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IMI
Preprint Series

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Abstract

Let

$$h(t,x) := \text{p.v.} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{\pi i(tn^2+2xn)}}{2\pi in} = \lim_{N\to\infty} \sum_{0<|n|\le N} \frac{e^{\pi i(tn^2+2xn)}}{2\pi in}$$

($i = \sqrt{-1}$; t, x – real variables). It is proved that in the rectangle $D := \left\{(t,x) : 0 < t < 1, |x| \le \frac{1}{2}\right\}$, the function h satisfies the following *functional inequality*:

$$|h(t,x)| \;\le\; \sqrt{t}\left|h\left(\frac{1}{t}, \frac{x}{t}\right)\right| + c,$$

where c is an absolute postive constant. Iterations of this relation provide another, more elementary, proof of the known global boundedness result

$$\|h;\ L^\infty(\mathrm{E}^2)\| := \text{ess sup}\ |h(t,x)| < \infty.$$

The above functional inequality is derived from a general duality relation, of theta-function type, for solutions of the Cauchy initial value problem for Schrödinger equation of a free particle.

Variation and complexity of solutions of Schrödinger equation are discussed.

1 Schrödinger equation, functional relations and L^∞-result

Consider the Cauchy initial value problem for time-dependent Schrödinger equation of a free particle

$$\frac{\partial\psi}{\partial t} = \frac{1}{4\pi i}\frac{\partial^2\psi}{\partial x^2}, \qquad \psi(t,x)\ |_{t=0} = f(x). \tag{1}$$

The *Green's function* of this problem is $\Gamma(t,x) = \sqrt{\frac{i}{t}}\ e^{-\frac{\pi i x^2}{t}}$, with $\sqrt{i} := e^{\frac{\pi i}{4}} = \frac{1+i}{\sqrt{2}}$ and $\sqrt{t} := i\sqrt{|t|}$, $t<0$; the solution operator $\psi(f;\ t,x)$ is represented by the convolution

Math. Subject Classification. 42A16, 35J10, 11L07, 11T24, 33B20.

Keywords and Phrases. Cauchy initial value problem, Schrödinger equation, Hilbert transform, Gauss' sum, continued fraction, Fresnel integral, curlicue, selfsimilarity.

Acknowledgements and Notes. The author was supported by DEPSCOR Grant N000149611003.

The author expresses his gratitude to Irina Mitrea, who read the manuscript and made a number of valuable remarks.

$$\psi(f;t,x) = f * \Gamma(t,\cdot)(x) = \int_{\mathrm{E}} f(\xi)\,\Gamma(t,x-\xi)\,d\xi\,. \tag{2}$$

This representation implies a *reciprocity type* relation between $\psi(f)$ and $\psi(\hat{f})$, where $\hat{f}$ denotes the Fourier transform of f:

$$\psi(f;t,x) = \Gamma(t,x)\,\psi\left(\hat{f};-\frac{1}{t},-\frac{x}{t}\right); \quad \hat{f}(y) := \int_{\mathrm{E}} f(x)\,e^{-2\pi i x y}\,dx. \tag{3}$$

Starting from (3) and using some very basic properties of continued fractions, cf. (16) below, we derive a simplified proof, cf. [17], [20] of the following statement concerning discrete Hilbert transforms of imaginary exponentials with real algebraic polynomial of second degree in the exponent.

Theorem 1 . *Let*

$$h(t,x) := \text{p.v.} \sum_{n\in \mathrm{Z}\setminus\{0\}} \frac{e^{\pi i(tn^2+2xn)}}{2\pi i n} := \lim_{N\to\infty} \sum_{0<|n|\le N} \frac{e^{\pi i(tn^2+2xn)}}{2\pi i n}\,.$$

Then h is essentially bounded on the real plane E^2*:*

$$\|h,\, L^\infty\| := \text{ess sup}\left\{|h(t,x)| :\, (t,x)\in \mathrm{E}^2\right\} < \infty. \tag{4}$$

First we deduce from (3) a functional inequality of theta type for the function h, see (10). Then we derive (4) from this inequality, using simple properties of continued fractions.

However, our main goal is not just another proof of (4). Rather, our intention is to demonstrate some deep relations which exist between objects of analytic number theory and partial differential equations of Schrödinger type with periodic initial data.

In section 2 we provide some comments. Section 3 contains a discussion of complexity of solutions of the problem (1).

Proof. Assume that the initial data function $f(x)$ in the problem (1) is smooth and rapidly decreasing as $|x|\to\infty$, say, $f(x)$ belongs to the Schwartz' space S of test functions. Via Fourier method of separation of variables, the solution $\psi(f;t,x)$ is given by

$$\psi(f;t,x) \;=\; \int_{\mathrm{E}} \hat{f}(y)\,e^{\pi i(ty^2+2xy)}\,dy\,. \tag{5}$$

If we take the initial data $f(x)$ to be Dirac's delta-function $\delta(x)$, so that $\hat{\delta}(y)\equiv 1$, we obtain the Green's function $\Gamma(t,x)$ of the problem (1):

$$\Gamma(t,x) = (\mathcal{R})\int_{\mathrm{E}} e^{\pi i(ty^2+2xy)}\,dy = e^{-\frac{\pi i x^2}{t}}\,(\mathcal{R})\int_{\mathrm{E}} e^{\pi i t y^2}\,dy = \sqrt{\frac{i}{t}}\,e^{-\frac{\pi i x^2}{t}}, \quad t\neq 0.$$

(($\mathcal{R}$) denotes improper Riemannian integration.) Thus,

$$\psi(f;t,x) = \; f*\Gamma(t,\cdot)(x) \;=\; \int_{\mathrm{E}} f(\xi)\,\Gamma(t,x-\xi)\,d\xi = \Gamma(t,x)\int_{\mathrm{E}} f(\xi)\,e^{-\frac{\pi i(\xi^2-2x\xi)}{t}}\,d\xi,$$

and (3) follows from (5) by inspection.

Let us introduce the *generalized* Θ-function:

$$\Theta(t,x) := \sum_{n\in \mathrm{Z}} e^{\pi i(tn^2+2xn)}.$$

If t is a fixed rational number, then the sequence of exponentials $e_n := e^{\pi i n^2 t}$, $n \in \mathrm{Z}$ is periodic in n, and the series is summable to a linear combination of shifted δ-functions, say, by $(C,1)$-means, cf. also (23) below. On the other hand, as it was observed by G.H. Hardy and J.E. Littlewood [10], for irrational values of t the series *is not summable* by regular methods.

We understand $\Theta(t,x)$ as a family of linear functionals, parametrized by $t \in \mathrm{E}$, over the Schwartz space S of test functions $\varphi(x)$, $x \in \mathrm{E}$. *By definition* for $\varphi \in S$ we have

$$\Theta(t,\cdot)\cdot\varphi(\cdot) := \sum_{n\in \mathrm{Z}} \int_{\mathrm{E}} \varphi(x)\, e^{\pi i(tn^2+2xn)}\, dx = \sum_{n\in \mathrm{Z}} \hat{\varphi}(-n)\, e^{\pi i n^2 t} = \sum_{n\in \mathrm{Z}} \hat{\varphi}(n)\, e^{\pi i n^2 t}.$$

The role of Θ is clear. If the initial data function $f(x)$ in the problem (1) is periodic, i. e. $f(x+1) \equiv f(x)$, then

$$\psi(f;t,x) \sim \sum_{n\in \mathrm{Z}} \hat{f}(n)\, e^{\pi i(tn^2+2xn)}, \quad \hat{f}(n) := \int_0^1 f(x)\, e^{-2\pi i n x}\, dx, \quad n \in \mathrm{Z}\,.$$

The solution ψ is represented by convolution of f with Θ on the period:

$$\psi(f;t,x) \;\sim\; \Theta(t,\,\cdot) * f\,(x) := \int_0^1 \Theta(t, x-\xi)\, f(\xi)\, d\xi. \tag{6}$$

In the other words, Θ is the Green's function of the problem (1) with periodic initial data. In the sense of linear functionals over S, Θ coincides with the *periodization* of Γ in x:

$$\Theta(t,x) \;:=\; \sum_{n\in \mathrm{Z}} \Gamma(t, x-n).$$

In particular, $h(t,x)$ represents the *generalized solution* of the problem (1) with $f(x) = \frac{1}{2} - \{x\}$, where $\{x\}$ denotes the fractional part of $x \in \mathrm{E}$:

$$\frac{\partial h}{\partial t} = \frac{1}{4\pi i}\frac{\partial^2 h}{\partial x^2}, \qquad h(t,x)\mid_{t=0} = \frac{1}{2} - \{x\} = \text{p.v.} \sum_{n\in \mathrm{Z}\setminus\{0\}} \frac{e^{2\pi i n x}}{2\pi i n}.$$

The following relation is a corollary of (3). It is a variant of the well known functional equation for the Jacobi theta-function $\vartheta(t,x)$, cf. (20), (21), and can be considered as a limiting case of the latter for $\Re\, t \to 0_+$ in the sence of linear functionals over S.

Lemma 1 . *For each fixed $t \neq 0$, $\Theta(t,x)$ satisfies, in the sense of linear functionals, the following equation:*

$$\Theta(t,x) = \Gamma(t,x)\, \Theta^*\left(\frac{1}{t}, \frac{x}{t}\right), \quad \textit{or} \quad \sum_{n\in \mathrm{Z}} e^{\pi i(tn^2+2xn)} = \sqrt{\frac{i}{t}}\, e^{-\frac{\pi i x^2}{t}} \sum_{n\in \mathrm{Z}} e^{-\frac{\pi i(n^2+2xn)}{t}}. \tag{7}$$

(Here and below, z^* denotes the conjugate of a complex number z.)

Indeed, let us take the initial data $f(x)$ in the problem (1) to be the periodized δ-function $\Delta(x) := \sum_{n\in Z} \delta(x-n)$. Then $\Delta(x) = \hat{\Delta}(x)$ (Poisson summation formula) and (7) follows from (3).

Lemma 2 . *The function h satisfies the following functional equation:*

$$x + h(t,x) = g(t,x) + \sqrt{it}\, e^{-\frac{\pi i x^2}{t}}\, h^*\left(\frac{1}{t}, \frac{x}{t}\right) + R(t,x), \quad t \neq 0, \tag{8}$$

where

$$g(t,x) := \int_0^x \Gamma(t,y)\, dy, \quad R(t,x) := -t \int_0^x h^*\left(\frac{1}{t}, \frac{y}{t}\right) d\Gamma(t,y).$$

Furthermore, for each fixed $t > 0$ the remainder term $R(t,x)$ is a Lipschitz' function of x in the interval $|x| \leq \frac{1}{2}$, and the following estimates hold

$$a)\ |R(t,x)| \leq c|x|\sqrt{t}; \quad b)\ \left|\frac{\partial R}{\partial x}\right| \leq \frac{c|x|}{\sqrt{t}}, \quad t > 0,\ |x| \leq \frac{1}{2}, \tag{9}$$

where c is an absolute postive constant. In particular,

$$|h(t,x)| \ \leq\ \sqrt{t}\left|h\left(\frac{1}{t}, \frac{x}{t}\right)\right| + c, \quad t > 0,\ |x| \leq \frac{1}{2}. \tag{10}$$

By term-wise differentiation of the series defining the function $h(t,x)$ we see that

$$\frac{\partial(x + h(t,x))}{\partial x} = \sum_{n\in Z} e^{\pi i(tn^2 + 2xn)} = \Theta(t,x).$$

Of course, here we keep the convention that the derivatives are understood as linear functionals on S.

Keeping this in mind, we can rewrite (7) in the following form:

$$\frac{\partial(x + h(t,x))}{\partial x} = \Gamma(t,x)\left[1 + t\frac{\partial}{\partial x} h^*\left(\frac{1}{t}, \frac{x}{t}\right)\right]. \tag{11}$$

Let us integrate both sides of (11) in the variable x and apply integration by parts to the righthand side. Since $h(t,x)$ satisfies the initial condition $h(t,x)|_{x=0} = 0$, we see that

$$\begin{aligned} x + h(t,x) &= \int_0^x \Gamma(t,y)\, dy + t\int_0^x \Gamma(t,y)\frac{\partial}{\partial y} h^*\left(\frac{1}{t}, \frac{y}{t}\right) dy \\ &= \int_0^x \Gamma(t,y)\, dy + t\Gamma(t,x) h^*\left(\frac{1}{t}, \frac{x}{t}\right) - t\int_0^x h^*\left(\frac{1}{t}, \frac{y}{t}\right) d\Gamma(t,y), \end{aligned}$$

whence (8) follows.

In (8), the function $g(t,x)$ is represented by the Fresnel integral Frl:

$$g(t,x) = \sqrt{i}\int_0^{\frac{x}{\sqrt{t}}} e^{-\pi i y^2}\, dy = \sqrt{i}\, \mathrm{Frl}^*\left(\frac{x}{\sqrt{t}}\right), \quad \text{where} \quad \mathrm{Frl}(x) := \int_0^x e^{\pi i y^2}\, dy. \tag{12}$$

Let us estimate the remainder term $R(t,x)$. By (8) and the definition of h we have

$$R(t,x)=\sum_{n\in \mathrm{Z}\setminus\{0\}} r_n(t,x),\quad r_n(t,x):=-t\int_0^x \left(\frac{e^{\frac{\pi i(n^2+2ny)}{t}}}{2\pi i n}\right)^* d\Gamma(t,y),$$

and it is easy to see that

$$\left(e^{\frac{\pi i(n^2+2ny)}{t}}\right)^* d\Gamma(t,y)=\frac{y}{y+n}d\Gamma(t,y+n).$$

Assume that $n\in \mathrm{Z}\setminus\{0\}$ and $|x|\le \frac{1}{2}$. Then we can integrate by parts

$$r_n(t,x)=\frac{t}{2\pi i n}\int_0^x \frac{y}{y+n}d\Gamma(t,y+n)=\frac{t}{2\pi i n}\left(\frac{x}{x+n}\Gamma(t,x+n)-\int_0^x \Gamma(t,y+n)\,d\left(\frac{y}{y+n}\right)\right).$$

and after it estimate the righthand side trivially:

$$|\Gamma(t,y)|\le\frac{1}{\sqrt{t}};\quad \left|\frac{x}{x+n}\right|+\int_0^x\left|d\frac{y}{y+n}\right|\le 2\left|\frac{x}{x+n}\right|.$$

Thus,

$$|r_n(t,x)|\le\frac{\sqrt{t}}{\pi}\left|\frac{x}{n(x+n)}\right|,\quad n\in \mathrm{Z}\setminus\{0\},\ |x|\le\frac{1}{2}.$$

These estimates imply

$$|R(t,x|\le\frac{\sqrt{t}}{\pi}\sum_{n\in \mathrm{Z}\setminus\{0\}}\left|\frac{x}{n(x+n)}\right|=\frac{2\sqrt{t}|x|}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2-x^2}\le\frac{4}{\pi}\sqrt{t}|x|,\quad |x|\le\frac{1}{2},$$

and (9)a) follows.

Further, (10) is a corollary of (8), (9)a) and (12), because $\mathrm{Frl}(x)$ is a bounded function on E. .

Now we can finish the proof of (4).

Given a real x, denote $[x]$, $\{x\}$ respectively, the integral and fractional parts of x. Let $\langle x\rangle$ stand for the distance from x to the nearest integer taken with its' sign, i.e. $\langle x\rangle:=\{2x\}-\{x\}$. It is easy to see that

$$e^{\pi i(tn^2+2nx)}\equiv e^{\pi i(\tau n^2+2\xi n)},\quad \text{where}\quad \tau:=\{t\},\ \xi:=\left\langle\frac{[t]}{2}+x\right\rangle;\quad t,x\in \mathrm{E};\ n\in \mathrm{Z}. \tag{13}$$

In particular, it is sufficient to prove that h is essentially bounded in the basic rectangle $D:=\left\{(t,x):0<t<1,|x|\le\frac{1}{2}\right\}$.

Let us iterate (10),using (13) and the following mapping M of D onto itself :

$$M:\quad (t,x)\mapsto M(t,x):=\left(\left\{\frac{1}{t}\right\},\left\langle\frac{1}{2}\left[\frac{1}{t}\right]+\frac{x}{t}\right\rangle\right).$$

This map is directly related with the *continued fraction* of $t \in (0,1)$:

$$t = [[q_1,\, q_2,\, \dots,\, q_m,\, \dots]] := \frac{1}{q_1+} : \frac{1}{q_2+\dots} : \frac{1}{q_m+\dots} := \frac{1}{q_1 + \frac{1}{q_2+\dots}}\,, \tag{14}$$

where the positive integers $q_j = q_j(t)$ are *partial quotients* of t.

Let us iterate (10), using the following properties of M, which follow from (13):

$$M(t,x) \in D; \quad h\left(\frac{1}{t}, \frac{x}{t}\right) \equiv h(M(t,x)).$$

Given a natural number j, and a point $(t,x) \in D$, after j steps we obtain the following inequality:

$$|h(t,x)| \le \sqrt{t_1 t_2 \cdots t_j}\,\left|h\left(M^j(t,x)\right)\right| + c(1 + \sqrt{t_1} + \sqrt{t_1 t_2} + \dots + \sqrt{t_1 t_2 \cdots t_{j-1}}), \tag{15}$$

where $M^{k+1}(t,x) := M\left(M^k(t,x)\right)$, $t_{k+1} = \left\{\frac{1}{t_k}\right\}$. Note that $\{t\}\left\{\frac{1}{t}\right\} \le \frac{1}{2}$, $\quad t \in \mathrm{E}$, so that in (15) we have $t_k t_{k+1} \le \frac{1}{2}$ for all $k = 1, \dots$. Thus,

$$1 + \sqrt{t_1} + \sqrt{t_1 t_2} + \dots + \sqrt{t_1 t_2 \cdots t_{j-1}} + \dots \le c. \tag{16}$$

Now, assume that t is a fixed *rational* number. Then iterations of (15) terminate when we reach the bottom of the corresponding finite continued fraction (14).

Namely, one has $t_2 = [[q_2, q_3, \dots, q_m]], \dots, t_m = [[q_m]] = \frac{1}{q_m}$, $t_{m+1} = 0$, and $M^m(t,x) = (0,\xi)$, where ξ is a point on the basic interval $[-1/2, 1/2)$, so that $h(M^m(t,x)) = h(0,\xi) = \frac{1}{2} - \{\xi\}$. Thus we see from (15) and (16) that for each fixed rational t, $h(t,x)$ is bounded for all $x \in \mathrm{E}$ by an absolute constant c.

By routine density arguing, this implies (4). Indeed, for a natural N denote $\sigma_N(t,x)$ the $(C,1)$-means of the trigonometric series defining h:

$$\sigma_N(t,x) := \sum_{1 \le |n| \le N} \left(1 - \frac{|n|}{N}\right) \frac{e^{\pi i(n^2 t + 2nx)}}{2\pi i n}.$$

The means $\sigma_N(t,x)$ are uniformly bounded in N, all real x and all *rational* t. Simply by continuity of $\sigma_N(t,x)$ in t one has $\sup_N \sup_{t,x \in \mathrm{E}} |\sigma_N(t,x)| < \infty$, i.e. the means σ are uniformly bounded on E, and (4) follows.

Now that (4) is established, the estimate (9)b) follows from (8), because

$$\frac{\partial R(t,x)}{\partial x} = 2\pi i x \Gamma(t,x) h^*\left(\frac{1}{t}, \frac{x}{t}\right).$$

One also has

$$g(t,x) = \text{p.v.} \int_{\mathrm{E}} \frac{e^{\pi i(ty^2 + 2xy)}}{2\pi i y}\, dy = \lim_{\varepsilon \to 0;\, Y \to \infty} \int_{\varepsilon \le |y| < Y} \frac{e^{\pi i(ty^2 + 2xy)}}{2\pi i y}\, dy,$$

which means that $g(t,x)$ coincides with the integral analog of the function $h(t,x)$.

2 Comments

Remark 1. An essentially more general assertion than (4) is also true. It was proved in [2], and independently by E.M. Stein and S. Wainger [22] that *the finite discrete Hilbert transforms*

$$H_N(\mathbf{x}) = H_N(x_r, \ldots, x_2, x_1) := \sum_{0<|n|\le N} \frac{e^{i(x_r n^r + \ldots + x_2 n^2 + x_1 n)}}{n}$$

are uniformly bounded in all natural $N = 1, 2, \ldots$ *and all real vectors* $\mathbf{x} = (x_r, \ldots, x_1) \in \mathrm{E}^r$:

$$\sup_{\mathbf{x}\in \mathrm{E}^r} \sup_N |H_N(\mathbf{x})| = \kappa_r < \infty. \tag{17}$$

The pointwise limit $H(\mathbf{x}) := \lim_{N\to\infty} H_N(\mathbf{x})$ exists everywhere in the space E^r, cf. [2].

However, the proof of (17) and the pointwise convergence required complicated techniques even in the case of $r = 2$. The main tool was a variant of I.M. Vinogradov's method, cf. [23] and [1], of estimates and asymptotic formulas for H. Weyl's exponential sums. More general series of the type

$$V(f;\, \mathbf{x}) := \sum_{n\in \mathrm{Z}} \hat{f}(n)\, e^{2\pi i(x_r n^r + \ldots + x_2 n^2 + x_1 n)}, \qquad \hat{f}(n) := \int_0^1 f(x)\, e^{-2\pi i n x}\, dx, \quad n \in \mathrm{Z},$$

were also considered. They were called by the author *Vinogradov series, or V-series, V-continuations of f* of r-th degree. It turned out subsequently that V-continuations have certain noteworthy applications to investigation of local and global properties of solutions of time-dependent Schrödinger type equations. In these investigations, Vinogradov's method played a decisive role, cf. [2], [17] – [20].

This justified the interest in an alternative, more elementary proof of (4), with possibly minimal references to methods of Analytic Number Theory. As mentioned above, the present proof of (4) is based on the duality relation (3), or simply the representation $\Gamma(t, x) = \sqrt{\frac{i}{t}}\, e^{-\frac{\pi i x^2}{t}}$ for the Green's function of the problem (1).

Remark 2. The global boundedness result

$$\sup_{N>0} \|h_N;\ L^\infty(\mathrm{E}^2)\| := \sup_{N>0} \max_{(t,x)\in \mathrm{E}^2} \left| \sum_{0<|n|\le N} \frac{e^{\pi i(tn^2 + 2xn)}}{2\pi i n} \right| < \infty,$$

can be deduced from L. Carleson's theorem [5] on almost everywhere convergence of trigonometric Fourier series of the class L^2. The following strong type $(2,2)$-estimate for the operator of maximal discrete Hilbert transform is sufficient:

$$\sum_{n\in \mathrm{Z}} \sup_{x\in \mathrm{E}} \left| \sum_{m\in \mathrm{Z}\setminus\{0\}} \frac{a_{m+n} e^{imx}}{m} \right|^2 \le c \sum_{n\in \mathrm{Z}} |a_n|^2.$$

The latter estimate was derived from Carleson's theorem by E. Makai [14]. The author learned about this way of estimating of h_N from E.M. Stein in 1990 (personal communication).

Remark 3. (3) is a reflection of the duality relation for the solution operator $u(f;t,x)$ of the Cauchy initial value problem for the heat transfer equation:

$$\frac{\partial u}{\partial t} = \frac{1}{4\pi}\,\frac{\partial^2 u}{\partial x^2}, \quad t > 0; \qquad u(t,x)\mid_{t=0} = f(x). \tag{18}$$

The Green's function of this problem is the Gaussian kernel $G(t,x) := \sqrt{\frac{1}{t}}\, e^{-\frac{\pi x^2}{t}}, \quad t > 0$, and the following identity is valid:

$$u(f;t,x) \equiv \sqrt{\frac{1}{t}}\, e^{-\frac{\pi x^2}{t}}\, u\left(\hat{f};\, \frac{1}{t},\, -\frac{ix}{t}\right), \quad t > 0,\ x \in \mathrm{E}. \tag{19}$$

The following classical functional equation for the elliptic Jacobi ϑ-function of real and positive argument t, cf. e.g. [12], Chapter 1:

$$\vartheta(t) = \frac{1}{\sqrt{t}}\,\vartheta\left(\frac{1}{t}\right), \quad \text{where} \quad \vartheta(t) := \sum_{n\in \mathrm{Z}} e^{-\pi n^2 t}, \qquad t > 0, \tag{20}$$

is a particular case of (19). Indeed, take as above $f(x) = \Delta(x)$ to be the periodized Dirac's δ-function. Then, on one hand, the corresponding solution $\vartheta(t,x) := u(\Delta;t,x)$ is given by

$$\vartheta(t,x) = \sum_{n\in \mathrm{Z}} e^{-\pi n^2 t + 2\pi i n x},$$

and, on the other,

$$\vartheta(t,x) = \Delta * G(t,\cdot)(x) = \sum_{n\in \mathrm{Z}} G(t, x-n) = \sqrt{\frac{1}{t}} \sum_{n\in \mathrm{Z}} e^{-\frac{\pi(x-n)^2}{t}}.$$

Thus, we obtain the well-known Fourier expansion of the *Green's function* $\vartheta(t,x)$ of the problem (18) with periodic initial data $f(x+1) \equiv f(x)$:

$$\vartheta(t,x) = \sqrt{\frac{1}{t}} \sum_{n\in \mathrm{Z}} e^{-\frac{\pi(x-n)^2}{t}} = \sum_{n\in \mathrm{Z}} e^{-\pi n^2 t + 2\pi i n x}. \tag{21}$$

Obviously, (20) follows from (21), if we take $x = 0$. Relations (20) – (21) admit an extension to complex t, with $\Re t > 0$, and (20), (21), which is an essential point in establishing the classical functional equation for Riemann's zeta-function, cf. [12], Chapter 1.

Remark 4. For fixed rational values of t, both functionals $\Theta(t,x)$ and $\Theta^*\left(\frac{1}{t}, \frac{x}{t}\right)$ are represented by linear combinations of shifted Dirac's δ-functions. Let us verify that for *rational t represented by reduced fractions $t = \frac{p}{q}$, where $pq \equiv 0 \pmod 2$, the relation (7) is equivalent to the known reciprocity of truncated Gauss' sums, cf. [10], p. 22*:

$$\sum_{n=1}^{q} e^{\frac{\pi i n^2 p}{q}} = \sqrt{\frac{iq}{p}} \sum_{m=1}^{p} e^{-\frac{\pi i m^2 q}{p}}. \tag{22}$$

In particular the boundedness result (4) can be considered as a corollary of (22).

Since $(p,q)=1,\ pq\equiv 0 \pmod 2$, the sequence of exponentials $e_n := e^{\pi i n^2 t},\quad n\in Z$, is periodic in n, with the period $=q$, i.e. $e_{n+q}\equiv e_n$. The series defining $\Theta(t,x)$ can be summed, say, by $(C,1)$-means, as follows:

$$\Theta(t,x)=\frac{1}{\sqrt{q}}\sum_{n\in \mathrm{Z}}\delta\left(x-\frac{n}{q}\right)\gamma\left(t,\frac{n}{q}\right),\quad \text{where}\quad \gamma\left(t,\frac{n}{q}\right):=\frac{1}{\sqrt{q}}\sum_{m=1}^{q}e^{\frac{\pi i(m^2p+2mn)}{q}}. \tag{23}$$

The factors $\gamma\left(t,\frac{n}{q}\right)=\gamma\left(\frac{p}{q},\frac{n}{q}\right)$ in (23) represent *shifted truncated* [1] Gauss' sums.

Let us use the following properties of delta-functions $\delta(x)$ and $\Delta(x)$ (q – a fixed positive number; $f(x)$ – a fixed continuous and bounded function on E; ξ – a fixed real number):

$$f(x)\,\delta(qx-\xi)\ \equiv\ \frac{1}{q}f\left(\frac{\xi}{q}\right)\delta\left(x-\frac{\xi}{q}\right);\quad \Delta(qx)\equiv\frac{1}{q}\sum_{n\in\mathrm{Z}}\delta\left(x-\frac{n}{q}\right);$$

$$\Delta(x)\ :=\ \sum_{n\in\mathrm{Z}}\delta(x-n)\equiv\sum_{m\in\mathrm{Z}}e^{2\pi i n x}\equiv\hat{\Delta}(x).$$

Then we obtain:

$$\Theta(t,x)=\sum_{n\in\mathrm{Z}}e_n\,e^{2\pi i n x}=\sum_{m=1}^{q}e_m\,e^{2\pi i m x}\sum_{n\in\mathrm{Z}}e^{2\pi i n q x}=\Delta(qx)\sum_{m=1}^{q}e_m\,e^{2\pi i m x}$$
$$=\frac{1}{q}\sum_{n\in\mathrm{Z}}\delta\left(x-\frac{n}{q}\right)\sum_{m=1}^{q}e_m\,e^{2\pi i m x}=\sum_{n\in\mathrm{Z}}\delta\left(x-\frac{n}{q}\right)\left(\frac{1}{q}\sum_{m=1}^{q}e_m\,e^{\frac{2\pi i m n}{q}}\right),$$

and (23) follows. Since $pq\equiv 0 \pmod 2$, the sequence $e^{\frac{\pi i n^2}{t}}=e^{\frac{\pi i n^2 q}{p}}$, $n\in\mathrm{Z}$ is again periodic, this time with the period equal p. Thus, (23) is applicable to $\Theta\left(\frac{1}{t},\frac{x}{t}\right)$, and we have:

$$\Theta\left(\frac{1}{t},\frac{x}{t}\right)=\frac{1}{\sqrt{p}}\sum_{n\in\mathrm{Z}}\delta\left(\frac{x}{t}-\frac{n}{p}\right)\gamma\left(\frac{1}{t},\frac{n}{p}\right)$$
$$=\frac{t}{\sqrt{p}}\sum_{n\in\mathrm{Z}}\delta\left(x-\frac{tn}{p}\right)\gamma\left(\frac{1}{t},\frac{n}{p}\right)=\frac{t}{\sqrt{p}}\sum_{n\in\mathrm{Z}}\delta\left(x-\frac{n}{q}\right)\gamma\left(\frac{1}{t},\frac{n}{p}\right).$$

Consequently, the righthand side of (7) can be rewritten as follows:

$$\sqrt{\frac{i}{t}}\,e^{-\frac{\pi i x^2}{t}}\,\Theta^*\left(\frac{1}{t},\frac{x}{t}\right)=\sqrt{\frac{it}{p}}\sum_{n\in\mathrm{Z}}e^{-\frac{\pi i x^2}{t}}\,\delta\left(x-\frac{n}{q}\right)\gamma^*\left(\frac{1}{t},\frac{n}{p}\right)$$
$$=\sqrt{\frac{i}{q}}\sum_{n\in\mathrm{Z}}\delta\left(x-\frac{n}{q}\right)e^{-\frac{\pi i n^2}{pq}}\gamma^*\left(\frac{q}{p},\frac{n}{p}\right). \tag{24}$$

[1] If p is even, say $p=2a$, then the sum

$$\gamma\left(\frac{p}{q},\frac{n}{q}\right)=\frac{1}{\sqrt{q}}\sum_{n=1}^{q}e^{\frac{2\pi i(n^2r+nm)}{q}}$$

is the usual (complete) shifted Gauss' sum with odd denominator q.

Let us compare the coefficients by $\delta\left(x-\frac{n}{q}\right)$ on the right of (23) and (24). To establish the equivalence of (7) and (22) we need to prove that the following relations for shifted Gauss' sums $\gamma\left(t,\, \frac{n}{q}\right)$ are valid:

$$\gamma\left(\frac{p}{q},\, \frac{n}{q}\right) = \sqrt{i}\, e^{-\frac{\pi i n^2}{pq}}\, \gamma^{\star}\left(\frac{q}{p}, \frac{n}{p}\right), \quad n \in \mathrm{Z}. \tag{25}$$

The shifted Gauss' sums $\gamma\left(t,\, \frac{n}{q}\right)$ can be expressed in terms of non-shifted ones, i.e. $\gamma(t,0)$. Indeed, assume as above that $t=\frac{p}{q}$ is a reduced fraction, and denote $T=T(t)$ one of the *Farey neighbors* to t, that is a rational number $T=\frac{P}{Q}$ satisfying

$$|t-T| = \left|\frac{p}{q}-\frac{P}{Q}\right| = \frac{1}{qQ}, \quad \text{i.}e. \quad pQ-qP=\hbar, \quad \text{where} \quad \hbar=\pm 1.$$

Then

$$(pQ)^2+(qP)^2 \equiv 1 \pmod{2pq}; \quad pm^2+2mn \equiv p(m^2+2\hbar Qmn) \pmod{2q}, \quad m,n\in \mathrm{Z}.$$

Consequently,

$$e^{\frac{\pi i(pm^2+2mn)}{q}} = e^{\frac{\pi i p(m+\nu)^2}{q}}\, e^{\frac{-\pi i p(Qn)^2}{q}}, \quad \text{with} \quad \nu := \hbar Q,$$

so that

$$\gamma\left(\frac{p}{q},\, \frac{n}{q}\right) = \frac{1}{\sqrt{q}}\, e^{\frac{-\pi i p(Qn)^2}{q}} \sum_{m=1}^{q} e^{\frac{\pi i p(m+\nu)^2}{q}} = \frac{1}{\sqrt{q}}\, e^{\frac{-\pi i p(Qn)^2}{q}} \sum_{m=1}^{q} e^{\frac{\pi i p m^2}{q}} = e^{\frac{-\pi i p(Qn)^2}{q}}\, \gamma\left(\frac{p}{q},\, 0\right), \tag{26}$$

and further

$$\gamma^{*}\left(\frac{q}{p}, \frac{n}{p}\right) = e^{\frac{\pi i q(Pn)^2}{p}}\, \gamma^{*}\left(\frac{q}{p}, 0\right).$$

Since

$$e^{\pi i\left(\frac{q(Pn)^2}{p}+\frac{p(Qn)^2}{q}-\frac{n^2}{pq}\right)} = e^{\pi i\frac{(qP)^2+(pQ)^2-1}{pq}n^2} = 1, \quad n\in \mathrm{Z},$$

it follows from (26) that (25) indeed reduces to (22).

Remark 5. To find a Farey neighbor $T(t)$ to $t=\frac{p}{q}$, $t\in(0,1)$, consider the representation $t=[[q_1,\, q_2,\, \ldots,q_m]]$ by (14). One can select $T(t)$ to be the last but one in the corresponding sequence of convergents, i. e. $T(t)=T^{-}(t)=[[q_1,\, q_2,\, \ldots,q_{m-1}]]$, or an arbitrary *following fraction* of the form $T(t)=T^{+}(t)=[[q_1,\, q_2,\, \ldots,q_m, q_{m+1}]]$, where q_{m+1} is a natural number. For more details, see e.g. [10].

Remark 6. G.H.Hardy and J.E.Littlewood (see [10], pp. 67–112, and pp. 113, 114) established the following *approximate functional equation* for finite partial sums of the series defining $\Theta(t,0)$: uniformly in $t\in(0,1)$ and $N>0$,

$$S_N(t) = \sqrt{\frac{i}{t}}\,\left(S^{*}_{Nt}\left(\frac{1}{t}\right)+O(1)\right), \quad \text{where} \quad S_N(t) := \sum_{0\le n<N} e^{\pi i n^2 t}.$$

The relations (22) represent a remarkable class of cases when the remainder term $O(1)$ equals 0, i.e. the approximate equation is *exact.*

Further, in [10], pp. 113 – 114 and [9] the iterations of the same kind as in (15) were used in estimates of incomplete Gauss' sums.

Remark 7. It is not hard to see that

$$\sup_t \{t_1 t_2 \cdots t_j\} = \frac{1}{F_j}, \quad j = 1, 2, \dots,$$

where F_j denotes the j-th Fibonacci's number, i.e. $F_1 := 1,\ F_2 := 2,\ F_j := F_{j-1} + F_{j-2}, \quad j \geq 3$. It can also be shown that the maximal value of the infinite sum

$$\max_{t \in (0,1)} \{1 + \sqrt{t_1} + \sqrt{t_1 t_2} + \dots\}$$

is attained for the *golden mean* $t_* = \frac{\sqrt{5}-1}{2}$, i. e. $\max_{t\in(0,1)}\{\dots\} = \frac{1}{1-\sqrt{t_*}}$. For this remark the author is indebted to B. Popov.

Remark 8. As a disadvantage of the above proof of (4), it should be noted that it does not provide the *existence* of h for all (t, x). The case of concrete irrational t has not been treated by the new aproach simply because pointwise convergence remained obscure, and a priori the series might be divergent on a certain set of points of zero Lebesgue measure.

The appoach based on Vinogradov's method provides more detailed information concerning local and global properties of h. For instance, if t is irrational, then the series defining h converges uniformly in x, so that for such t, $h(t, x)$ is contionuous in x. Also, the traces of $h(t, x)$ are continuous on every line on the plane E^2 not parallel to the x-axis; in particular, for each fixed x, $h(t, x)$ as a function of $t \in \mathrm{E}$ is everywhere continuous.

For more details concerning local and global properties of V-continuations of higher degree and implications to the Cauchy initial value problem for Schrödinger type equations, see [17]. This includes, in particular, the degenerate (linearized) Korteweg – deVries equation

$$\frac{\partial u}{\partial t} = a \frac{\partial^3 u}{\partial x^3}, \quad u(t, x)|_{t=0} = f(x).$$

3 On variation and complexity features. Curlicues

The function $h(t, x)$ is related with a wide class of solutions of (1) with periodic $f(x)$ of bounded variation on the period $[0, 1)$. The solution operator for such initial data can be represented by Stieltjes convolution

$$\psi(f;\ t, x) \sim \sum_{n \in \mathrm{Z}} \hat{f}(n)\, e^{\pi i (n^2 t + 2nx)} = \hat{f}(0) + \int_{\mathrm{T}} h(t, x - \xi)\, df(\xi) \tag{27}$$

(here and below, T indicates that the functions are periodic, of period equal 1, and that the integral is taken over the period).

As mentined above (cf. [17]), for fixed irrational t, $h(t,x)$ is continuous in x and the integral is Riemann – Stieltjes. If t is a rational number, then $x + h(t,x)$ is piecewise constant, with equidistant jumps on T, and for a general $f(x)$ of bounded variation, the integral can be understood as Lebesgue – Stieltjes.

Denote $\mathrm{Var}_1(\mathrm{T})$ the space of functions $f(x)$, $f(x+1) \equiv f(x)$ of bounded variation $\mathrm{var}_1(f;\ \mathrm{T})$ on T. We assume that $f(x) \equiv \frac{f(x_-)+f(x_+)}{2}$ for all x, and denote $\|f, \mathrm{Var}_1(\mathrm{T})\| := \|f, L^\infty(\mathrm{T})\| + \mathrm{var}_1(f;\ \mathrm{T})$ the norm in V_1^{T}.

Theorem 1 implies that

$$\|\psi : \quad \mathrm{Var}_1(\mathrm{T}) \mapsto L^\infty(\mathrm{E}^2)\| < \infty, \tag{28}$$

i.e. the solution operator ψ is bounded from $\mathrm{Var}_1(\mathrm{T})$ into $L^\infty(\mathrm{E}^2)$.

This statement can be strengthened, see Theorem 2 below. Namely, the functional equation (8) with size- and smoothness estimates (9) of the remainder term $R(t,x)$ can be used to analyze *variational* features of $h(t,x)$, and consequently those of $\psi(f;\ t,x)$ with $f \in \mathrm{Var}_1(\mathrm{T})$. In particular, one can provide an alternative proof, cf. [20], of the fact that *for each fixed* t, $h(t,x)$ *as a function of* x *is of bounded weak quadratic variation on the period* $[0,1)$, *and that the latter property holds uniformly in* $t \in \mathrm{E}$:

$$\sup_t \mathrm{war}_2(h(t,x) :\ x \in [0,1)) < \infty. \tag{29}$$

Let us recall the corresponding definitions of generalized strong and weak α-variation, where $\alpha \geq 1$.

The strong α-variation $\mathrm{var}_\alpha(f;\ I)$ of a bounded function $f(x)$ on an interval $I \subset \mathrm{E}$ is defined by

$$\mathrm{var}_\alpha(f;\ I) := \sup_{\{I_j\} \in \mathcal{I}} \left\{ \sum_j \mathrm{osc}^\alpha(f;\ I_j) \right\}^{\frac{1}{\alpha}},$$

where $\mathrm{osc}(f;\ I) := \sup\{|f(x) - f(y)| :\ x, y \in I\}$, and $\mathcal{I}$ denotes the class of all partitions $\{I_j\}$ of the basic interval I into unions of pairwise disjoint subintervals I_j. The notion of strong 2-variation was introduced by N. Wiener, and usefulness of α-variation in Fourier analysis has been thoroughly studied, cf.[3], Ch. 4.

The notion of weak α-variation $\mathrm{war}_\alpha(f;\ I)$ is a modification of $\mathrm{var}_\alpha(f;\ I)$ in the general key of weak type estimates. Namely,

$$\mathrm{war}_\alpha(f;\ I) := \sup_{\varepsilon > 0} \sup_{\{I_j\} \in \mathcal{I}} \varepsilon (\mathrm{card}\,\{j :\ \mathrm{osc}(f;\ I_j) > \varepsilon\})^{\frac{1}{\alpha}},$$

where card denotes the number of elements in a (finite) set. Weak variations are more handy in applications, than strong ones, and they are easier to compute (see below, (30) and (31)).

Denote $\mathrm{Var}_\alpha(I)$, $(\mathrm{War}_\alpha(I))$ the corresponding classes of all functions $f(x)$, $x \in I$ with $f(x) \equiv \frac{f(x_-)+f(x_+)}{2}$ such that $\mathrm{var}_\alpha(f;\ I) < \infty$ or, respectively, $\mathrm{war}_\alpha(f;\ I) < \infty$. Then $\mathrm{Var}_\alpha(I) \subset \mathrm{Var}_\beta(I)$, $\mathrm{War}_\alpha(I) \subset \mathrm{War}_\beta(I)$ for $\beta > \alpha$, i.e. the classes are expanding along with the growing α. Further, by Chebyshev's inequality, $\mathrm{war}_\alpha(f;\ I) \leq \mathrm{var}_\alpha(f;\ I)$, and the imbedding $\mathrm{Var}_\alpha(I) \subset \mathrm{War}_\alpha(I)$ is obvious. One has (cf. [7], Ch. 12, Theorem 4.3) $\mathrm{Var}_1(I) = \mathrm{War}_1(I)$, while for $\alpha > 1$ the class $\mathrm{War}_\alpha(I)$ is essentially wider than $\mathrm{Var}_\alpha(I)$. Also, if $1 < \alpha < \beta$, then $\mathrm{War}_\alpha(I) \subset \mathrm{Var}_\beta(I)$, i.e. *if a function $f(x)$ is of bounded weak α-variation, then it is also of bounded strong β-variation for every $\beta > \alpha$.*

There are two alternative equivalent definitions of the class $\mathrm{War}_\alpha(I)$: a) in terms of the rates of non-linear approximation by piecewise constant functions (splines with free nodes), and b) via interpolation – in terms of J. Peetre's functionals.

Namely, for a given natural n denote $\mathcal{P}_n$ the class of all piecewise constant functions $P(x)$ on I, such that I can be represented as a union $I = \bigcup_1^m I_j$ of $m \le n$ pairwise disjoint subintervals I_j and $P(x)$ is constant on each I_j. Further, denote

$$\mathcal{E}_n(f;\, I) := \inf\,\{\|f - g;\, L^\infty(I)\| :\ g \in \mathcal{P}_n\}$$

the n-th best uniform piecewise constant approximation of $f(x)$ on I. Then

$$f \in \mathrm{War}_\alpha(I) \iff \mathcal{E}_n(f;\, I) = O\left(n^{-\frac{1}{\alpha}}\right), \quad n \to \infty. \tag{30}$$

In interpolatory terms, the definition of the class $\mathrm{War}_\alpha(I)$ is given by

$$f \in \mathrm{War}_\alpha(I) \iff \inf\,\{\|g;\, V_1(I)\| :\ \|f - g;\, L^\infty(I)\| \le \varepsilon\} = O(\varepsilon^{-\alpha}), \quad \varepsilon \to 0. \tag{31}$$

In the other words, $f(x)$ belongs to the class $\mathrm{War}_\alpha(I)$ if and only if for all (small) $\varepsilon > 0$ it can be uniformly approximated with the accuracy ε by a function $g(x) = g_\varepsilon(x)$ whose ordinary total variation $\mathrm{var}_1(g;\, I)$ is of order $O(\varepsilon^{-\alpha})$.

The next simple lemma is useful in applications to (1).

Lemma 3 . *Assume that $f(x) \in \mathrm{Var}_1(\mathrm{T})$ and $h(x) \in \mathrm{War}_\alpha(\mathrm{T})$, where $\alpha \ge 1$, and let*

$$\psi(x) := (h * df)(x) = \int_{\mathrm{T}} h(x - \xi)\, df(\xi).$$

Then $\psi(x) \in \mathrm{War}_\alpha(\mathrm{T})$.

For the proof, let us represent $h(x)$, in accordance with (31), as $h = g_\varepsilon + r_\varepsilon$, where $g_\varepsilon \in \mathrm{Var}_1(\mathrm{T})$, and $\|g_\varepsilon;\, V_1(I)\| \le \mathrm{const}\cdot\varepsilon^{-\alpha}$, $\|r_\varepsilon;\, L^\infty(I)\| \le \varepsilon$. Then it is easy to see that the function $\psi_\varepsilon(x) := (h_\varepsilon * df)(x)$ is in $\mathrm{Var}_1(\mathrm{T})$, and

$$\|\psi - \psi_\varepsilon;\, L^\infty(\mathrm{T})\| = O(\varepsilon), \quad \mathrm{var}_1(\psi_\varepsilon;\, \mathrm{T}) = O(\varepsilon^{-\alpha}), \quad \varepsilon \to 0.$$

Thus, the next statement follows from [20] and (29), (31). It is a refinement of (28).

Theorem 2 . *The following property holds uniformly in $t \in \mathrm{E}$ for the solution operator ψ of the problem (1):*

$$\psi : \quad \mathrm{Var}_1(\mathrm{T}) \mapsto \mathrm{War}_2(\mathrm{T}), \tag{32}$$

and in particular, for all $\alpha > 2$

$$\psi : \quad \mathrm{Var}_1(\mathrm{T}) \mapsto \mathrm{Var}_\alpha(\mathrm{T}), \quad \alpha > 2. \tag{33}$$

For possible generalizations to a wider class of problems of type (1) involving sufficiently smooth potentials $p(t,x)$ (periodic in x), i.e.

$$\frac{\partial\psi}{\partial t} = i\left(\frac{\partial^2\psi}{\partial x^2} + p(t,x)\psi\right), \quad \psi(t,x)|_{t=0} = f(x),$$

the reader may be referred to [17] and [18].

Remark 9. In the limiting case $\alpha = 2$, the statement (33) of Theorem 2 is not true. There exist such values of t that $h(t,x)$ is not of bounded strong 2-variation in x, see [20].

The above variational results provide the first insight into *complexity features* of the solutions of the problem (1). As noted above, for each fixed irrational value of t, the function $h(t,x)$ is everywhere continuous; however, it is *nowhere differentiable in* x. For irrational t, uniform smoothness of $h(t,x)$ in the variable x is "best possible" if the sequence of partial quotients $\{q_j\}$ in the continued fraction (14) is bounded. In such cases, $h(t,\cdot) \in \mathrm{Lip}\,\frac{1}{2}$, i.e. $|h(t,x) - h(t,y)| \leq c(t)|x-y|^{\frac{1}{2}}$ for all $x, y \in \mathrm{E}$. A wide set of such values of t is provided by *quadratic surds* (irrationalities), e.g. $t = \sqrt{2}$, $t = \frac{\sqrt{5}-1}{2}$, etc. These results indicate on a complicated character of the corresponding *trajectories*, which resemble those of Brownian particles on the plane.

Let us establish some preliminary facts concerning these objects, and their relationship with the so-called *curlicues*. For a fixed $t \in \mathrm{E}$ consider the following set of points on the complex plane C:

$$\mathcal{H}_t := \{z \in \mathrm{C}: \quad z = x + h(t,x),\ x \in \mathrm{T}\}. \tag{34}$$

One can understand (34) as parametric equation of the set $\mathcal{H}_t$ on the real plane E^2:

$$\mathcal{H}_t := \{\mathbf{x} = \langle x_1,\, x_2\rangle \in \mathrm{E}^2: \quad x_1 = x + \Re\, h(t,x),\ x_2 = \Im\, h(t,x), \quad x \in \mathrm{T}\}.$$

If t is a rational number, the function $x + h(t,x)$ is piecewise constant, so that $\mathcal{H}_t$ is a discrete set. The values of h at rational points on E^2 are computable as finite discrete Hilbert transforms, cf. [20], [17]:

$$h\left(\frac{2a}{q},\, \frac{b}{q}\right) = \frac{1}{2qi}\sum_{n=1}^{q-1}\frac{e^{2\pi i\frac{an^2+bn}{q}}}{\tan\frac{\pi n}{q}}.$$

where $a,\, b \in \mathrm{Z}$, $(a,q) = 1$. On the other hand, if t is irrational, then $x + h(t,x)$ is continuous and nondifferetiable, and $\mathcal{H}_t$ is a continuous and nonrectifiable curve.

Curlicues were studied by M.V. Berry and J. Goldberg in [4]. They represent a peculiar class of curves on C resulting from computation and plotting of the values of *incomplete Gauss' sums.* Such computations and analysis were seemingly initiated by D.H. Lehmer [13], and later continued by several authors, cf. e.g. [4], [16], with emphasis on possible applications as models in Optics and Thermodynamics. A curlicue is defined for a fixed real parameter τ as the broken line on C resulting from computations of quadratic exponential sums

$$z_n(\tau) := \sum_{m=0}^{n} e^{\pi i m^2 \tau}, \quad n = 0,\, 1,\, \ldots, \tag{35}$$

and joining each pair of consecutive points by line segments $[z_n,\, z_{n+1}]$. For "complicated" rational τ, i. e. when the the number of "levels" m in the continued fraction (14) is large, the corresponding curlicue represents a rather spectacular combination of several hierarchies of coiling and uncoiling *Cornu spirals.* In a curlicue of such kind, features of *selfsimilarity* are present.

The Cornu spiral is parametrically represented by the incomplete Fresnel integral, cf. (12):

$$\mathcal{F} := \left\{ z = \sqrt{i}\,\mathrm{Frl}^*(x) = \sqrt{i}\int_0^x e^{-\pi i y^2}\,dy, \quad x \in \mathrm{E} \right\}.$$

Geometrically, such a spiral uncoils counterclockwise from the point $z = -\frac{1}{2}$, passes through the origin and then coils clockwise towards $z = \frac{1}{2}$.

The function $x + h(t,x)$ is a *generating function* of all curlicues with rational parameters of the form $\tau = \frac{p}{q}$, $(p,q) = 1$, where p is even, say, $p = 2a$, $a \in \mathrm{Z}$ and q odd. In this case, the corresponding numbers z_n are of the form

$$z_n = z_n\left(\frac{2a}{q}\right) = \sum_{m=0}^{n} e^{\frac{2\pi i a m^2}{q}},$$

i.e. they are *incomplete Gauss' sums with odd denominators.* Indeed, introduce the following sum-function of continuous argument x:

$$Z(\tau,\, x) = Z\left(\frac{2a}{q},\, x\right) := \sum_{0 \le m \le qx} e^{\frac{2\pi i a m^2}{q}},$$

where the first term and, in case of integral value of qx, also the last one have to be taken with the factor $\frac{1}{2}$. The relation between curlicues and the function $x + h(t,x)$ can be seen from the following identity:

$$Z(\tau,x) \equiv G(t)(x + h^*(2t,x)), \quad \text{where} \quad t = t(\tau) := \frac{(4a)^{-1}}{q} = \frac{(q+1)^2 a^{-1}}{4q}. \tag{36}$$

For the proof, see [20]. In (36), a^{-1} denotes the unique modulo q solution of the congruence $aa^{-1} \equiv 1 \pmod{q}$, and we keep the assumptions $q \equiv 1 \pmod{2}$, $(a,q) = 1$. Further, $G(t)$ denotes the complete Gauss' sum:

$$G\left(\frac{a}{q}\right) := \sum_{n=1}^{q} e^{\frac{2\pi i a n^2}{q}}.$$

The values of $G\left(\frac{a}{q}\right)$ are given by classical relations due to K.F. Gauss:

$$\left|G\left(\frac{a}{q}\right)\right| = \sqrt{q}, \quad \text{and if } q \text{ is a prime,} \quad G\left(\frac{a}{q}\right) = \left(\begin{array}{c} a \\ q \end{array}\right)\sqrt{q}\begin{cases} 1, & \text{if } q \equiv 1 \pmod{4}, \\ i, & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Here $\left(\begin{array}{c} a \\ q \end{array}\right)$ denotes the Legendre symbol modulo prime the number q:

$$\left(\begin{array}{c} a \\ q \end{array}\right) := \begin{cases} \quad 1, & \text{if } a \text{ is a quadratic residue} \pmod{q}, \\ -\ 1 & \text{in the opposite case.} \end{cases}$$

Thus, it follows from (36) that complete Gauss' sums $G(t) = G(\tau)$ play the role of *scaling factors* in the curlicue parametrized by $x + h^*(2t(\tau), x)$. Moreover, the truncated Cornu spirals and the hierarchical nature of curlicues reflect the functional equation (8).

Remark 10. The Cornu spiral $\mathcal{F}$ is a nontrivial set from the point of view of the theory of comlexity. Let us conclude the paper by proving that $\operatorname{ent}\mathcal{F} = \frac{4}{3}$, where $\operatorname{ent}\mathcal{F}$ denotes *the metric entropy* of $\mathcal{F}$.

Recall the notion of Kolmogorov - Schnirelman entropy introduced in [11], [21], and studied later along with *Hausdorff dimension* $\dim_H$ in the literature on fractal sets, see e.g. [6], [8], [15].

Given a set $\mathcal{F}$ in a metric space $\mathcal{E}$ (in our case, $\mathcal{F}$ is the Cornu spiral, $\mathcal{E} := \mathrm{C}$) and a number $\varepsilon > 0$, denote $N_{\mathcal{F}}(\varepsilon)$ the smallest number of balls of radius $\leq \varepsilon$ in $\mathcal{D}$ which are needed to cover $\mathcal{F}$. Then

$$\operatorname{ent}\mathcal{F} := \limsup_{\varepsilon \to 0} \frac{\ln N_{\mathcal{F}}(\varepsilon)}{\ln \frac{1}{\varepsilon}}.$$

The proof of the relation $\operatorname{ent}\mathcal{F} = \frac{4}{3}$ consists of two parts: the estimate of $N_{\mathcal{F}}(\varepsilon)$ from above, and the estimate of this number from below. As common in theory of coplexity, the estimate from above is easier, because any reasonable covering works. In our case, let $0 < X < \frac{1}{\varepsilon}$, and represent $\mathcal{F}$ as a union $\mathcal{F} = \mathcal{F}_1(X) \bigcup \mathcal{F}_2(X)$ where

$$\mathcal{F}_1(X) := \{z \in C: \ z = \sqrt{i}\mathrm{Frl}^*(x), \ |x| \leq X\}, \quad \mathcal{F}_2(X) := \{z \in C: \ z = \sqrt{i}\mathrm{Frl}^*(x), \ |x| > X\}.$$

The part $\mathcal{F}_1(X)$ is a curve on C of length $2X$. This set be covered by $\leq c\frac{X}{\varepsilon}$ discs of radius ε. The part $\mathcal{F}_2(X)$ is contained in two discs with the centers at $z = \pm\frac{1}{2}$ and radii $\leq \frac{c}{X}$. The latter follows from the estimate

$$\int_X^\infty e^{\pi i y^2}\, dy = O\left(\frac{1}{X}\right)$$

of the tails of the Fresnel integral for large X. Thus, $\mathcal{F}_2(X)$ can be covered by $\leq c\frac{1}{(X\varepsilon)^2}$ discs of radius ε, and we have

$$N_{\mathcal{F}}(\varepsilon) \leq c\left(\frac{X}{\varepsilon} + \frac{1}{X^2\varepsilon^2}\right).$$

To minimize the expression on the right in X, we choose $X \sim \varepsilon^{-\frac{1}{3}}$. This implies the estimate $N_{\mathcal{F}}(\varepsilon) \leq c\varepsilon^{-\frac{4}{3}}$, and consequently $\operatorname{ent}\mathcal{F} \leq \frac{4}{3}$.

To prove the estimate from below $N_{\mathcal{F}}(\varepsilon) \geq c_0\varepsilon^{-\frac{4}{3}}$, one applies the asymptotic formula

$$\int_X^\infty e^{\pi i y^2}\, dy = \int_X^\infty \frac{e^{\pi i y^2}(2\pi i y)}{2\pi i y}\, dy = -\frac{e^{\pi i X^2}}{2\pi i X} + O\left(\frac{1}{X^3}\right), \quad X \to \infty.$$

We omit the details.

The author hopes to return to complexity problems of solutions of Schrödinger equation of a free particle, such as estimates of Hausdorff dimensions of trajectories. Although the problem (1) with periodic $f(x)$ is linear, the above considerations show that the solutions may be chaotic even in the case of simple initial data, i.e. possess features rather typical for non-linear problems in PDE. The author believes that a combination of Vinogradov's method and the functional equations of the type (8) may be useful in this direction. It seems likely, that the Hausdorff dimension of curves $\mathcal{H}_t$ is non-trivial e. g. when t is a quadratic irrationality.

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