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A new extension of Lubell's inequality to the lattice of divisors

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Abstract

P. L. Erdős and G. O. H. Katona gave an inequality involving binomial coefficients summed over an antichain in the product of two chains. Here we present the common generalization of this inequality and Lubell's famous inequality for the Boolean lattice to an arbitrary product of chains (lattice of divisors). We also describe the connection between this inequality and the LYM property.

1 Introduction

Let X be an $n-$set provided with a partition in M subsets X_i, called color classes, for $1 \leq i \leq M$. Let $n_i = |X_i|$ for all i. Associated with this coloring, we consider the poset $R(n_1, \ldots, n_m) = \{0 < \cdots < n_1\} \times \cdots \times \{0 < \cdots < n_M\}$, which consists of the product of M chains with ranks n_i. This poset is isomorphic to the lattice of divisors of $N = p_1^{n_1} \cdots p_M^{n_M}$, where the p_i's are distinct primes.

P. L. Erdős and G. O. H. Katona [3] discovered the following inequality for the product of just two chains in connection with their study of more-part Sperner families of subsets: For every antichain $I \subseteq R(n_1, n_2)$,

$$\sum_{(i_1,i_2)\in I} \frac{\binom{n_1}{i_1}\binom{n_2}{i_2}}{\binom{n_1+n_2}{i_1+i_2}} \leq 1. \tag{1}$$

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Their arguments were somehow lengthy, and a proof of a generalization for M colors was not apparent. We present such a generalization here along with some related observations.

Theorem 1.1 *If $I \subseteq R(n_1, \ldots, n_M)$ is an antichain, then*

$$\sum_{(i_1,\ldots,i_M)\in I} \frac{\binom{n_1}{i_1}\cdots\binom{n_M}{i_M}}{\binom{n_1+\cdots+n_M}{i_1+\cdots+i_M}} \leq 1.$$

Notice that this extends Lubell's familiar inequality [7] for the Boolean lattice B_M of all subsets of an M-set, which is the case that all $n_i = 1$. In the next section we present two different proofs, both simpler than the original one in [3] for $M = 2$. The first is by counting chains, an argument that just extends Lubell's proof of Sperner's theorem [7]. We recently discovered the same proof, for $M = 2$ only, in a paper [1] of Ahlswede and Zhang.

It is also stated in [1] that (1) is just the LYM inequality for the poset (evidently, $R(n_1, n_2)$), which is not quite true. Let P be a ranked poset, with rank function $r : P \to \{0, 1, \ldots\}$. Let P_k denote the set of elements with rank k. Let $N_P(x)$ denote the number of elements of rank $r(x)$. We recall that P is said to be LYM provided that for every antichain $I \subseteq P$,

$$\sum_{x\in I} \frac{1}{N_P(x)} \leq 1.$$

It is well-known that $R(n_1, \ldots, n_M)$ is LYM. (See [4] for a survey). Note that the contribution of an element $x \in I$ to the sum in the LYM inequality depends only on its rank, which is not the case for inequality (1).

Our second proof of Theorem 1.1 shows that it is in fact the LYM inequality for a weighted poset obtained naturally as a quotient of the Boolean lattice B_n of all subsets of X.

We must mention that (1) is in fact just a special case of an earlier inequality which lies at the heart of the proof of the product theorem for LYM posets, as presented in the survey by Greene and Kleitman ([4], p. 42). They show that for LYM, rank-log-concave posets P_1 and P_2 and maximum chains $C_1 \subseteq P_1$ and $C_2 \subseteq P_2$, every antichain $I \subseteq P_1 \times P_2$ satisfies

$$\sum_{(i_1,i_2)\in I\cap(C_1\times C_2)} \frac{N_{P_1}(i_1)N_{P_2}(i_2)}{N_{P_1\times P_2}(i_1, i_2)} \leq 1. \tag{2}$$

We obtain (1) when we take P_i to be the Boolean lattice B_{n_i} for $i = 1, 2$ in (2). Restricting the proof of Greene and Kleitman to this instance gives another proof of (1), although we cannot yet see how to extend it to prove Theorem 1.1 for general M. However, looking at (2) and Theorem 1.1 together, a common generalization is suggested, with (2) extended to general M and Theorem 1.1 extended to arbitrary LYM, rank-log-concave posets.

Theorem 1.2 *If $P_1, \ldots, P_m$ are LYM and rank-log-concave posets, and $C_i \subseteq P_i$ are maximum chains ($i = 1, \ldots, m$), then for any antichain $I \subseteq P_1 \times \cdots \times P_m$,*

$$\sum_{(i_1,\ldots,i_m)\in I\cap(C_1\times\cdots\times C_m)} \frac{N_{P_1}(i_1)\cdots N_{P_m}(i_m)}{N_{P_1\times\cdots\times P_m}(i_1,\ldots,i_m)} \leq 1.$$

We use the LYM Product Theorem of Harper, for weighted posets, to derive this result in Section 3. Note that it restricts to yet another proof of Theorem 1.1 when $P_i = B_{n_i}$.

2 Two Proofs of Theorem 1.1

First Proof of Theorem 1.1

Suppose that I is an antichain as stated in Theorem 1.1. The total number of maximal chains in the product poset $\{0,\ldots,n_1\} \times \cdots \times \{0,\ldots,n_M\}$ is given by

$$\binom{n_1+\cdots+n_M}{n_1,\ldots,n_M} := \frac{(n_1+\cdots+n_M)!}{n_1!\cdots n_M!}.$$

For any vector $(i_1,\ldots,i_M)$, the number of maximal chains that pass through it is given by

$$\binom{i_1+\cdots+i_M}{i_1,\ldots,i_M}\binom{n_1-i_1+\cdots+n_M-i_M}{n_1-i_1,\ldots,n_M-i_M}.$$

Finally, since I is an antichain,

$$\sum_{(i_1,\ldots,i_M)\in I} \binom{i_1+\cdots+i_M}{i_1,\ldots,i_M}\binom{n_1-i_1+\cdots+n_M-i_M}{n_1-i_1,\ldots,n_M-i_M} \leq \binom{n_1+\cdots+n_M}{n_1,\ldots,n_M},$$

and (1.1) follows after rewriting this last expression. □

Second Proof of Theorem 1.1

We need to recall a well-known result derived from Lubell's proof of Sperner's Theorem.

Theorem 2.1 *The Boolean lattice B_n of subsets of X has the LYM property.* $\square$

A *weighted poset* is a pair (P, v), with P a finite ranked poset and v a function that assigns a positive real number to each element of P. A weighted poset (P, v) satisfies the *LYM inequality* if for any antichain $I \subseteq P$,

$$\sum_{x \in I} \frac{v(x)}{v(P_{r(x)})} \leq 1.$$

If P is a poset and G is a group of automorphisms of P, then the *quotient poset* P/G consists of the orbits of P under G ordered by $A \leq B$ in P/G whenever there exist $x \in A$ and $y \in B$ with $x \leq y$ in P.

We will use the following theorem due essentially to Harper (1974) [6]. (See [2] for a complete treatment.)

Theorem 2.2 *A finite ranked poset P has the LYM property if and only if $(P/G, v)$ has the LYM property, where G is any subgroup of the group of automorphisms of P and $v(A)$ is the size $|A|$ of the class $A \in P/G$.* $\square$

Now consider the subgroup G of permutations of X that are color preserving, that is, if $\sigma \in G$, $\sigma(X_i) \subseteq X_i$, for each $1 \leq i \leq M$. Clearly G induces a subgroup of the group of automorphisms of 2^X, which we will still call G. It is immediate to check that the quotient poset $2^X/G$ with the canonical weight function as described in Theorem 2.2 is isomorphic to the weighted poset

$$P = (\{0, \ldots, n_1\} \times \cdots \times \{0, \ldots, n_M\},\ v) \quad ,$$

where $v((i_1, \ldots, i_M)) = \binom{n_1}{i_1} \cdots \binom{n_M}{i_M}$.

Now, since 2^X is LYM, by Theorem 2.2 (we are using the 'easy direction'), P is LYM. Hence, if $I \subseteq \{0, \ldots, n_1\} \times \cdots \times \{0, \ldots, n_M\}$ is an antichain, the LYM inequality ensures that

$$\sum_{(i_1, \ldots, i_M) \in I} \frac{v((i_1, \ldots, i_M))}{v(P_{r((i_1, \ldots, i_M))})} \leq 1.$$

Finally, the stated inequality follows from

$$\begin{aligned} v(P_{r((i_1, \ldots, i_M))}) &= \sum_{\substack{x_1 + \cdots + x_M = i_1 + \cdots + i_M \\ 0 \leq x_i \leq n_i}} \binom{n_1}{x_1} \cdots \binom{n_M}{x_M} \\ &= \binom{n_1 + \cdots + n_M}{i_1 + \cdots + i_M}. \square \end{aligned}$$

3 The Proof of Theorem 1.2

A weighted poset (P, v) is said *weight-log-concave* if the sequence $\{v(P_k)\}$ is log-concave. We recall the following Product Theorem due to Harper [6].

Theorem 3.1 *If (P_1, v_1) and (P_2, v_2) are weight-log-concave and satisfy the LYM inequality, then $(P_1 \times P_2, v_1 v_2)$ also satisfies the LYM inequality and is weight-log-concave.* $\square$

By induction we obtain the following.

Corollary 3.2 *If $(P_1, v_1), \ldots, (P_M, v_M)$ are weight-log-concave and satisfy the LYM inequality, then $(P_1 \times \cdots \times P_M, v_1 \cdots v_M)$ also satisfies the LYM inequality and is weight-log-concave.* $\square$

To prove (1.2) we consider the weighted posets $(C_1, N_{P_1}), \ldots, (C_M, N_{P_M})$ and apply the corollary. The inequality in Theorem 1.2 is just the LYM inequality for $(C_1 \times \cdots \times C_M, N_{P_1} \cdots N_{P_M})$. $\square$

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