + kt + 2 = (kt - k + 1)q + k\alpha + \beta$, we can write $q$ in terms of other variables and obtain the formula for $g(t)$.

(3) From the conditions $0 < k\alpha + \beta \leq kt - k + 1$ and $0 < \beta \leq k$, we have $0 \leq \alpha \leq t - 1$. Thus $0 \leq \frac{-\alpha+(t-1)\beta}{kt-k+1} < 1$ and the formula for $g(t)$ is verified. □

**Lemma 2.2.** Assume $2 \leq m < n$ and $n = km + 1$. Then 

$$f(m, n) = \min\{g(t) : 1 \leq t \leq m\}.$$ 

**Proof.** Let $M = [M_{i,j}]$ be a $2m \times 2n$ $(0, 1)$-matrix with Property Z. By Proposition 2.1(1), it suffices to show that the number of zeroes in $M$ is not less than $g(t)$ for some $t$, $1 \leq t \leq m$.

Let $R_0 = \emptyset$. For $i = 1, \ldots, 2m$, let $R_i = \{j : M_{i,j} = 0\}$ and $r_i = |R_i|$. Without loss of generality, we may assume $0 \leq r_1 \leq r_2 \leq \cdots \leq r_{2m}$. Choose the integer $t$ as small as possible such that $1 \leq t \leq m$ and $|R_0 \cup R_1 \cup \cdots \cup R_{m-t}| \leq k(m - t)$. We consider three cases:

**Case (1)** $t = 1$: Since $M$ has Property Z, we have $|R_1 \cup \cdots \cup R_m| \geq km + 2$. Then the condition “$t = 1$” forces $r_m \geq k + 2$. Thus the number of zeroes in $M \geq (km + 2) + m(k + 2) = g(1)$.

**Case (2)** $2 \leq t \leq m - 1$ and $|R_0 \cup R_1 \cup \cdots \cup R_{m-t}| < k(m - t)$: Note that $g(t) \leq (2k + 2)m + t$, since we have Proposition 2.1(3) and $t \geq 2$. Let $|R_1 \cup \cdots \cup R_{m-t}| = p$. Then the choice of $t$ implies $r_{m-t+1} \geq k(m - t + 1) - p + 1$, and hence the number of zeroes in $M \geq p + (k(m - t + 1) - p + 1)(m + t)$. Replacing $p$ with $k(m - t) - 1$, we obtain that the number of zeroes in $M \geq (2k + 2)m + 2t - 1 > (2k + 2)m + t \geq g(t)$.

**Case (3)** $2 \leq t \leq m$ and $|R_0 \cup R_1 \cup \cdots \cup R_{m-t}| = k(m - t)$: For $i = m - t + 1, m - t + 2, \ldots, 2m$, let $R'_i = R_i \setminus (R_0 \cup R_1 \cup \cdots \cup R_{m-t})$ and $r'_i = |R'_i|$. Then the choice of $t$ implies $r'_{m-t+1} \geq k + 1$. Write $km + kt + 2 = (kt - k + 1)q + k\alpha + \beta$, where $0 < k\alpha + \beta \leq kt - k + 1$ and $0 < \beta \leq k$. Comparing $M$ with $M_t$, we note that it is enough to show $r'_{m-t+(t-1)q+\alpha+1} \geq k + 2$.

Assume the contrary. Then $r'_{m-t+1} = \cdots = r'_{m-t+(t-1)q+\alpha+1} = k + 1$. Divide the index set $I = \{m - t + 1, \ldots, m - t + (t - 1)q + \alpha + 1\}$ into as many disjoint subsets.
We note that this inequality is equivalent to (1).

By Lemma 2.2, it suffices to show that \( \min_{t \in I} R'_t \) gives \( g \) and the proof is complete.

Proof. We have \( \sum_{i=1}^{P} (k|I_i| + 1) = k \sum_{i=1}^{P} |I_i| + p \geq k|I| + \lceil |I|/(t-1) \rfloor \geq k|I| + q + 1 \); on the other hand, we note that \( \min_{t \in I} R'_t \leq k((t-1)q + \alpha) + q + \beta \leq k(|I| - 1) + q + k \leq k|I| + q \), a contradiction. \( \blacksquare \)

Lemma 2.2 will facilitate our search for \( f(m, km + 1) \). It allows us to confine our analysis to the values of \( g(t) \) only. Using some fundamental Calculus, we obtain the minimum of \( g(t) \):

**Theorem 2.3.** Assume \( 2 \leq m < n \) and \( n = km + 1 \). Let \( t_0 = \frac{k^{-1} + \sqrt{km + k + 1}}{k} \). Then

\[
 f(m, n) = \min\{g([t_0]), g([t_0])\}. 
\]

**Proof.** It is easy to verify that \( 1 < t_0 \leq m \). So \( g([t_0]) \) and \( g([t_0]) \) are well-defined. By Lemma 2.2, it suffices to show that \( \min\{g(t) : 1 \leq t \leq m \} = \min\{g([t_0]), g([t_0])\} \). Consider a continuous function \( h(x) = (2k + 1 + \frac{1}{k^x-k-1})m + x + \frac{-x^2}{k^x-k-1} \), where \( x \in (1 - \frac{1}{k}, m + 1) \). Then \( h(t) = g(t) \) for \( t = 1, \ldots, m \), since we have Proposition 2.1(3). By taking the first and second derivatives for \( h(x) \), we verify that \( h(t_0) \) is a minimum and the proof is complete. \( \blacksquare \)

We provide in next theorem an alternative formula for \( f(m, km + 1) \).

**Theorem 2.4.** Assume \( 2 \leq m < n \), \( n = km + 1 \), and \( i \) is the largest integer such that \( \lfloor i^2/4 \rfloor k + i - 1 < m \). Then

\[
 f(m, n) = g(\lfloor (i+3)/2 \rfloor) = 2n + m + i. 
\]

**Proof.** We assume that \( i \) is an odd number and let \( i = 2\ell - 1 \) for some integer \( \ell \). (The proof of the other case “\( i \) is even” is similar.)

First, we prove \( g(\lfloor (i+3)/2 \rfloor) = 2n + m + i \), i.e., \( g(\ell+1) = 2n + m + 2\ell - 1 \). By the choice of \( i \), we have \( (\ell^2-\ell)k+2\ell-2 < m \leq \ell^2k+2\ell-1 \). Then \( (k\ell+1)(\ell+2) < km + k(\ell+1)+2 \leq (k\ell+1)(\ell+2) \). So we can write \( km+k(\ell+1)+2 = (k\ell+1)q + k\alpha + \beta \), where \( 0 < k\alpha + \beta \leq k\ell + 1 \), \( 0 < \beta \leq k \), and \( k\ell - k + 2 \leq q \leq k\ell + 1 \). Therefore, \( \frac{q+\beta-2}{k} = \ell \) and \( g(\ell+1) = k(m-\ell-1)+(k+2)\frac{q+\beta-2}{k} + (k+1)(2m-(m-\ell-1) - \frac{q+\beta-2}{k}) = 2n + m + 2\ell - 1 \).

By Lemma 2.2, it remains to prove that for \( 1 \leq t \leq m \), \( g(t) \geq 2n + m + i \). Indeed, by Proposition 2.1(3), we only need to show \( t + \frac{m-t+2}{k\ell-k+1} > 2\ell \). Since the choice of \( i \) gives \( m > (\ell^2-\ell)k+2\ell-2 \), it is enough to show that \( k\ell^2 - (2\ell k + k)t + \ell^2 k + \ell k \geq 0 \). We note that this inequality is equivalent to \( (t-\ell)(t-(\ell+1)) \geq 0 \), which is verified for all integers \( t \) and \( \ell \). \( \blacksquare \)

For general \( m, n \) with \( n = km + 1 \), Theorem 1.2(2) gives \( f(m, n) \leq 2n + 2m \). Now we can improve this upper bound.
Corollary 2.5. Assume $2 \leq m < n$ and $n = km + 1$. Then

$$2n + m + 1 \leq f(m, n) \leq 2n + m + 2 \left\lfloor \sqrt{m} \right\rfloor.$$ 

Proof. Let $i = 2 \left\lfloor \sqrt{m} \right\rfloor$. By Theorem 2.4, it suffices to show that $m \leq \left\lfloor (i + 1)^2 / 4 \right\rfloor k + i$. Let $\ell = \left\lfloor \sqrt{m} \right\rfloor$. Then $\left\lfloor (i + 1)^2 / 4 \right\rfloor k + i = (\ell^2 + \ell)k + 2\ell \geq (\ell + 1)^2 > (\sqrt{m})^2 = m$. 

Section 3. An Upper Bound on $f(m, n)$ for General $m, n$

When $n$ is a multiple of $m$, Theorem 1.1 gives $f(m, n) = 2n + m + 1$. So we assume in this section that $n$ is not a multiple of $m$.

We have constructed the matrix $M_t$ for the case $n = km + 1$ in Section 2. Now we consider the following extension for general $m, n$: Let $2 \leq m < n$ and $n = km + r$, where $0 < r < m$. For any integer $t$ with $1 \leq t \leq m$ and $t = r\ell + 1$ for some integer $\ell$, we construct a $2m \times 2n$ $(0, 1)$-matrix $M_t$ illustrated in Figure 4. In this construction, $q, \alpha,$ and $\beta$ are the integers satisfying

$$km + kt + 2r = (k\ell + 1)q + k\alpha + \beta,$$

where $0 < k\alpha + \beta \leq k\ell + 1$ and $0 < \beta \leq k$. For example, when $m = 4$ and $n = 6$, Figure 5 displays the matrices $M_1$ and $M_3$.

In particular, when $k + r \geq m$, $M_{r+1}$ is the same matrix as shown in Figure 1 that achieves $f(m, n) = 2n + m + 1$.

Denote the number of zeroes in $M_t$ by $g(t)$. Similar to Proposition 2.1 and Theorem 2.4, we can prove the following results:

Proposition 3.1. Assume $2 \leq m < n$ and $n = km + r$, where $0 < r < m$. Let $t$ be an integer such that $1 \leq t \leq m$ and $t = r\ell + 1$. Then

1. The matrix $M_t$ has Property Z;
2. $g(t) = (2k + 1 + \frac{r}{kt+1})m + r\ell + 1 + \frac{r^2\ell + r}{kt+1} + \frac{-r\alpha + r\ell\beta}{kt+1}$. 

Theorem 3.2. Assume $2 \leq m < n$ and $n = km + r$, where $0 < r < m$. Let $i$ be the largest integer such that $\left\lfloor \frac{i^2}{4} \right\rfloor k + \left\lfloor \frac{i}{2} \right\rfloor r + \left\lfloor \frac{i-1}{2} \right\rfloor < m$.

1. If $1 \leq i \leq 2 \left\lfloor \frac{m-1}{r} \right\rfloor$, then

$$f(m, n) \leq g \left( \left\lfloor \frac{i+1}{2} \right\rfloor r + 1 \right) \leq 2n + m + 1 + (i - 1)r;$$

2. If $i > 2 \left\lfloor \frac{m-1}{r} \right\rfloor$, i.e., $g \left( \left\lfloor \frac{i+1}{2} \right\rfloor r + 1 \right)$ is not defined, let $\ell = \left\lfloor \frac{m-1}{r} \right\rfloor$, then

$$f(m, n) \leq g(r\ell + 1) \leq 2n + m + 1 + \left( \ell - 1 + \left\lfloor \frac{kr + k - 1}{k\ell + 1} \right\rfloor / k \right) r.$$
Figure 4. The matrix $M_t$ for $n = km + r$, $r \neq 0$.

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \ \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \ \end{bmatrix}$$

Figure 5. The matrices $M_1$ and $M_3$ for $(m, n) = (4, 6)$. 
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**Proof.** The proof of (1) is similar to that of Theorem 2.4. To prove (2), we note that
\[ km + k(r\ell + 1) + 2r \leq (k\ell + 1)(2r) + kr + k, \] since \( m \leq r(\ell + 1). \) So we can write
\[ km + k(r\ell + 1) + 2r = (k\ell + 1)q + k\alpha + \beta, \] where \( 0 < k\alpha + \beta \leq k\ell + 1, 0 < \beta \leq k, \) and \( q \leq 2r + \left\lceil \frac{kr + k - 1}{k\ell + 1} \right\rceil. \) Therefore, \( g(r\ell + 1) = k(m - r\ell - 1) + (k + r + 1)\frac{q + \beta - 2r}{k} + (k + 1)(2m - (m - r\ell - 1) - \frac{q + \beta - 2r}{k}) \leq 2n + m + 1 + \left( \ell - 1 + \left\lceil \frac{kr + k - 1}{k\ell + 1} \right\rceil/k \right) r. \]

Note that each of Theorem 1.2 and Theorem 3.2 does not always provide a sharp bound for given \( m, n. \) For example, when \( m = 4 \) and \( n = 6, \) both theorems give \( f(4, 6) \leq 19; \) however, the matrix in Figure 6 shows \( f(4, 6) \leq 18. \) (Then it follows from Theorem 1.1 that \( f(4, 6) = 18.) \) We will check the performance of these two theorems for some small \( m, n \) in next section.

![Figure 6. A matrix giving \( f(4, 6) \leq 18. \)](image)

By Theorem 1.1 and Theorem 1.2(2), Griggs and Ouyang [4] observed that \( 3m + 4 \leq f(m, m + 1) \leq 4m + 2 \) and proposed determining \( \lim_{m \to \infty} \{ f(m, m + 1)/m \}. \) From Corollary 2.5, we can show that this limit is 3. In general, we have the following extension:

**Theorem 3.3.** For fixed positive integers \( k \) and \( r, \)

\[ \lim_{m \to \infty} \frac{f(m, km + r)}{m} = 2k + 1. \]

**Proof.** Note that if \( m \geq r^2 + 2, \) then \( i = 2 \left\lfloor \sqrt{m} \right\rfloor \leq 2 \left\lfloor (m - 1)/r \right\rfloor \) in Theorem 3.2(1) gives the upper bound \( f(m, km + r) \leq (2k + 1)m + 2r \left\lfloor \sqrt{m} \right\rfloor + r + 1. \) On the other hand, Theorem 1.1 gives the lower bound \( f(m, km + r) \geq (2k + 1)m + 2r + 1. \) Thus \( f(m, km + r)/m \to 2k + 1 \) as \( m \to \infty. \]

**Section 4. The Actual Value of \( f(m, n) \) for Small \( m, n \)**

By Theorems 1.1, 1.2, 2.4, 3.2, and tedious analysis of several cases, we have obtained in Figure 7 the actual value of \( f(m, n) \) for \( m \leq 7 \) and \( n \leq 20. \) In this figure, \( B \) denotes the general lower bound \( 2n + m + 1. \)
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| $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ | $n=8$ | $n=9$ | $n=10$ | $n=11$ | $n=12$ | $n=13$ | $n=14$ | $n=15$ | $n=16$ | $n=17$ | $n=18$ | $n=19$ | $n=20$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $m=1$ | 4     | 6     | 8     | 10    | 12    | 14    | 16    | 18    | 20    | 22    | 24    | 26    | 28    | 30    | 32    | 34    | 36    | 38    | 40    |
| $m=2$ | 7     | 9     | 11    | 13    | 15    | 17    | 19    | 21    | 23    | 25    | 27    | 29    | 31    | 33    | 35    | 37    | 39    | 41    | 43    |
| $m=3$ | $B_{+0}$ | $B_{+1}$ | 10    | 13    | 16    | 18    | 20    | 22    | 24    | 26    | 28    | 30    | 32    | 34    | 36    | 38    | 40    | 42    | 44    |
| $m=4$ | $B_{+0}$ | $B_{+1}$ | 13    | 16    | 18    | 19    | 21    | 24    | 25    | 27    | 29    | 31    | 33    | 35    | 37    | 39    | 41    | 43    | 45    |
| $m=5$ | $B_{+0}$ | $B_{+1}$ | $B_{+2}$ | 20    | 22    | 24    | 26    | 28    | 30    | 32    | 34    | 36    | 38    | 40    | 42    | 44    | 46    | 48    | 50    |
| $m=6$ | $B_{+0}$ | $B_{+1}$ | $B_{+2}$ | $B_{+3}$ | 23    | 25    | 27    | 29    | 31    | 33    | 34    | 36    | 38    | 40    | 42    | 44    | 46    | 48    | 50    |
| $m=7$ | $B_{+0}$ | $B_{+1}$ | $B_{+2}$ | $B_{+3}$ | $B_{+4}$ | 26    | 28    | 30    | 32    | 34    | 36    | 38    | 40    | 42    | 44    | 46    | 48    | 50    | 52    |

Figure 7. The actual value of $f(m, n)$ for $m \leq 7$ and $n \leq 20$.

Note that $f(5, 6) > f(6, 6)$ and $f(7, 18) > f(7, 19)$. Thus increasing $m$ or $n$ may actually decrease $f$.

When $n = km + r$ with $r \neq 0$, $r \neq 1$, and $k + r < m$, we may use Theorem 1.2 or Theorem 3.2 to find an upper bound for $f(m, n)$. For small $m, n$, the performance of these two theorems is displayed in Figure 8.

<table>
<thead>
<tr>
<th>$m=4$</th>
<th>$m=5$</th>
<th>$m=6$</th>
<th>$m=7$</th>
<th>$m=8$</th>
<th>$m=9$</th>
<th>$m=10$</th>
<th>$m=11$</th>
<th>$m=12$</th>
<th>$m=13$</th>
<th>$m=14$</th>
<th>$m=15$</th>
<th>$m=16$</th>
<th>$m=17$</th>
<th>$m=18$</th>
<th>$m=19$</th>
<th>$m=20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{+0}$</td>
<td>$B_{+1}$</td>
<td>$B_{+2}$</td>
<td>$B_{+3}$</td>
<td>$B_{+4}$</td>
<td>$B_{+5}$</td>
<td>$B_{+6}$</td>
<td>$B_{+7}$</td>
<td>$B_{+8}$</td>
<td>$B_{+9}$</td>
<td>$B_{+10}$</td>
<td>$B_{+11}$</td>
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<td>$B_{+13}$</td>
<td>$B_{+14}$</td>
<td>$B_{+15}$</td>
<td>$B_{+16}$</td>
</tr>
</tbody>
</table>

Figure 8. The performance of two upper bound theorems.

Section 5. Conclusion

We summarize the results concerning the value of $f(m, n)$ here: Assume $m \leq n$ and write $n = km + r$, where $0 \leq r < m$.

Case (1) If $r = 0$ or $k + r \geq m$, then $f(m, n) = 2n + m + 1$;
Case (2) If $r = 1$, $f(m, n)$ can be evaluated by Theorem 2.4 (or Theorem 2.3);  
Case (3) If $m \leq 7$ and $n \leq 20$, the value of $f(m, n)$ is given in Figure 7 in Section 4.
The Half-Half Problem

If $(m, n)$ is not in any of these three cases, then $2n + m + 2 \leq f(m, n) \leq u$, where $u$ is an upper bound obtained from Theorem 1.2 or 3.2. So the value of $f(m, n)$ for general $m$, $n$ is still undetermined.

For Case (1), Griggs and Ouyang described in [4] all extremal matrices, i.e., the matrices attaining $f(m, n)$. In this study we obtain the actual value of $f(m, n)$ for Case (2). So the extremal matrices for Case (2) deserve further investigation.

As we mentioned in Section 1, the problem of determining $f(m, n)$ is a special case of the famous problem of Zarankiewicz [5]. See [4] for more related open problems.

References


