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Nonlinear Approximation and the Space $\mathrm{BV}(\mathbb{R}^2)$[1]

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Abstract. Given a function $f \in L_2(Q)$, $Q := [0,1)^2$ and a real number $t > 0$, let $U(f,t) := \inf_{g \in \mathrm{BV}(Q)} \|f - g\|_{L_2(I)}^2 + t\,\mathrm{V}_Q(g)$, where the infimum is taken over all functions $g \in \mathrm{BV}$ of bounded variation on I. This and related extremal problems arise in several areas of mathematics such as interpolation of operators and statistical estimation, as well as in digital image processing. Techniques for finding minimizers g for $U(f,t)$ based on variational calculus and nonlinear partial differential equations have been put forward by several authors ([DMS], [LOR], [MS], [CL]). The main disadvantage of these approaches is that they are numerically intensive. On the other hand, it is well-known that more elementary methods based on wavelet shrinkage solve related extremal problems, for example, the above problem with BV replaced by the Besov space $B_1^1(L_1(I))$ (see e.g. [CDLL]). However, since BV has no simple description in terms of wavelet coefficients, it is not clear that minimizers for $U(f,t)$ can be realized in this way. We shall show in this paper that simple methods based on Haar thresholding provide near minimizers for $U(f,t)$. Our analysis of this extremal problem brings forward many interesting relations between Haar decompositions and the space BV.

1. Introduction.

Nonlinear approximation has recently played an important role in several problems of image processing including compression, noise removal, and feature extraction. We have in mind techniques such as wavelet compression [DJL], wavelet shrinkage or thresholding [DJKP1], wavelet packets [CW], and greedy algorithms [MZ, DT]. There has also been an impressive contribution of techniques based on variational calculus and nonlinear partial differential equations (see e.g. [DMS], [LOR], [MS], [CL]) especially to the problems of noise removal and image segmentation. The common point between these two approaches is their ability to adapt to the composite nature of images: edge, textures and smooth regions should be treated adaptively, a requirement which is certainly not fulfilled by the classical linear filtering techniques.

One problem which plays an important role in the latter approach is the the following extremal problem introduced in [LOR]:

Given a function (image) f defined on the unit square, $Q := [0,1)^2$, and a parameter $t > 0$, find the function $g \in \mathrm{BV}(Q)$ which attains the infimum

$$U(f,t) := \inf_{g \in \mathrm{BV}(Q)} \|f - g\|_{L_2(Q)}^2 + t\,\mathrm{V}_Q(g). \tag{1.1}$$

Here $\mathrm{BV}(Q)$ is the space of functions of bounded variation on Q (see §2 for the definition of this space) and $\mathrm{V}_Q(f) = |f|_{BV}$ is the associated semi-norm, i.e. the total variation of f.

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In the practice of noise removal, f represents the noisy image and t is usually chosen to be proportional to the noise level. The minimizer g then appears as a denoised image. The functional in (1.1) can also be viewed as a variant of the Mumford and Shah functional introduced in their celebrated paper [MS] on image processing.

A minimization problem close to (1.1) is also familiar in the context of interpolation of linear operators: the expression

$$K(f,t) := K(f,t,L_2(Q),\mathrm{BV}(Q)) := \inf_{g\in \mathrm{BV}(Q)} \|f-g\|_{L_2(Q)} + t\,\mathrm{V}_Q(g), \tag{1.2}$$

called the K-functional of f for the pair $(L_2(Q),\mathrm{BV}(Q))$, is the basic tool for generating interpolation spaces between these two spaces by the so-called real method.

Numerical techniques for solving (1.1) based on partial differential equations have been developed and successfully applied to image processing. The advantage of these techniques is high performance. Their disadvantage is they are numerically intensive, and require in practise the approximation of the BV term in $U(f,t)$ by a quadratic term (e.g. $\int(\epsilon + |\nabla f|^2)^{1/2}$) in order to find a solution in reasonable computational time (see [VO] for a discussion on numerical methods for solving (1.1)).

In comparison, wavelet thresholding methods simply amount to the application of multiscale decomposition and reconstruction algorithms on the image, and of a thresholding procedure, which can all be performed in $\mathcal{O}(N)$ operations, where N is the number of pixels in the image. These methods can be made translation invariant by a cyclic averaging technique introduced in [CD], which seems to bring significant visual improvement, while only raising the complexity to $\mathcal{O}(N\log N)$. On a more theoretical point of view, thresholding procedures have been proved to be optimal, in the minimax sense of asymptotical statistics, in various non-parametric contexts where the images are typically modelized by their regularity in Sobolev and Besov classes (see [DJKP2]).

A striking remark (see [CDLL]) is that wavelet thresholding also provides the exact solution to an extremal problem which is very close to (1.1), namely

$$\tilde U(f,t) := \inf_{g\in B_1^1(L_1(Q))} \|f-g\|^2_{L_2(Q)} + t|g|_{B_1^1(L_1(Q))}, \tag{1.3}$$

where the Besov space $B_1^1(L_1(Q))$ is taken in place of the (larger) space $\mathrm{BV}(Q)$. Both $\mathrm{BV}(Q)$ and $B_1^1(L_1(Q))$ are smoothness spaces of order one in $L_1(Q)$, e.g. the space $\mathrm{BV}(Q)$ is the same as $\mathrm{Lip}(1,L_1(Q))$ (see [M], or [DP1] for the definition of the Besov spaces). In contrast to BV, the $B_1^1(L_1)$ norm has a simple equivalent expression as the ℓ_1 norm of the coefficients in a wavelet basis decomposition $f=\sum_{\lambda\in\Lambda} f_\lambda\psi_\lambda$ (where Λ denotes the set of indices for the wavelet basis). One can thus use this decomposition to obtain an equivalent discrete problem

$$\tilde U((f_\lambda),t) := \inf_{(g_\lambda)\in\ell_1} \sum_{\lambda\in\Lambda} [|f_\lambda - g_\lambda|^2 + t|g_\lambda|], \tag{1.4}$$

whose solution (obtained by minimizing separately on each index λ) is exactly given by a "soft thresholding" procedure at level $t/2$:

$$g_\lambda = \mathrm{sgn}(f_\lambda)\max\{0, |f_\lambda| - t/2\}. \tag{1.5}$$

The minimization problem (1.3) can thus be solved (up to a constant related to the equivalence between continuous and discrete norms), by a simple wavelet-based procedure.

One could argue that the distinction between the two problems (1.1) and (1.2) is slight. However, BV seems more adapted to model real images, since it allows sharp edges (i.e. discontinuities on a line), which cannot occur in a bivariate function that belongs to the smaller space $B^1_1(L_1)$. This fact is confirmed in the practice of image processing: the performance of (1.1) for noise removal, for example, seems slightly better than that of (1.3), at least in aesthetic terms.

We call a family of functions g_t a *near minimizer* for (1.1) if

$$\|f-g_t\|^2_{L_2(Q)} + t\,\mathrm{V}_Q(g_t) \le C \inf_{g\in \mathrm{BV}(Q)} \|f-g\|^2_{L_2(Q)} + t\,\mathrm{V}_Q(g) \tag{1.6}$$

with C an absolute constant (not depending on t or f). A similar definition applies to (1.2). The question arises whether one could find a near minimizer to (1.1) and (1.2), using simple non-linear approximation techniques such as wavelet thresholding. Note that in contrast to $B^1_1(L_1)$, we are then allowed to use approximations that have line discontinuities, such as the multidimensional Haar basis or, more generally, piecewise constant functions. The main point of this paper is to develop such techniques and to prove that they indeed yield near minimizers for the problems (1.1) and (1.2).

Our main result in this paper is to show that either of the extremal problems (1.1-2) has a near minimizer taken from certain "non-linear" spaces Σ_N, $N \ge 1$, whose elements are piecewise constants that can be described by N parameters. In the case of wavelet thresholding, the space Σ_N is simply the set of all linear combinations $\sum f_\lambda H_\lambda$ with at most N terms and H_λ the bivariate Haar functions.

In order to prove that a given family Σ_N provides the solution to (1.1) or (1.2), we shall make use of several ingredients, among which are two types of inequalities that are frequently used in numerical analysis and approximation theory:

(i) A *direct or Jackson type estimate*

$$\inf_{g\in\Sigma_N} \|f-g\|_{L_2(Q)} \le CN^{-1/2}|f|_{\mathrm{BV}(Q)}, \tag{1.7}$$

that describes the approximation power of Σ_N for functions in BV.
(ii) An *inverse or Bernstein type estimate*

$$|f|_{\mathrm{BV}(Q)} \le CN^{1/2}\|f\|_{L_2(Q)} \ \text{ if } \ f\in\Sigma_N, \tag{1.8}$$

that describes the smoothness properties of the approximation spaces Σ_N.

When BV is replaced by $B^1_1(L_1)$ and Σ_N is the set of N-terms linear combination in a sufficiently smooth wavelet basis, these inequalities reduce to simple considerations on sequences. Since the BV norm has no simple equivalent expression in terms of the wavelet coefficients (it is actually known that BV is nonseparable), (1.7) and (1.8) (in particular the direct estimate) are by far less obvious, and will require more involved arguments.

We shall now give a more precise formulation of our results. We shall denote by Σ_N^w the non-linear spaces associated with N-term approximation in the Haar system, i.e.

$$\Sigma_N^w := \{\sum_{\lambda\in E} c_\lambda H_\lambda \; ; \; E \subset \Lambda, \; |E| \leq N\}, \tag{1.9}$$

where $|E|$ denotes the cardinality of the discrete set E (in the case of a continuous set Ω of $\mathbb{R}^d$, $|\Omega|$ will stand for its volume), and where $(H_\lambda)_{\lambda\in\Lambda}$ is the bivariate Haar system derived from the univariate system of $L_2[0,1]$ by the usual tensor-product construction: from $H^0 = \chi_{[0,1)}$ and $H^1 := \chi_{[0,1/2)} - \chi_{[1/2,1)}$, one defines the multivariate functions

$$H^e(x) := H^{e_1}(x_1)H^{e_2}(x_2), \quad e = (e_1, e_2) \in V, \tag{1.10}$$

where V is the set consisting of the nonzero vertices of Q. The bivariate Haar system for $L_2(Q)$ consists of the constant function 1 and of all functions

$$H^e_{j,k}(x) = 2^j H^e(2^j x - k), \;\; e \in V, \; j \geq 0, \; k \in \mathbb{Z}^2 \cap 2^j Q. \tag{1.11}$$

We refer to [D] for a general introduction to wavelet bases.

We shall prove that the wavelet thresholding, which is equivalent to approximation by the elements Σ_N^w, gives a near minimizer to the extremal problems (1.1) and (1.2) (§9). However, our proofs are neither direct nor simple. Rather, we prove these results by considering various types of nonlinear approximation by piecewise constants. Note that the functions in Σ_N^w are piecewise constant taking at most $2N$ values.

To describe the other spaces of piecewise constant functions which we shall use in this paper we introduce the following notation which will be used throughout the paper. If Ω is a set of $\mathbb{R}^2$, we denote by φ_Ω its characteristic function, and by

$$a_\Omega(f) = |\Omega|^{-1} \int_\Omega f, \tag{1.12}$$

the average of an L_1-function f on Ω. By definition, a *dyadic cube* I is the tensor product of two dyadic intervals, i.e. $I = I(j,k,l) = [2^{-j}k, 2^{-j}(k+1)) \times [2^{-j}l, 2^{-j}(l+1))$. We shall denote by $\mathcal{D} := \mathcal{D}(Q)$ the set of all dyadic cubes contained in Q, and by $\mathcal{D}_k(Q)$ the set of all dyadic cubes in $\mathcal{D}(Q)$ with sidelength 2^{-k} (measure 2^{-2k}). We denote by $\mathcal{S}_k := \mathcal{S}_k(Q)$ the space of piecewise constants on the partition $\mathcal{D}_k(Q)$. This is a linear space spanned by the functions φ_I, $I \in \mathcal{D}_k(Q)$.

We define the family of non-linear spaces of piecewise constant functions:

$$\Sigma_N^c = \{\sum_{I\in E} c_I \varphi_I \; ; \; E \subset \mathcal{D}, \; |E| \leq N\}, \tag{1.13}$$

i.e. all linear combinations of at most N characteristic functions of dyadic cubes.

A natural procedure to approximate in Σ_N^w is the simple thresholding of wavelet coefficients. In order to obtain approximations in Σ_N^c, one can think of different procedures. The simplest one is based on a *quadtree splitting algorithm*: given a tolerance $\epsilon > 0$ and a function $f \in L_2(Q)$, one builds an adaptive partition of Q into dyadic cubes by splitting into four subcubes each cube I such that the *residual*

$$R(I) := \|f - a_I(f)\|_{L_2(I)},$$

is larger than ϵ. The procedure is initiated from the unit cube Q, and stops when all residuals are smaller than ϵ, and f is then approximated by $f_\epsilon := \sum_{I \in \mathcal{P}_\epsilon} a_I \varphi_I$, where $\mathcal{P}_\epsilon$ is the final partition of Q.

The approximation properties of such adaptive algorithms have been studied in [DY]. However, this algorithm does not exploit the full approximation properties of Σ_N^c since it imposes that the cubes involved in the definition of f_ϵ are disjoint. One can actually show by simple counterexamples that this procedure does not yield the direct estimate we desire in proving (1.1) or (1.2), i.e. too many cubes could be generated to achieve a certain accuracy in the approximation of certain BV functions.

A more efficient procedure should thus not only involve splitting, but also *merging* of cubes, which will amount in using non-disjoint cubes in the definition of a suitable approximation. In this paper, we shall introduce a "split and merge" algorithm that produces an approximation of f based on disjoint partitions of Q into *dyadic rings*. By definition a dyadic ring is the difference between two embedded dyadic cubes, i.e. any set of the type

$$K := I \setminus J, \quad J \subset I, \quad I, J \in \mathcal{D}. \tag{1.14}$$

We also consider a dyadic cube to be a degenerate case of a dyadic ring for which J is empty. Throughout this paper, a "cube" will always stand for a dyadic cube, and a "ring" for a dyadic ring. Our third family of approximation space Σ_N^r is the set of all functions of the form

$$f = \sum_{\Omega \in \mathcal{P}} c_\Omega \varphi_\Omega, \tag{1.15}$$

where $\mathcal{P}$ is a set of at most N dyadic rings, that form a *partition of* Q, i.e. the rings are disjoint and union to Q. Note that (1.11) means that $\varphi_\Omega = \varphi_I - \varphi_J$ so that $\Sigma_N^r \subset \Sigma_{2N}^c$. We can thus use Σ_N^r to prove results on approximation by Σ_N^c.

An important point that should be mentioned here is that the nonlinearity of the three families Σ_N^w, Σ_N^c and Σ_N^r, is "controlled" in the sense that they all satisfy

$$\Sigma_N + \Sigma_M \subset \Sigma_{a(M+N)}, \tag{1.16}$$

with a an absolute constant. This is obvious in the case of Σ_N^w and Σ_N^c, with $a = 1$. It can also be proved for Σ_N^r (with a larger value of a).

The outline of our paper is the following:

In §2, we define the spaces $\mathrm{BV}(\Omega)$ for domains $\Omega \subset \mathbb{R}^2$ and recall certain basic properties of these spaces. In §3, we prove inverse estimates of the type (1.8) for the spaces Σ_N^w, Σ_N^r and Σ_N^c.

In order to study the process of approximation for Σ_N^r, we prove in §4 the projection error estimate

$$\|f - a_\Omega\|_{L_2(\Omega)} \le C_1 |f|_{\mathrm{BV}(\Omega)}, \tag{1.17}$$

where C_1 is independent of the ring Ω. We then prove in §5 the stability estimate

$$|\sum_{\Omega\in\mathcal{P}} a_\Omega(f)\varphi_\Omega|_{\mathrm{BV}(Q)} \le C_2 |f|_{\mathrm{BV}(Q)}, \tag{1.18}$$

where C_2 does not depend on the partition $\mathcal{P}$ of Q into disjoint rings. The uniformity of C_1 and C_2 is ensured by the the controlled shape of a dyadic ring which cannot be very anisotropic.

In §6, we introduce our algorithm for approximation by the elements of Σ_N^r and use it to prove the Jackson inequality. This algorithm relies on a general result concerning the existence of partitions of Q into rings which are well balanced with respect to a super-additive cost function. We prove in §7 that this algorithm is also a near best solution to the extremal problem (1.2). We anticipate therefore that this algorithm will be useful in image processing but this will not be addressed in the present paper which mostly concentrates on the theoretical issues.

In §8, we prove the direct estimate for (Haar) wavelet shrinkage, i.e. approximation by Σ_N^w, and we show in §9 that this procedure is stable in BV and provide solutions for the two extremal problems (1.1) and (1.2). It should be pointed out that the results of these two sections make important use of the results that we establish for Σ_N^r, and that so far we do not know how to prove them in a more direct way.

Finally, we use our results in §10 to identify the interpolation spaces between $L_2(Q)$ and $BV(Q)$.

Throughout the paper, we give explicit constants for all important inequalities. Most of them (in particular $(C_0, C_1, \cdots, C_6)$ that appear in the end of the paper), can probably be improved using more refined arguments.

2. The space $\mathrm{BV}(\Omega)$.

In this section, we shall define for certain domains $\Omega \subset \mathbb{R}^2$, the spaces $\mathrm{BV}(\Omega)$ of functions of bounded variation on Ω and recall some basic properties of this space. While $\mathrm{BV}(\Omega)$ can be defined for general domains, in this paper, we shall primarily be interested in *rings* $\Omega = I \setminus J$, where I and $J \subset I$ are in $\mathcal{D}(Q)$.

For a vector $\mu \in \mathbb{R}^2$, we define the difference operator Δ_μ in the direction μ by

$$\Delta_\mu(f, x) := f(x + \mu) - f(x). \tag{2.1}$$

Let Ω be any domain in $\mathbb{R}^2$. For functions f defined on Ω, $\Delta_\mu(f, x)$ is defined whenever $x \in \Omega(\mu)$, where $\Omega(\mu) := \{x : [x, x + \mu] \subset \Omega\}$ and $[x, x + \mu]$ is the line segment connecting x and $x + \mu$. Note that if Ω is bounded and μ is large enough then $\Omega(\mu)$ is empty. Let e_j, $j = 1, 2$, be the two coordinate vectors in $\mathbb{R}^2$. We say that a function $f \in L_1(\Omega)$ is in $\mathrm{BV}(\Omega)$ if and only if

$$\mathrm{V}_\Omega(f) := \sup_{0<h} h^{-1} \sum_{j=1}^{2} \|\Delta_{he_j}(f, \cdot)\|_{L_1(\Omega(he_j))} = \lim_{h\to 0} \sum_{j=1}^{2} \|\Delta_{he_j}(f, \cdot)\|_{L_1(\Omega(he_j))} \tag{2.2}$$

is finite. Here, the last equality in (2.2) follows from the fact that $\|\Delta_{he_j}(f,\cdot)\|_{L_1(\Omega(he_j))}$ is subadditive (see e.g. Theorem 7.11.1 in [HP]). By definition, the quantity $\mathrm{V}_\Omega(f)$ is the *variation* of f over Ω. It provides a semi-norm and norm for $\mathrm{BV}(\Omega)$:

$$(2.3) \qquad |f|_{\mathrm{BV}(\Omega)} := \mathrm{V}_\Omega(f); \quad \|f\|_{\mathrm{BV}(\Omega)} := |f|_{\mathrm{BV}(\Omega)} + \|f\|_{L_1(\Omega)}.$$

Let $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint sets. Then for any $h > 0$ and $j = 1, 2$, one has the inclusion $\Omega_1(he_j) \cup \Omega_2(he_j) \subset \Omega(he_j)$. Hence, for $j = 1, 2$,

$$(2.4) \qquad \|\Delta_{he_j}(f,\cdot)\|_{L_1(\Omega_1(he_j))} + \|\Delta_{he_j}(f,\cdot)\|_{L_1(\Omega_2(he_j))} \le \|\Delta_{he_j}(f,\cdot)\|_{L_1(\Omega(he_j))}.$$

Summing over j and taking the the limit as h tends to 0, we obtain

$$(2.5) \qquad V_{\Omega_1}(f) + \mathrm{V}_{\Omega_2}(f) \le \mathrm{V}_\Omega(f).$$

By induction, the analogue of (2.5) holds for any finite union of disjoint sets.

We recall the L_1-modulus of continuity $\omega(f,t)_\Omega$ which is defined by

$$(2.6) \qquad \omega(f,t)_\Omega := \sup_{|\mu| \le t} \|\Delta_\mu(f,\cdot)\|_{L_1(\Omega(\mu))}.$$

Here and later $|x| := \sqrt{x_1^2 + x_2^2}$ is the Euclidean metric. For any ring, we have that $\mathrm{BV}(\Omega)$ is identical with $\mathrm{Lip}(1, L_1(\Omega))$, where the latter set consists of all functions such that

$$(2.7) \qquad |f|'_{\mathrm{BV}(\Omega)} := \sup_{t>0} t^{-1}\omega(f,t)_\Omega$$

is finite. We also have

$$(2.8) \qquad |f|'_{\mathrm{BV}(\Omega)} \le |f|_{\mathrm{BV}(\Omega)} \le 2|f|'_{\mathrm{BV}(\Omega)}.$$

Indeed, the right inequality in (2.8) is obvious from the definition of the two semi-norms. The left inequality follows from the fact for any point $x \in \Omega(\mu)$, $\mu = (\mu_1, \mu_2)$, either $[x, x+\mu_1 e_1]$ and $[x+\mu_1 e_1, x+\mu]$ are both contained in Ω or $[x, x+\mu_2 e_2]$ and $[x+\mu_2 e_2, x+\mu]$ are both contained in Ω.

For a ring $\Omega = I \setminus J$, we define $\mathcal{D}(\Omega)$ to be the set of all $I \in \mathcal{D}$ which are contained in Ω and similarly, we define $\mathcal{D}_k(\Omega)$ the subset of $\mathcal{D}(\Omega)$ that consists of the cubes of sidelength 2^{-k}. If $2^{-2k} \le |J|$, when J is non empty or if $2^{-2k} \le |I|$ when $\Omega = I$ is a cube, we can define $\mathcal{S}_k(\Omega)$ to be the restriction of $\mathcal{S}_k$ to Ω. For any $f \in L_1(\Omega)$, we define the $P_k(f)$ to be the orthogonal projection of f onto $\mathcal{S}_k(\Omega)$. Then,

$$(2.9) \qquad P_k(f) = \sum_{I \in \mathcal{D}_k(\Omega)} a_I(f)\varphi_I.$$

It is easy to prove that whenever $f \in \mathrm{BV}(\Omega)$

$$(2.10) \qquad \|f - P_k(f)\|_{L_1(\Omega)} \le 2^{-k} V_\Omega(f)$$

and

$$\mathrm{V}_\Omega(P_k(f)) \le \mathrm{V}_\Omega(f). \tag{2.11}$$

For a proof of these results see [L, Chapter 3, Lemma 3.2] for the case when Ω is a cube (the same proof also works for rings).

It is also easy to calculate the BV norm of functions $S \in \mathcal{S}_k(\Omega)$. For any set $A \subset \mathbb{R}^2$, let $\mathcal{L}_k(A)$ denote the edges L of the cubes $I \in \mathcal{D}_k(\bar{Q})$ which are contained in A. We also denote by Ω^o the interior of Ω, and by J_L, $L \in \mathcal{L}_k(\Omega^o)$, the jump in f across L. Then, (see again [L, Chapter 3, Lemma 3.1])

$$\mathrm{V}_\Omega(f) = 2^{-k} \sum_{L \in \mathcal{L}_k(\Omega^o)} |J_L|. \tag{2.12}$$

3. Inverse estimates.

In the introduction, we have introduced three families of non-linear spaces (Σ_N^w, Σ_N^c and Σ_N^r). We begin our study of these spaces in this section by proving (1.8) for any ring Ω. We shall obtain specific constants in (1.8) although this is not important for the theoretical results that follow.

We first treat the space Σ_N^w which appears in wavelet thresholding.

Theorem 3.1. *For each $f \in \Sigma_N^w$, we have*

$$\mathrm{V}_Q(f) \le 8\, N^{1/2} \|f\|_{L_2(Q)}. \tag{3.1}$$

Proof. We first observe that any Haar basis function ψ_λ (see (1.11)) satisfies

$$V_Q(\psi_\lambda) \le 8 = 8\|\psi_\lambda\|_{L_2}. \tag{3.2}$$

Indeed, if the support of ψ_λ is a square I of side length $h = 2^{-k}$, then it takes the values $\pm h^{-1}$ on I. We can calculate $\mathrm{V}_Q(\psi_\lambda)$ by (2.12). The jumps across the outer boundary of I give $h^{-1}4h = 4$ and those across the inner boundary give at most $2h^{-1}2h = 4$. Thus, (3.2) is proved.

If $f = \sum_{\lambda \in E} f_\lambda \psi_\lambda$ is in Σ_N^w, then

$$V_Q(f) \le 8 \sum_{\lambda \in E} |f_\lambda| \le 8|E|^{1/2} [\sum_{\lambda \in E} |f_\lambda|^2]^{1/2} \le 8N^{1/2}\|f\|_{L_2}, \tag{3.3}$$

by the Cauchy-Schwarz inequality. $\Box$

Remark 3.1. Using that $V_Q(f) \le 8\sum_{\lambda\in E}|f_\lambda|$, we also obtain the following variant of the inverse inequality (3.1): Let $t > 0$ and $f = \sum_{\lambda \in E} f_\lambda \psi_\lambda$ be a linear combination of Haar wavelets such that $|f_\lambda| \ge t$ for all $\lambda \in E$, then

$$|f|_{\mathrm{BV}} \le \frac{8}{t}\|f\|_{L_2}^2. \tag{3.4}$$

We now prove the Bernstein inequality for Σ_N^r by a very similar argument.

Theorem 3.2. *For each $f \in \Sigma_N^r$, we have*

$$\mathrm{V}_Q(f) \le \frac{4\sqrt{5}}{\sqrt{3}} N^{1/2} \|f\|_{L_2(Q)}. \tag{3.5}$$

Proof. We first prove that if $\Omega = I \setminus J$ is any ring contained in Q, then

$$|\varphi_\Omega|_{\mathrm{BV}} \le \frac{4\sqrt{5}}{\sqrt{3}} \|\varphi_\Omega\|_{L_2}. \tag{3.6}$$

To prove this, let ℓ be the side length of I and $h\ell$ be the side length of J. Then, $\|\varphi_\Omega\|_{L_2(Q)}^2 = \ell^2(1-h^2)$. We consider two cases. In the first case, we assume that J is in the interior of I. Then necessarily, $h \le 1/4$. In this case $\mathrm{V}_Q(\varphi_\Omega) \le 4\ell + 4\ell h = 4\ell(1+h)$ where the first term comes from the jump across the outer boundary and the second the jump across the inner boundary. Since $\frac{(1+h)^2}{1-h^2} \le \frac{5}{3}$, we have verified (3.6) in this case. In the second case, we assume that J shares an edge with I. Then $\mathrm{V}_Q(\varphi_\Omega) \le (4\ell - \ell h) + 3\ell h = 4\ell(1+h/2)$. Since $\frac{(1+h/2)^2}{1-h^2} \le 25/12$ for $0 \le h \le 1/2$, (3.6) follows in this case as well.

If $f \in \Sigma_N^r$, then $f = \sum_{\Omega \in \mathcal{P}} f_\Omega \varphi_\Omega$ with $\mathcal{P}$ a partition of Q into rings, then

$$V_Q(f) \le \frac{4\sqrt{5}}{\sqrt{3}} \sum_{\Omega \in \mathcal{P}} |f_\Omega| \|\varphi_\Omega\|_{L_2} \le \frac{4\sqrt{5}}{\sqrt{3}} N^{1/2} \|f\|_{L_2}, \tag{3.7}$$

by the Cauchy-Schwarz inequality. $\square$

We close this section by using ideas from [DP] to prove the Bernstein inequality for Σ_N^c. If E is a finite collection of dyadic cubes, then for each $I \in E$ we define $B_I(E)$ to be the set of all cubes J that are maximal in I, i.e., $J \subset I$, $J \in E$, and J is not contained in another cube with these properties. It was shown in Lemma 6.1 of [DP] that any set E can be embedded in a set E' with $|E'| \le 4|E|$ and

$$|B_I(E')| \le 4, \quad \text{for all } I \in E'. \tag{3.8}$$

Theorem 3.3. *For each $f \in \Sigma_N^c$, we have*

$$\mathrm{V}_Q(f) \le \frac{28}{\sqrt{3}} N^{1/2} \|f\|_{L_2(Q)}. \tag{3.9}$$

Proof. If $f \in \Sigma_N^c$, we can write $f = \sum_{I \in E} f_I \varphi_I$, where $E \subset \mathcal{D}(Q)$ and $|E| \le N$. Let E' be a set which contains E, satisfies (3.8), and such that $|E'| \le 4N$. Then, we can also represent f as

$$f = \sum_{I \in E'} d_I \varphi_I. \tag{3.10}$$

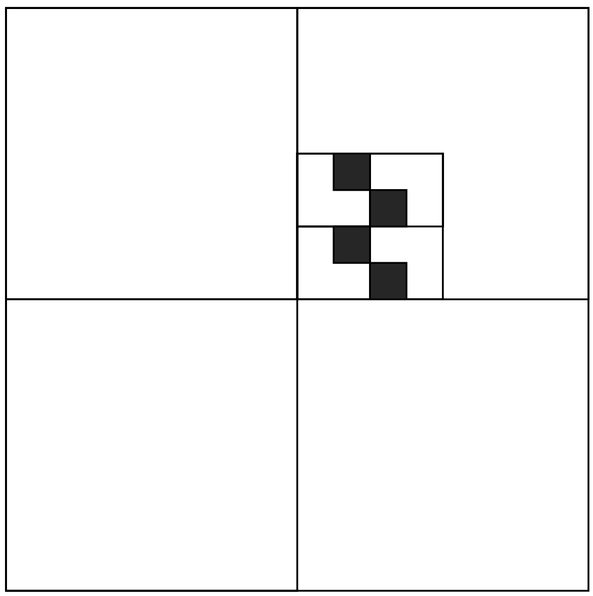

FIGURE 1

If $I \in E'$, we define $I' := I \setminus \cup\{J : J \in \mathcal{B}_I(E')\}$. The functions $\varphi_{I'}$, $I \in E'$, have disjoint supports and

$$f = \sum_{I \in E'} c_I \varphi_{I'}, \tag{3.11}$$

with $c_I := \sum_{J \supseteq I, J \in E'} d_J$. We can assume that all $\varphi_{I'}$ appearing in (3.11) are nonzero.

For each of these functions, we have a basic inverse estimate

$$\mathrm{V}_Q(\varphi_{I'}) \leq \frac{14}{\sqrt{3}} \|\varphi_{I'}\|_{L_2}. \tag{3.12}$$

The proof of (3.12) is similar to that of (3.2) and (3.6) except that we have to check more cases. The quotient

$$\frac{\mathrm{V}_Q(\varphi_{I'})}{\|\varphi_{I'}\|_{L_2}}$$

takes its largest value for the configuration in Figure 1 which gives the constant $\frac{14}{\sqrt{3}}$. We leave this verification to the reader.

Using the Cauchy-Schwarz inequality, we find

$$\begin{aligned} \mathrm{V}_Q(f) &\le \sum_{I\in E'} |c_I|\,\mathrm{V}_Q(\varphi_{I'}) \\ &\le \frac{14}{\sqrt{3}} \sum_{I\in E'} |c_I| \|\varphi_{I'}\|_{L_2} \\ &\le \frac{28}{\sqrt{3}} N^{1/2} \left(\sum_{I\in E'} |c_I|^2 \|\varphi_{I'}\|_{L_2}^2 \right)^{1/2} = \frac{28}{\sqrt{3}} N^{1/2} \|f\|_{L_2(Q)}. \quad \square \end{aligned}$$

4. Approximation by a constant on a ring-shaped domain.

In this section, we shall give bounds for the L_2-error of approximation of a BV function by a constant on a ring-shaped domain. At first, we shall make certain preliminary constructions which will be used in the proofs of these results as well as those of the next section.

Let Ω be a ring contained in Q: $\Omega := I_1 \setminus I_0$, $I_0, I_1 \in \mathcal{D}(Q)$, $I_0 \subset I_1$. We shall consider piecewise constant functions in $\mathcal{S}_k(\Omega)$. We assume that k is large enough that $2^{-2k} \le |I_1|$ and $2^{-2k} \le |I_0|$ if I_0 is not empty. We can therefore write $|I_1| = m_1^2 2^{-2k}$ and $|I_0| = m_0^2 2^{-2k}$ with m_0, m_1 positive integers and $m_0 < m_1$.

Let $\mathcal{B}_k(\Omega)$ denote the external layer of boundary cubes for Ω, i.e. the set of cubes $I \in D_k(\mathbb{R}^2)$ such that I is not in $\mathcal{D}_k(\Omega)$ but $\bar{I} \cap \bar{\Omega}$ contains a line segment. Let (a, b) be the lower left vertex of I_1. We index each cube $I \in \mathcal{D}_k(I_1)$ by the pair of integers (i, j), $1 \le i, j \le m_1$, such that $(a, b) + 2^{-k}(j - 1/2, i - 1/2)$ is in I (we have purposefully reversed i and j in the indexing so that i will now correspond to a row and j to a column). Boundary cubes can be indexed in the same way with i, j now allowed to take the values 0 and $m_1 + 1$. Note that, in general, there are two types of boundary cubes: the *interior boundary cubes* (which are contained in I_0) and the *exterior boundary cubes* which are outside of I_1. If I is indexed by (i, j), we say that I is in row i and column j. We say a row i (respectively column j) is *unobstructed* if all cubes $I \in \mathcal{D}_k(I_1)$ from row i (respectively column j) are in $\mathcal{D}_k(\Omega)$.

By an *admissible path* ρ for Ω, we shall mean a piecewise linear path with the following properties. Each segment of ρ is parallel to a coordinate axis and connects a center of a cube $I \in \mathcal{D}_k(\Omega) \cup \mathcal{B}_k(\Omega)$ to the center of another cube $J \in \mathcal{D}_k(\Omega) \cup \mathcal{B}_k(\Omega)$. Each edge $L \in \mathcal{L}_k(\Omega \cup \partial\Omega)$ is transversed at most once by ρ and each edge not in this set is never transversed by ρ.

For each $i = 1, \dots, m_1$, there are either two or four boundary cubes in $\mathcal{B}_k(\Omega)$ which are in row i. For each distinct pair of these cubes (I, J), we shall construct an admissible path $\rho_i(I, J)$ which connects I to J as follows.

If there are exactly two such boundary cubes for row i, we take the strictly horizontal path which connects the center of I to the center of J.

Consider next the case where there are four boundary cubes in row i. The indices of these cubes are (i, j), $j = j_0, j_1, j_2, j_3$, where $j_0 = 0 < j_1 < j_2 < j_3 = m_1 + 1$. Moreover,

$j_1 > m_0$ and $j_3 - j_2 > m_0$. Let I and J be two of these boundary cubes with indices (i,j) and (i,j') and $j < j'$. If $j = j_0$ and $j' = j_1$, we take the path $\rho_i(I,J)$ to again be the strictly horizontal path connecting the center of I to the center of J. We proceed similarly if $j = j_2$ and $j' = j_3$.

We now consider the remaining cases. Let $j(i) \in [1, m_0]$ be congruent to i mod m_0. Then, the column with index $j(i)$ is unobstructed. Similarly, the column with index $j'(i) := m_1 - j(i) + 1$ is unobstructed. Also, for one of the two choices $i_1 := i \pm m_0$, the row with index i_1 is unobstructed.

If I, J are a pair for which we have not yet constructed $\rho_i(I,J)$, then we construct this path as the concatenation of the the five segments which connect the centers of the cubes with the following indices in the specified order: (i,j), $(i,j(i))$, $(i_1,j(i))$, $(i_1,j'(i))$, $(i,j'(i))$, (i,j'). It follows that $\rho_i(I,J)$ is an admissible path.

We shall need one last type of row path that occurs only in the case that row i is obstructed but there are only two boundary cubes. This case occurs when I_0 touches the boundary of I_1. Let I be the boundary cube in row i which touches the boundary of I_1. We assume that I has index $(i,0)$ (the case when I has index (i,m_1+1) is handled in a symmetric manner). We let $j(i)$ and i_1 be as above. We let $\rho(I)$ be the admissible path which consists of the three segments which connect the centers of the cubes with indices $(i,0)$, $(i,j(i))$, $(i_1,j(i))$ and (i_1,m_1+1) in that order.

We make the analogous construction of paths which connect the boundary cubes in column j and denote these paths by $\gamma_j(I,J)$.

We shall now use these paths to prove the error estimate (1.17) for rings. Before proceeding to the proof of (1.17), we remark that this inequality holds for general Lipschitz domains Ω. Indeed, using the known embedding of $\mathrm{BV}(\Omega)$ into $L_2(\Omega)$: we have

$$\|f-a\|_{L_2(\Omega)} \le C\|f-a\|_{\mathrm{BV}(\Omega)}, \tag{4.1}$$

for any function f and constant a. Therefore, taking the infimum over a, we obtain

$$\|f-a_\Omega(f)\|_{L_2(\Omega)} \le C\inf_{a\in\mathbb{R}}\|f-a\|_{\mathrm{BV}(\Omega)} \le C_1|f|_{\mathrm{BV}(\Omega)} = C_1\,\mathrm{V}_\Omega(f). \tag{4.2}$$

The last inequality in (4.2) follows for example from elementary results in approximation (see e.g. Theorem 3.5 in [DS]). It is to see that the constant C_1 is invariant by isotropic scaling of Ω, but grows by anisotropic (e.g. one directional) scaling. This reveals that C_1 strongly depends on the shape of Ω. Our goal is to directly prove (1.17) with a constant C_1 that is uniform for rings $\Omega = I_1 \setminus I_0$.

Let $S \in \mathcal{S}_k(\Omega)$ be a piecewise constant function on Ω with k such that 2^{-2k} is less than $|I_1|$ and 2^{-2k} is less than $|I_0|$ in the case where I_0 is not empty. Given a path ρ, let

$$J(\rho) := \sum_L |J_L|, \tag{4.3}$$

where the sum is taken over all edges $L \in \mathcal{L}_k(\Omega^o)$ which are crossed by ρ. Here and later, we use the notation K^o to denote the interior of a set $K \subset \mathbb{R}^2$.

For each i, we define

$$r_i := \sum_{\rho_i} J(\rho_i), \tag{4.4}$$

where the sum is taken over all the paths ρ_i associated to the row index i (recall there are one or six such paths) and

$$R := \sum_{i=1}^{m_1} r_i. \tag{4.5}$$

Similarly, we define

$$c_j := \sum_{\gamma_j} J(\gamma_j), \tag{4.6}$$

where the sum is taken over all the paths γ_j associated to the column index j and

$$C := \sum_{j=1}^{m_1} c_j. \tag{4.7}$$

Lemma 4.1. *For any ring Ω and any $S \in \mathcal{S}_k(\Omega)$, we have*

$$2^{-k}(R + C) \leq 9\,\mathrm{V}_\Omega(f). \tag{4.8}$$

Proof. We shall first estimate how often $|J_L|$, with L a fixed vertical edge, $L \in \mathcal{L}_k(\Omega^o)$, appears in the sum $R + C$. Suppose first that L is in an unobstructed row i. Then L appears exactly once for paths ρ_i. The row i is used at most four times for paths $\rho_{i'}$, with $i \neq i'$. The row i is also used at most four times for paths γ_j. Hence J_L appears at most 9 times in the sum $R + C$. Consider next the case when i is obstructed. Then, J_L appears exactly once for paths ρ_i and it never appears for any other paths $\rho_{i'}$ or γ_j. The same estimate holds for J_L when L is a horizontal edge. Thus,

$$2^{-k}(R + C) \leq 9 \sum_{L \in \mathcal{L}_k(\Omega^o)} 2^{-k}|J_L| = 9V_\Omega(f), \tag{4.9}$$

where the last equality is given by (2.12). $\square$

Remark 4.1 In the case Ω is a cube, the constant 9 in (4.8) can be replaced by 1.

Theorem 4.1. *For any ring $\Omega = I_1 \setminus I_0$ and any function $f \in \mathrm{BV}(\Omega)$, we have*

$$\|f - a_\Omega(f)\|_{L_2(\Omega)} \leq 6\sqrt{3}V_\Omega(f) \tag{4.10}$$

Proof. Let us first observe that it is sufficient to prove this estimates for the special case of functions $S \in \mathcal{S}_k(\Omega)$. Indeed, if this has been shown, then we have

$$\|f - a_\Omega(f)\|_{L_2(\Omega)} \le \|f - P_k(f)\|_{L_2(\Omega)} + \|P_k(f) - a_\Omega(f)\|_{L_2(\Omega)}, \tag{4.11}$$

where P_k is the projector onto $\mathcal{S}_k(\Omega)$. The first term tends to zero with k and the second would provide our estimate since $a_\Omega(P_k(f)) = a_\Omega(f)$ and since by (2.11) $\mathrm{V}_\Omega(P_k(f)) \le \mathrm{V}_\Omega(f)$ if k is sufficiently large.

Henceforth, we consider $f \in \mathcal{S}_k$, with k such that 2^{-2k} is less than $|I_1|$ and 2^{-2k} is less than $|I_0|$ in the case where I_0 is not empty. Let $p_I = p_{i,j}$ denote the value of f on the cube I with I in row i and column j. (with similar notation for I') and let Λ denote the set of (i,j) such that the cube with index (i,j) is contained in Ω and let $N := |\Lambda|$. Then, $A := a_\Omega(f) = \frac{1}{N}\sum_{(i',j')\in\Lambda} p_{i',j'}$. Therefore,

$$|p_{i,j} - A| \le N^{-1} \sum_{(i',j')\in\Lambda} |p_{i,j} - p_{i',j'}|. \tag{4.12}$$

We can construct an admissible path ρ which connects the center of I to the center of I' using portions of the paths ρ_i and $\gamma_{j'}$. Indeed, it is easy to see from our constructions there is a path ρ_i associated to row i which passes through I and a path γ_j associated to column j which passes through j such that ρ_i intersects γ_j. We take ρ as the shortest path contained in $\rho_i \cup \gamma_j$ which connects the center of I to the center of J. It follows that $|p_{i,j} - p_{i',j'}|$ does not exceed the sum of the J_L crossed by this path. Hence,

$$|p_{i,j} - p_{i',j'}| \le r_i + c_{j'}. \tag{4.13}$$

By a symmetric argument, we obtain that

$$|p_{i,j} - p_{i',j'}| \le r_{i'} + c_j. \tag{4.14}$$

By (4.13) we obtain

$$|p_{i,j} - A| \le N^{-1} \sum_{(i',j')\in\Lambda} (r_i + c_{j'}) \le r_i + \frac{m_1 C}{N}, \tag{4.15}$$

and by (4.14)

$$|p_{i,j} - A| \le c_j + \frac{m_1 R}{N}. \tag{4.16}$$

Hence

$$|p_{i,j} - A|^2 \le (r_i + \frac{m_1 C}{N})(c_j + \frac{m_1 R}{N}) = r_i c_j + \frac{m_1}{N} r_i R + \frac{m_1}{N} c_j C + \frac{m_1^2}{N^2} RC.$$

We note that $N 2^{-2k} = |\Omega| \ge \frac{3}{4}|I_1| = \frac{3}{4} m_1^2 2^{-2k}$. In other words, $m_1^2 \le \frac{4}{3}N$. Therefore, summing over i,j we obtain

$$\begin{aligned}\|S - A\|_{L_2(\Omega)}^2 = 2^{-2k} \sum_{(i,j)\in\Lambda} |p_{i,j} - A|^2 &\le 2^{-2k}(RC + \frac{m_1^2}{N} R^2 + \frac{m_1^2}{N} C^2 + \frac{m_1^2}{N} RC) \\ &\le \frac{4}{3} 2^{-2k} (R + C)^2 \le \frac{4}{3} 9^2 \,\mathrm{V}_\Omega(f)^2,\end{aligned}$$

where we have used Lemma 4.1. This proves (4.10). $\square$

Remark 4.2 In the case Ω is a cube, the constant $6\sqrt{3}$ in (4.10) can be replaced by 1.

5. Projections onto piecewise constant functions.

In this section, we shall prove the BV stability of projections onto a space of piecewise constant functions related to a partition of Q into rings.

We denote by $\mathcal{P}$ a partition of Q into a finite number of rings. This means that the elements of $\mathcal{P}$ are rings K which are pairwise disjoint and union to Q. For each such partition $\mathcal{P}$, we define

$$P_{\mathcal{P}}(f) := \sum_{K \in \mathcal{P}} a_K(f)\varphi_K, \tag{5.1}$$

where we recall that $a_K(f)$ is the average of f over K and φ_K is the characteristic function of K.

Theorem 5.1. *For any finite partition $\mathcal{P}$ of Q into rings and any $f \in \mathrm{BV}(Q)$, we have*

$$\mathrm{V}_Q(P_{\mathcal{P}}(f)) \le 10\,\mathrm{V}_Q(f). \tag{5.2}$$

Proof. Let k be large enough so that for any $K \in \mathcal{P}$, $K = I_1 \setminus I_0$, we have $|I_1| \ge 2^{-2k}$ and $|I_0| \ge 2^{-2k}$ if I_0 is not empty. Then $P_{\mathcal{P}}(f) = P_{\mathcal{P}}(P_k(f))$. Thus, in view of (2.11), it is enough to show that (5.2) holds for any $f \in \mathcal{S}_k$. We consider only such f in the remainder of this proof.

If $L \in \mathcal{L}_K(Q)$, we denote by $J_L := J_L(f)$ the jump in f across L and by $J_L(P_{\mathcal{P}}(f))$ the jump in $P_{\mathcal{P}}(f)$ across L. For any set $R \subset Q$, we define

$$\Sigma(f,R) := \sum_{L \in \mathcal{L}_k(R)} |J_L|. \tag{5.3}$$

Fix one set K from $\mathcal{P}$ and let f_0 be obtained from f by redefining f to be $a_K(f)$ on K. Note that the jumps in f_0 are the same as those of f except for those inside K (which will be 0 in f_0) and those on ∂K, the boundary of K. We shall prove that

$$\Sigma(f_0,Q) \le \Sigma(f,Q) + 9\Sigma(f, K \setminus \partial K). \tag{5.4}$$

Assume for the moment, we have proven (5.4). Then, repeating successively for each $K \in \mathcal{P}$ the process that constructs f_0 from f, we arrive at

$$\Sigma(P_{\mathcal{P}}(f),Q) \le \Sigma(f,Q) + 9\sum_{K \in \mathcal{P}} \Sigma(f, K \setminus \partial K) \le (1+9)\Sigma(f,Q). \tag{5.5}$$

Since $V_Q(f) = 2^{-k}\Sigma(f,Q)$, (5.5) implies (5.2).

We finish the proof by proving (5.4). We shall use the paths that were constructed in §4. We fix a ring $K \in \mathcal{P}$ and we index the cubes $I \in \mathcal{D}_k(K) \cup \mathcal{B}_k(K)$ as in §4. Let $p_I = p_{i,j}$ denote the value of f on I when I has index (i,j). Let $J'_L := J_L(f_0)$ be the jump in f_0 across $L \in \mathcal{L}_k(Q^o)$. We need to estimate J'_L for those L contained in the boundary of K. To each such L, there is an $I = I(L) \in \mathcal{B}_k(K)$ which contains L as one of its sides.

We let (i,j) denote the index of I. Then, we have

$$|J'_L| \le \frac{1}{N} \sum_{(i',j')\in\Lambda} |p_{i,j} - p_{i',j'}|, \tag{5.6}$$

where as before Λ denotes the set of (i,j) such that the cube with index (i,j) is contained in K, and $N = |\Lambda|$. Let I' have index (i',j'). As in the proof of Theorem 4.1, using a subpath of one of the ρ_i and a subpath of one of the $\gamma_{j'}$ (in the case $1 \le i \le m_1$) or from $\rho_{i'}$ and γ_j (in the case $1 \le j \le m_1$), we can construct an admissible path $\rho(i,j,i',j')$ for K which connects the center of I to the center of I'. Let $\Gamma(i,j,i',j')$ denote the collection of all of the $M \in \mathcal{L}_k(Q)$ which intersect this path. Then,

$$|J'_L| \le \frac{1}{N} \sum_{(i',j')\in\Lambda} \sum_{M\in\Gamma(i,j,i',j')} |J_M|. \tag{5.7}$$

Thus,

$$\sum_{L\in\partial K} |J'_L| \le \frac{1}{N} \sum_{M\in\mathcal{L}_k(Q^o)} n_M |J_M|, \tag{5.8}$$

where n_M is the total number of times M appears in all of the sets $\Gamma(i,j,i',j')$, with (i,j) the index of a cube in $\mathcal{B}_k(K)$ and (i',j') the index of a cube in $\mathcal{D}_k(K)$. We shall complete the proof by showing that

(i) $n_M = 0$, if M is not contained in $L_k(K) \cup L_k(\partial K)$,

(ii) $n_M = N$, if $M \in \mathcal{L}_k(\partial K)$,

(iii) $n_M \le 9N$, if $M \in \mathcal{L}_k(K^o)$.

Clearly, these three estimates used in (5.8) prove (5.4).

Now, statement (i) is obvious because all the paths $\rho(i,j,i',j')$ are admissible for K. Statement (ii) is also obvious because J_M, $M \in \mathcal{L}_k(\partial K)$ is crossed only by the paths that emanate from $I(M)$ and there are exactly N of these (one for each cube I' in $\mathcal{D}_k(K)$). To prove (iii), consider for example a vertical segment $M \in \mathcal{L}_k(K \setminus \partial K)$. If M is in an obstructed row, then for each (i',j'), M will appear in exactly one $\Gamma(i,j,i',j')$; namely for one pair (i,j) with i the index of the row which contains M. So for these M, we have $n_M = N$. On the other hand, if M is in an unobstructed row i^*, then for each (i',j'), M will appear in only one of the $\Gamma(i^*,j,i',j')$ for the two values of j corresponding to boundary cubes. At the same time, M can appear at most four times in the sets $\Gamma(i,j,i',j')$, $1 \le i \le m_1$, $i \ne i^*$; namely for the one possible obstructed row with index i which is congruent to i^* mod m_0. Similarly, for each (i',j'), M can appear at most four times in the sets $\Gamma(i,j,i',j')$, $1 \le j \le m_1$. Thus $n_M \le 9N$ in this case. We have proved (i-iii) and completed the proof of the theorem. □

6. A partition algorithm and a direct estimate for Σ_N^r.

In this section, we shall prove the direct estimate (1.7) for Σ_N^r. Our proof is based on two ingredients:

(i) The projection error inequality (1.17) for ring-shaped domains that was established in

§4.
(ii) A general result on the partitioning of Q into rings with respect to a super-additive function.

The proof of this second result will actually provide a concrete algorithmic procedure that builds adaptive partitions of Q into rings for the approximation of a given function f.

If $f \in L_2(Q)$, we define

$$\sigma_N^r(f) := \inf_{g \in \Sigma_N^r} \|f - g\|_{L_2(Q)} \tag{6.1}$$

which is the error of approximation by the elements of Σ_n^r.

In the following, we let Φ denote a positive set function defined on the algebra $\mathcal{A}(Q)$ generated by the rings $K \subset Q$. That is, $\mathcal{A}(Q)$ consists of all subsets of Q which can be formed by finite unions and intersections of rings $K \subset Q$ and their complements. We make the following assumptions on Φ:
(i) Φ is super-additive: if K_1 and K_2 are disjoint sets in $\mathcal{A}(Q)$, we have

$$\Phi(K_1) + \Phi(K_2) \le \Phi(K_1 \cup K_2). \tag{6.2a}$$

(ii) Φ applied to cubes of decreasing size goes uniformly to zero, i.e.

$$\lim_{k \to \infty} \sup_{K \in \mathcal{D}_k(Q)} \Phi(K) = 0. \tag{6.2b}$$

Note that an immediate consequence of (6.2a) is that $\Phi(K_1) \le \Phi(K_2)$ when $K_1 \subset K_2$.

We shall prove a general partitioning result with respect to such functions. In practice, we shall be interested in applying this result in the case where

$$\Phi(K) = \Phi_f(K) = \|f - a_K(f)\|^2_{L_2(K)}, \tag{6.3}$$

for $f \in L_2(Q)$, and also in the case where

$$\Phi(K) = \mathrm{V}_K(f) = |f|_{\mathrm{BV}(K)}, \tag{6.4}$$

for $f \in \mathrm{BV}(Q)$. It is easy to see that properties (i) and (ii) are satisfied in both of these cases (see [Z] for a proof of (ii) for the second example using a slight modification of the BV norm).

We next make some preliminary remarks which will be useful for stating and proving our main result (Theorem 6.1) of this section. Recall that each dyadic cube I has four *children* J; these are the dyadic cubes $J \subset I$ with $|J| = |I|/4$ and one *parent*. Given a function Φ as above and a parameter $\epsilon > 0$, we define $\mathcal{T}_\epsilon$ to be the set of cubes $I \in \mathcal{D}(Q)$ such that $\Phi(I) > \epsilon$. The collection of cubes in $\mathcal{T}_\epsilon$ form a *tree* which means that whenever $I \in \mathcal{T}_\epsilon$ and $I \ne Q$, then its parent also belongs to $\mathcal{T}_\epsilon$. We also remark that $\mathcal{T}_\epsilon$ has finite cardinality, due to (6.2b).

In what follows, we shall assume that $\Phi(Q) \neq 0$ and that ϵ is small enough so that $\mathcal{T}_\epsilon$ is not empty. In the tree $\mathcal{T}_\epsilon$, we shall make the distinction between several types of cubes:
(i) The set of *final cubes* $\mathcal{F}_\epsilon$ consists of the elements $I \in \mathcal{T}_\epsilon$ with no child in $\mathcal{T}_\epsilon$.
(ii) The set $\mathcal{N}_\epsilon$ of *branching cubes* consists of the elements $I \in \mathcal{T}_\epsilon$ with more than one child in $\mathcal{T}_\epsilon$.
(iii) The set $\mathcal{C}_\epsilon$ of *chaining cubes* consists of the elements $I \in \mathcal{T}_\epsilon$ with exactly one child in $\mathcal{T}_\epsilon$.

¿From the fact that a branching cube always contains at least two final cubes, one easily derives

$$|\mathcal{N}_\epsilon| \leq |\mathcal{F}_\epsilon| - 1. \tag{6.5}$$

The set $\mathcal{C}_\epsilon$ can be partitioned into *maximal chains* C_q. That is, $\mathcal{C}_\epsilon = \cup_{q=1}^n C_q$, where each C_q is a sequence of $m = m(q)$ embedded cubes:

$$C_q = (I_0, \cdots, I_{m-1}), \tag{6.6}$$

where I_{k+1} is a child of I_k, and where I_0 (resp. I_{m-1}) is not a child (resp. parent) of a chaining cube.

The last cube I_{m-1} of a chain C_q, always contains exactly one cube I_m from $\mathcal{T}_\epsilon$ and this cube is either a final cube or branching cube. The cube I_m is uniquely associated to this chain. This shows that the number of chains $n = n(\epsilon)$ satisfies

$$n \leq |\mathcal{N}_\epsilon| + |\mathcal{F}_\epsilon| - 1 \leq 2|\mathcal{F}_\epsilon| - 1. \tag{6.7}$$

Our next theorem gives our main result of this section. It algorithmically constructs a partition $\mathcal{P}_\epsilon$ of Q into rings K with $\Phi(K) \leq \epsilon$. It also describes a second partition $\tilde{\mathcal{P}}_\epsilon$ whose sole purpose is to help count the number of rings in $\mathcal{P}_\epsilon$.

Theorem 6.1. *Let $\epsilon > 0$ be such that $\mathcal{T}_\epsilon \neq \emptyset$. Then, there exist a partition $\mathcal{P}_\epsilon$ of Q into disjoint rings such that*

$$\Phi(K) \leq \epsilon \ \text{ if } \ K \in \mathcal{P}_\epsilon, \tag{6.8}$$

and a set $\tilde{\mathcal{P}}_\epsilon = \tilde{\mathcal{P}}_\epsilon^1 \cup \tilde{\mathcal{P}}_\epsilon^2$ of pairwise disjoint sets K which are cubes (in the case $K \in \tilde{\mathcal{P}}_\epsilon^1$) or rings (in the case $K \in \tilde{\mathcal{P}}_\epsilon^2$) such that

$$\Phi(K) > \epsilon, \ \text{ if } \ K \in \tilde{\mathcal{P}}_\epsilon, \tag{6.9}$$

and

$$|\mathcal{P}_\epsilon| \leq 8|\tilde{\mathcal{P}}_\epsilon^1| + 3|\tilde{\mathcal{P}}_\epsilon^2| \leq 8|\tilde{\mathcal{P}}_\epsilon|. \tag{6.10}$$

Proof. We define $\mathcal{P}_\epsilon = \mathcal{P}_\epsilon^1 \cup \mathcal{P}_\epsilon^2 \cup \mathcal{P}_\epsilon^3$, with
(i) $\mathcal{P}_\epsilon^1$: all children J of the final cubes $I \in \mathcal{F}_\epsilon$.
(ii) $\mathcal{P}_\epsilon^2$: the children J of the branching cubes $I \in \mathcal{N}_\epsilon$, such that $J \notin \mathcal{T}_\epsilon$.

(iii) $\mathcal{P}_\epsilon^3$: rings and cubes obtained from the chains of $\mathcal{T}_\epsilon$ by an algorithm that we now describe.

If $C_q = (I_0, \dots, I_{m-1})$ is a maximal chain ($1 \le q \le n$), and I_m is as above, then we associate a *chain ring* $K_q = I_0 \setminus I_m$ to each chain C_q. Note that

$$\mathcal{P}_\epsilon^1 \cup \mathcal{P}_\epsilon^2 \cup \{K_q \ : \ q = 1, \dots, n\} \tag{6.11}$$

is a partition of the cube Q. We next partition each chain ring K_q, $q = 1, \dots, n$, according to

$$K_q = (I_{j_0} \setminus I_{j_1}) \cup (I_{j_1} \setminus I_{j_2}) \cup \dots \cup (I_{j_{p-1}} \setminus I_{j_p}), \tag{6.12}$$

where $0 = j_0 < j_1 < \dots < j_p = m$ ($p = p(q)$) are uniquely defined by the following recursion algorithm: assuming that j_k is defined, and that $j_k < m$, we choose j_{k+1} as follows:
(i) if $\Phi(I_{j_k} \setminus I_m) \le \epsilon$, then $j_{k+1} := m$, i.e. $p := k+1$ and the algorithm terminates.
(ii) if $\Phi(I_{j_k} \setminus I_{j_k+1}) > \epsilon$, then $j_{k+1} := j_k + 1$.
(iii) if neither (i) or (ii) apply, then j_{k+1} is chosen such that $\Phi(I_{j_k} \setminus I_{j_{k+1}}) \le \epsilon$ and $\Phi(I_{j_k} \setminus I_{j_{k+1}+1}) > \epsilon$. In other words, j_{k+1} is the largest $j > j_k$ such that $\Phi(I_{j_k} \setminus I_j) \le \epsilon$.

We can now define the set $\mathcal{P}_\epsilon^3$. For each chain ring K_q, $q = 1, \dots, n$, we include in $\mathcal{P}_\epsilon^3$: (i) all rings $I_{j_k} \setminus I_{j_{k+1}}$ such that $\Phi(I_{j_k} \setminus I_{j_{k+1}}) \le \epsilon$, (ii) the children of I_{j_k} $(J_{j_k}^1, J_{j_k}^2, J_{j_k}^3)$ that differ from $I_{j_{k+1}}$, for all k such that $\Phi(I_{j_k} \setminus I_{j_{k+1}}) > \epsilon$ (in this case $j_{k+1} = j_k + 1$, i.e. $I_{j_{k+1}}$ is a child of I_{j_k}). Note that the cubes $(J_{j_k}^1, J_{j_k}^2, J_{j_k}^3)$ are not in $\mathcal{T}_\epsilon$.

Because of (6.11), $\mathcal{P}_\epsilon$ is a partition which clearly satisfies (6.8).

Next, we define $\tilde{\mathcal{P}}_\epsilon := \tilde{\mathcal{P}}_\epsilon^1 \cup \tilde{\mathcal{P}}_\epsilon^2$, where
(i) $\tilde{\mathcal{P}}_\epsilon^1$ is the set of all of the final cubes of $\mathcal{T}_\epsilon$.
(ii) $\tilde{\mathcal{P}}_\epsilon^2$ is a set of rings constructed by an algorithm that we now describe.

For each chain ring K_q, $q = 1, \dots, n$, we recall its decomposition according to $K_q = (I_{j_0} \setminus I_{j_1}) \cup \dots \cup (I_{j_{p-1}} \setminus I_{j_p})$, and we construct a new decomposition

$$K_q = (I_{s_0} \setminus I_{s_1}) \cup (I_{s_1} \setminus I_{s_2}) \cup \dots \cup (I_{s_{r-1}} \setminus I_{s_r}), \tag{6.13}$$

where $0 = s_0 < s_1 < \dots < s_r = m$ ($r = r(q)$) constitute a subset of $(j_0, \dots, j_p)$ uniquely defined by the following recursion algorithm: assuming $s_k = j_l < m$ is defined,
(i) if $j_{l+1} = m$, we take $s_{k+1} := m$ and $r := k+1$ and terminate the algorithm.
(ii) if $j_{l+1} < m$, and if $\Phi(I_{j_l} \setminus I_{j_{l+1}}) \le \epsilon$, we take $s_{k+1} = j_{l+2}$. In the case that $j_{l+2} = m$, we terminate the algorithm.
(iii) if $j_{l+1} < m$, and if $\Phi(I_{j_l} \setminus I_{j_{l+1}}) > \epsilon$, we take $s_{k+1} = j_{l+1}$.

For each chain ring K_q, $q = 1, \dots, n$, we then include in $\tilde{\mathcal{P}}_\epsilon^2$ the rings $I_{s_k} \setminus I_{s_{k+1}}$, $k = 0, \cdots, r-2$, for which we have $\Phi(I_{s_k} \setminus I_{s_{k+1}}) > \epsilon$ (by the construction of $\mathcal{P}_\epsilon^3$) and we also include the last ring $I_{s_{r-1}} \setminus I_{s_r}$ only if it satisfies $\Phi(I_{s_{r-1}} \setminus I_{s_r}) > \epsilon$. This means that we do not include any ring from the chain ring K_q if $\Phi(K_q) \le \epsilon$.

We now claim that

$$|\mathcal{P}_\epsilon^3| \le 3|\tilde{\mathcal{P}}_\epsilon^2| + n \le 3|\tilde{\mathcal{P}}_\epsilon^2| + 2|\mathcal{F}_\epsilon| - 1 = 3|\tilde{\mathcal{P}}_\epsilon^2| + 2|\tilde{\mathcal{P}}_\epsilon^1| - 1, \tag{6.14}$$

Indeed, each ring $I_{s_k} \setminus I_{s_{k+1}}$ of $\tilde{\mathcal{P}}_\epsilon^2$ contains (as subsets) at most three rings of $\mathcal{P}_\epsilon$ and in each chain $\mathcal{C}_q$, $q = 1, \ldots, n$, at most one ring of $\mathcal{P}_\epsilon^3$ is not contained in some element of $\tilde{\mathcal{P}}_\epsilon^2$.

Finally, we prove the estimate (6.10). First, we clearly have

$$|\mathcal{P}_\epsilon^1| \le 4|\tilde{\mathcal{P}}_\epsilon^1| \tag{6.15}$$

and

$$|\mathcal{P}_\epsilon^2| \le 2|\mathcal{N}_\epsilon| \le 2(|\mathcal{F}_\epsilon| - 1) = 2(|\tilde{\mathcal{P}}_\epsilon^1| - 1). \tag{6.16}$$

Using these last two estimates with (6.14), we obtain

$$|\mathcal{P}_\epsilon| \le 3|\tilde{\mathcal{P}}_\epsilon^2| + 8|\tilde{\mathcal{P}}_\epsilon^1| - 3 \le 8|\tilde{\mathcal{P}}_\epsilon^1| + 3|\tilde{\mathcal{P}}_\epsilon^2| \le 8|\tilde{\mathcal{P}}_\epsilon|. \tag{6.17}$$

This proves (6.10) and completes the proof of the theorem. □

We shall now use Theorem 4.1 to prove a direct estimate for approximation by the elements of Σ_n^r. To do so, we fix $f \in L_2(Q)$ which is not constant and we take for Φ the L_2-error function defined by (6.3). For each $\epsilon > 0$, the algorithm described in the proof of Theorem 6.1 gives a partition $\mathcal{P}_\epsilon = \mathcal{P}_\epsilon(f)$ adapted to f. We then consider the piecewise constant approximation

$$A_\epsilon f := P_{\mathcal{P}_\epsilon} f, \tag{6.18}$$

where $P_{\mathcal{P}_\epsilon}$ is defined by (5.1).

Theorem 6.2. *If $f \in \mathrm{BV}(Q)$ is not constant and if $\epsilon > 0$, then the algorithm of Theorem 6.1, with Φ given by (6.3), produces a partition $\mathcal{P}_\epsilon$ that satisfies*

$$|\mathcal{P}_\epsilon| \le \frac{M}{\sqrt{\epsilon}} \mathrm{V}_Q(f), \quad M := 18\sqrt{3}, \tag{6.19}$$

and an approximation $A_\epsilon f$ that satisfies

$$\|f - A_\epsilon f\|_{L_2(Q)}^2 \le M\sqrt{\epsilon}\,\mathrm{V}_Q(f). \tag{6.20}$$

Consequently, one has the Jackson estimate

$$\sigma_N^r(f) \le M N^{-1/2} V_Q(f). \tag{6.21}$$

Proof. We consider the set $\tilde{\mathcal{P}}_\epsilon$ with the properties indicated in the statement of Theorem 6.1. Using the error estimate (4.10) with constant $6\sqrt{3}$ for rings and 1 for cubes (see Remark 4.2) together with (6.10) we obtain

$$\begin{aligned}
\sqrt{\epsilon}|\mathcal{P}_\epsilon| &\le \sqrt{\epsilon}[8|\tilde{\mathcal{P}}_\epsilon^1| + 3|\tilde{\mathcal{P}}_\epsilon^2|] \\
&\le 8 \sum_{K \in \tilde{\mathcal{P}}_\epsilon^1} [\Phi(K)]^{1/2} + 3 \sum_{K \in \tilde{\mathcal{P}}_\epsilon^2} [\Phi(K)]^{1/2} \\
&\le 8 \sum_{K \in \tilde{\mathcal{P}}_\epsilon^1} V_K(f) + 18\sqrt{3} \sum_{K \in \tilde{\mathcal{P}}_\epsilon^2} V_K(f) \\
&\le 18\sqrt{3} \sum_{K \in \tilde{\mathcal{P}}_\epsilon} \mathrm{V}_K(f) \le 18\sqrt{3} V_Q(f).
\end{aligned}$$

Dividing by $\sqrt{\epsilon}$, we obtain (6.19).

The approximation error (6.20), is then obtained from

$$\|f - A_\epsilon f\|_{L_2(Q)}^2 = \sum_{K \in \mathcal{P}_\epsilon} \Phi(K) \le |\mathcal{P}_\epsilon| \epsilon,$$

and (6.19). If we take $\sqrt{\epsilon} := \frac{M V_Q(f)}{N}$, then (6.19) and (6.20) imply (6.21). $\square$

We can also obtain (6.21) by using the function $\Phi(K) = \mathrm{V}_K(f)$. We now denote by $\mathcal{P}_\epsilon(f)$ the resulting partition and $A_\epsilon^* f := P_{\mathcal{P}_\epsilon(f)} f$ the resulting partition when the tolerance is chosen as ϵ.

Theorem 6.3. *If $f \in \mathrm{BV}(Q)$, $\mathrm{V}_Q(f) \neq 0$, $N > 0$ and $\epsilon := 8N^{-1}\,\mathrm{V}_Q(f)$, then the algorithm of Theorem 6.1, with Φ given by (6.4), produces a partition $\mathcal{P}_\epsilon$ that satisfies*

$$|\mathcal{P}_\epsilon| \le N \tag{6.22}$$

and an approximation $A_\epsilon^* f$ that satisfies

$$\|f - A_\epsilon^* f\|_{L_2(Q)} \le 48\sqrt{3} N^{-1/2}\,\mathrm{V}_Q(f). \tag{6.23}$$

Proof. The proof is similar to the previous theorem. We consider the sets $\mathcal{P}_\epsilon$ and $\tilde{\mathcal{P}}_\epsilon$ of Theorem 6.1. Using (6.9) and (6.10), we have

$$\epsilon|\mathcal{P}_\epsilon| \le 8\epsilon|\tilde{\mathcal{P}}_\epsilon| \le 8 \sum_{K \in \tilde{\mathcal{P}}_\epsilon} \Phi(K) = 8 \sum_{K \in \tilde{\mathcal{P}}_\epsilon} \mathrm{V}_K(f) \le 8\,\mathrm{V}_Q(f),$$

which gives (6.22).

We use the error estimate (4.10) and (6.22) to obtain

$$\begin{aligned}\|f - A_\epsilon^* f\|_{L_2(Q)}^2 &= \sum_{K \in \mathcal{P}_\epsilon} \|f - a_K(f)\|_{L_2(K)}^2 \le (6\sqrt{3})^2 \sum_{K \in \mathcal{P}_\epsilon} \mathrm{V}_K(f)^2 \\ &\le (6\sqrt{3})^2 |\mathcal{P}_\epsilon| \epsilon^2 \le (48\sqrt{3})^2 N^{-1}\,\mathrm{V}_Q(f)^2,\end{aligned}$$

which proves (6.23) $\square$

We close this section with the following simple remark about existence of best approximants from Σ_n^r.

Lemma 6.1. *For each $f \in L_2(Q)$ and $N > 0$ there exists $g \in \Sigma_N^r$ such that*

$$\|f - g\|_{L_2(Q)} = \sigma_N^r(f).$$

Proof. By the definition of $\sigma_N^r(f)$ (see (6.1)), there exist $g_1, g_2, \ldots$ such that $g_j \in \Sigma_N^r$ and

$$\|f - g_j\|_{L_2(Q)} \le \sigma_N^r(f) + j^{-1}.$$

Let $\mathcal{P}_j$ ($|\mathcal{P}_j| = N$) be the partition for g_j and furthermore let $K_m^j = I_m^j \setminus J_m^j \in \mathcal{P}_j$, $m = 1, 2, \dots, N$, be the rings of $\mathcal{P}_j$ with the indices selected such that $|K_1^j| \geq |K_2^j| \geq \cdots \geq |K_N^j|$.

By selecting a subsequence from (g_j), we can find an $\eta > 0$ and an $N_0 \leq N$ such that $|K_m^j| \geq \eta$, $1 \leq m \leq N_0$, $j = 1, 2, \dots$, and $|K_m^j| \to 0$, $j \to \infty$, $N_0 < m \leq N$. It follows that for each m, either the $|I_m^j| \geq \eta$ for all j or $|I_m^j| \to 0$, $j \to \infty$. A similar statement applies to the J_m^j. Since there are only a finite number of dyadic cubes with measure $\geq \eta$, by again extracting a subsequence, we can assume that for each m, either I_m^j does not change with j or $|I_m^j| \to 0$. A similar statement applies to the J_m^j.

It follows that there exist disjoint rings K_m^*, $m = 1, \dots, N$, such that $|K_i^j \setminus K_i^*| + |K_i^* \setminus K_i^j| \to 0$, $j \to \infty$ and $K_m^* = \emptyset$, $N_0 < m \leq N$. It is now easy to see that $\|g - g_j\|_{L_2(Q)} \to 0$, $j \to \infty$, for

$$g := \sum_{m=1}^{N_0} a_{K_m^*} \phi_{K_m^*}.$$

Therefore, g satisfies the conclusions of the theorem. $\square$

7. Minimization of the K-functional by piecewise constant approximation.

In this section, we shall use the Jackson and Bernstein estimates that we have proved for Σ_N^r to show that a near minimizer for the problem (1.2), i.e. the K-functional, can be taken from some space Σ_N^r. We shall also show how the algorithm of the previous section can be used to find a near minimizer.

We begin with the following simple result.

Theorem 7.1. *For each $f \in L_2(Q)$ and $N > 0$, and for each $\delta > 0$, there exists a function $h \in \Sigma_N^r$ such that*

$$\|f - h\|_{L_2(Q)} + N^{-1/2} V_Q(h) \leq (1 + \delta) 18\sqrt{3} K(f, N^{-1/2}). \tag{7.1}$$

Proof. If $K(f, N^{-1/2}) = 0$ then f is constant and (7.1) follows by taking $h = f$. If $K(f, N^{-1/2}) \neq 0$ and $\delta > 0$, let $g \in \mathrm{BV}(Q)$ satisfy

$$\|f - g\|_{L_2(Q)} + N^{-1/2} V_Q(g) \leq (1 + \delta) K(f, N^{-1/2}). \tag{7.2}$$

Then, according to (6.21) of Theorem 6.2, for each N, there exists a function $g_N \in \Sigma_N^r$ such that

$$\|g - g_N\|_{L_2} \leq 18\sqrt{3} N^{-1/2}\, \mathrm{V}_Q(g). \tag{7.3}$$

We take $h := g_N$ so that

$$\begin{aligned} \|f - h\|_{L_2(Q)} &\leq \|f - g\|_{L_2(Q)} + \|g - h\|_{L_2(Q)} \\ &\leq \|f - g\|_{L_2(Q)} + 18\sqrt{3} N^{-1/2}\, \mathrm{V}_Q(g) \\ &\leq 18\sqrt{3}(1 + \delta) K(f, N^{-1/2}). \end{aligned} \tag{7.4}$$

We can estimate the variation of h by Theorem 5.1. Since $h = P_{\mathcal{P}} g$ with $\mathcal{P}$ the partition for h, this gives

$$V_Q(h) \le 10 V_Q(g) \le 10(1+\delta) N^{1/2} K(f, N^{-1/2}). \tag{7.5}$$

Then, (7.4) together with (7.5) proves the theorem. $\square$

We say that an element $g \in \Sigma_M^r$ is a *near best* approximation to $f \in L_2(Q)$ (with parameters $a \ge 1$, and $N \le M$) if

$$\|f - g\|_{L_2(Q)} \le a \sigma_N^r(f). \tag{7.6}$$

We next show that any such near best approximation is a near minimizer for (1.2).

Corollary 7.1. *If $f \in L_2(Q)$ and $g \in \Sigma_N^r$ is a near best approximation with parameter a, then g satisfies*

$$\|f - g\|_{L_2(Q)} + N^{-1/2} V_Q(g) \le C_0 a K(f, N^{-1/2}), \tag{7.7}$$

with $C_0 \le 2016 + 18\sqrt{3}$.

Proof. Let $h \in \Sigma_N^r$ be the function of Theorem 7.1. Then,

$$\|f - g\|_{L_2(Q)} \le a \sigma_N^r(f) \le a \|f - h\|_{L_2(Q)}. \tag{7.8}$$

Also, since $g - h \in \Sigma_{4N}^c$, from the Bernstein estimate (3.9), we conclude that

$$\begin{aligned} N^{-1/2} V_Q(g) &\le N^{-1/2} V_Q(h) + N^{-1/2} V_Q(g-h) \le N^{-1/2} V_Q(h) + \frac{56}{\sqrt{3}} \|g - h\|_{L_2(Q)} \\ &\le N^{-1/2} V_Q(h) + \frac{56}{\sqrt{3}} (\|f - g\|_{L_2(Q)} + \|f - h\|_{L_2(Q)}) \\ &\le N^{-1/2} V_Q(h) + \frac{56}{\sqrt{3}} (1+a) \|f - h\|_{L_2(Q)}. \end{aligned}$$

Combining this with (7.8) gives that the left side of (7.7) does not exceed

$$(a + \frac{56}{\sqrt{3}}(1+a)) \|f - h\|_{L_2(Q)} + N^{-1/2} V_Q(h) \le (a + \frac{56}{\sqrt{3}}(1+a))(\|f - h\|_{L_2(Q)} + N^{-1/2} V_Q(h)).$$

We now use (7.2) to arrive at (7.7). $\square$

While Theorem 7.1 and Corollary 7.1 both provide near minimizers of (1.2) they are not of practical interest since they are not constructive. Yet, they show that a near minimizer for (1.2) can be taken from Σ_N^r when N is chosen so that $N^{-1/2}$ has the same order of magnitude as t.

We shall use the remainder of this section to prove that a near minimizer can also be obtained by applying the algorithm of the previous section to the function f. Recall that this algorithm is controlled by the parameter $\epsilon > 0$: by decreasing ϵ, we increase the number of rings in the partition $\mathcal{P}_\epsilon$ and we decrease the approximation error $\|f - \mathcal{A}_\epsilon f\|_{L_2(Q)}$. We thus have $\mathcal{A}_\epsilon f \in \Sigma_N^r$ with $N = N(\epsilon)$ increasing as ϵ goes to zero. In practice, we would like to directly control the number of rings. This leads to the following question: given $N > 0$, can we find $\epsilon(N)$ such that $|\mathcal{P}_\epsilon| = N$, or equivalently does the function $N(\epsilon)$ reach all possible values of $N \in \mathbb{N}$? Strictly speaking, the answer to this question is negative. However, we can circumvent this difficulty as we shall now describe.

For a given f, and a given $N \in \mathcal{N}$, we define

$$\epsilon(N) := \min\{\epsilon \geq 0 \; ; \; |\mathcal{P}_\epsilon| \leq N\}, \tag{7.9}$$

$$\mathcal{P}_N^* = \mathcal{P}_{\epsilon(N)}, \tag{7.10}$$

and

$$\tilde{\mathcal{A}}_N f = A_{\epsilon(N)} f = P_{\mathcal{P}_N^*} f. \tag{7.11}$$

If $\epsilon(N) > 0$, the minimum is attained in (7.9). Indeed, the construction of $\mathcal{T}_\epsilon$, $\mathcal{P}_\epsilon$ and $\mathcal{A}_\epsilon f$ described in the previous section ensures that, for any given $\epsilon > 0$, there exists $\tilde{\epsilon} > 0$ small enough so that $\mathcal{T}_{\epsilon+s} = \mathcal{T}_\epsilon$, $\mathcal{P}_{\epsilon+s} = \mathcal{P}_\epsilon$ and $\mathcal{A}_{\epsilon+s} f = \mathcal{A}_\epsilon f$, for all $0 \leq s < \tilde{\epsilon}$.

If $\epsilon(N) = 0$, then from Lemma 6.1, $f \in \Sigma_N^r$. We can therefore apply the algorithm with $\epsilon = 0$ since the tree $\mathcal{T}_0$ will be finite. With this choice, the algorithm gives $\mathcal{A}_0 f = f$ and therefore $\tilde{\mathcal{A}}_N f = f$ as well.

In order to prove that $\tilde{\mathcal{A}}_N f$ is a near minimizer for the K-functional, we first need two lemmas that will be used to compare the partition $\mathcal{P}_N$ produced by the algorithm and the partition that is associated to the element $g \in \Sigma_N$ which is a known minimizer.

Lemma 7.1. *If $\mathcal{P}$ is a finite set of pairwise disjoint rings and $\mathcal{P}'$ a partition of Q into a finite number of rings, then for each $K' \in \mathcal{P}'$, there are at most two sets $K \in \mathcal{P}$ such that $K \cap K' \neq \emptyset$ but K is not contained in K'.*

Proof. Let $K' = I' \setminus J'$ where $J' \subsetneq I'$ and J' may possibly be empty. If $K = I \setminus J$ is in $\mathcal{P}$ and $K \cap K' \neq \emptyset$, then $I \cap I' \neq \emptyset$. Hence either $I \subset I'$ or $I' \subset I$. We shall show there is at most one K of each of these types that intersects K' but is not contained in K'.

(i) Case 1: $I' \subset I$. Suppose that there were two sets $K_1 = I_1 \setminus J_1$ and $K_2 = I_2 \setminus J_2$ from $\mathcal{P}$ with $I' \subset I_1, I_2$. Then, obviously $I_1 \cap I_2 \neq \emptyset$ and hence without loss of generality $I' \subset I_1 \subset I_2$. For K_1 and K_2 to be disjoint (as they must be since both are in $\mathcal{P}$) we must have $I_1 \subset J_2$. But this means K_2 does not intersect K', which is a contradiction. Thus, we have shown there is only one set K of this type.
(ii) Case 2: $I \subset I'$. Suppose again that there were two sets $K_1 = I_1 \setminus J_1$ and $K_2 = I_2 \setminus J_2$ from $\mathcal{P}$ with $I' \supset I_1, I_2$. Then, $I_i \cap J'$, $i = 1, 2$, since otherwise $K_i \subset K'$. Hence, $J' \subset I_1, I_2$. Obviously, $I_1 \cap I_2 \neq \emptyset$ and hence without loss of generality $I_1 \subset I_2 \subset I'$. Since $K_1 \cap K_2 = \emptyset$, we have

$$J_1 \subset I_1 \subset J_2 \subset I_2 \subset I'$$

Since $J' \subset I_1 \subset J_2$, this is a contradiction since it implies that $K_2 \subset K'$. $\square$

Lemma 7.2. *If $\mathcal{P}$ is a finite set of pairwise disjoint rings and $\mathcal{P}'$ a partition of Q into a finite number of rings, and if $|\mathcal{P}'| \le N$ and $|\mathcal{P}| \ge 2N$, then the subset $\mathcal{P}^1$ of all $K \in \mathcal{P}$ contained in some $K' \in \mathcal{P}'$ satisfies $|\mathcal{P}^1| \ge N$.*

Proof. Let denote by $\mathcal{P}^2$ the set of all $K \in \mathcal{P}$ that are not contained in any $K' \in P'$, and by $\mathcal{P}^3$ the set of $K' \in \mathcal{P}'$ such that there exist $K \in \mathcal{P}_2$ having a non-empty intersection with K'.

By the previous lemma, each $K' \in \mathcal{P}^3$ is associated with at most two $K \in \mathcal{P}^2$ such that K and K' are not disjoint. On the other hand, each $K \in \mathcal{P}^2$ is associated to at least two $K' \in \mathcal{P}^3$ such that K and K' are not disjoint. We thus have necessarily

$$|\mathcal{P}^2| \le |\mathcal{P}^3| \le |\mathcal{P}'| \le N,$$

so that $|\mathcal{P}^1| = |\mathcal{P}| - |\mathcal{P}^2| \ge 2N - N = N$. $\square$

We are now ready to prove the main result of this section.

Theorem 7.2. *Let $f \in L_2(Q)$ and $N \ge 1$ be an integer and $M := 16N$. The function $\tilde{\mathcal{A}}_M f = \mathcal{A}_{\epsilon(M)} f$ is a near best approximation to f in the sense of (7.6) and satisfies*

$$\|f - \tilde{\mathcal{A}}_M f\|_{L_2(Q)} + N^{-1/2}\,\mathrm{V}_Q(\tilde{\mathcal{A}}_M f) \le C_0' K(f, N^{-1/2}), \tag{7.12}$$

with $C_0' = 8C_0$ and C_0 the constant of Corollary 7.1.

Proof. We consider first the case that $\epsilon := \epsilon(M) > 0$. Let g be a best approximation to f from Σ_N^r and $\mathcal{P}$ be the partition associated to g. Fix an arbitrary $0 < \eta < \epsilon$ and let $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}_\eta$ be the partition of Theorem 6.1. Then, using the fact that $\eta < \epsilon$ together with Theorem 6.1, we find $M \le |\mathcal{P}_\eta(f)| \le 8|\tilde{\mathcal{P}}|$. Hence $|\tilde{\mathcal{P}}| \ge 2N$ and we can apply Lemma 7.2 to find a set $\mathcal{P}^1 \subset \tilde{\mathcal{P}}$ with $|\mathcal{P}^1| \ge N$ and each element $K \in \mathcal{P}^1$ is contained in some ring of $\mathcal{P}$. It follows that

$$N\eta \le \sum_{K \in \mathcal{P}^1} \|f - a_K(f)\|_{L_2(K)}^2 \le \|f - g\|_{L_2(Q)}^2 = \sigma_N^r(f)^2.$$

Since $\eta < \epsilon$ is arbitrary, we have

$$N\epsilon \le \sigma_N^r(f)^2.$$

Therefore,

$$\|f - \tilde{\mathcal{A}}_M f\|_{L_2(Q)}^2 = \sum_{K \in \mathcal{P}_\epsilon} \Phi(K) \le M\epsilon \le 16\sigma_N^r(f)^2.$$

Thus $\tilde{\mathcal{A}}_M f$ is a near best approximation to f with parameter $a = 4$ and (7.12) follows from Corollary 7.1.

In the second case, where $\epsilon(M) = 0$, we have $\tilde{A}_M f = A_0 f = f$ and $f \in \Sigma_M^r$. The left side of (7.12) does not exceed $N^{-1/2}\,\mathrm{V}_Q(f)$. Let h be the function of Theorem 7.1. Since $f - h \in \Sigma_{2(N+M)}^c = \Sigma_{34N}^c$, we have from the Bernstein inequality (3.9)

$$\|f-h\|_{L_2(Q)} + N^{-1/2} V_Q(h) \ge \frac{\sqrt{3}}{28\sqrt{34}} N^{-1/2} V_Q(f-h) + N^{-1/2} V_Q(h) \ge \frac{\sqrt{3}}{28\sqrt{34}} N^{-1/2} V_Q(f).$$

Hence, the left side of (7.12) does not exceed

$$\frac{28\sqrt{34}}{\sqrt{3}}(\|f-h\|_{L_2(Q)}+N^{-1/2}V_Q(h))$$

and the proof is completed by invoking inequality (7.1). □

8. Direct estimates for Haar thresholding.

In this section, we fix a function f in BV and show that its Haar coefficients are in weak ℓ_1. That is, we shall show that when the Haar coefficients are put in decreasing order according to the absolute value of their size, then the n-th rearranged coefficient is in absolute value less than $C|f|_{\mathrm{BV}}/n$, with the C an absolute constant. We shall see that this also yields the Jackson estimate (1.7) for Σ_N^w.

In the next section, we shall then use this result to show that the extremal problems (1.1) and (1.2) have near minimizers which can be obtained by wavelet thresholding of the coefficients with respect to the Haar basis.

Associated to each dyadic cube $I=[2^{-j}k_1,2^{-j}(k_1+1))\times[2^{-j}k_2,2^{-j}(k_2+1))$, there are three Haar coefficients $c_{j,k}^e=\langle f,H_{j,k}^e\rangle$, $e\in V$, $k=(k_1,k_2)$ with V the nonzero vertices of the square $Q=[0,1]^2$ (see (1.10-11). In this section as well as in §9, we shall denote any of these by $c_I=c_I(f)$ and the corresponding Haar function by H_I: when we state a property about c_I, we mean any of these three coefficients and similarly for H_I.

We shall assume without loss of generality that f has mean value zero so that the coefficient of φ_Q is zero. We shall denote by $\gamma_n(f)$ the the n-th largest of the absolute values of the Haar coefficients c_I^e of H_I^e, $I\in\mathcal{D}(Q)$, $e\in V$.

We begin with the following well-known lemma.

Lemma 8.1. *If $f\in BV(Q)$ and $\epsilon>0$, then there exists a continuous function f_ϵ which is piecewise continuously differentiable on Q such that*

$$\|f-f_\epsilon\|_{L_2(Q)}<\epsilon \tag{8.1}$$

and

$$V_Q(f_\epsilon)\le V_Q(f). \tag{8.2}$$

Proof. This can be proved in many ways by mollification; for example using Steklov averages. We shall prove this by using piecewise bilinear interpolants. We recall (see (2.11))that

$$\mathrm{V}_Q(P_kf)\le \mathrm{V}_Q(f), \tag{8.3}$$

where P_k is the projector onto $\mathcal{S}_k$. Since $\|f-P_kf\|_{L_2(Q)}$ goes to zero as k tends to infinity, it is sufficient to prove the result assuming that f is in $\mathcal{S}_k$.

For such an f, and $0<\epsilon<2^{-k-1}$, we define a tensor product grid

$$\Gamma_\epsilon:=\Gamma_\epsilon^1\otimes\Gamma_\epsilon^1 \tag{8.4}$$

where the univariate grid Γ_ϵ^1 is defined by

$$\Gamma_\epsilon^1 := \{0,1\} \cup \{2^{-k}n+\epsilon\,;\ n=0,\dots,2^k-1\} \cup \{2^{-k}n-\epsilon\,;\ n=1,\dots,2^k\}. \tag{8.5}$$

The f is well defined at each point in Γ_ϵ. Let f_ϵ be the the function which is piecewise bilinear relative to Γ_ϵ and interpolates f at each grid point in Γ_ϵ. That is f_ϵ is the unique continuous function, which is piecewise bilinear (i.e. of the form $a+bx+cy+dxy$) on each rectangular patch defined by Γ_ϵ and equal to f on Γ_ϵ.

One easily checks that by construction,

$$\mathrm{V}_Q(f_\epsilon) \le \mathrm{V}_Q(f). \tag{8.6}$$

On the other hand, it is clear that f_ϵ tends to f in $L_2(Q)$ as ϵ goes to zero. $\square$

In view of Lemma 8.1, in going further, we can assume without loss of generality that f is continuous and piecewise continuously differentiable on Q. Then,

$$V_K(f) = \int_K [|f_{x_1}| + |f_{x_2}|], \tag{8.7}$$

for any ring K. Therefore, $\mathrm{V}(K) := \mathrm{V}_K(f)$ is set additive on rings, i.e. $\mathrm{V}(K_1 \cup K_2) = \mathrm{V}(K_1) + \mathrm{V}(K_2)$ for any two disjoint rings K_1 and K_2.

Theorem 8.1. *For each $f \in \mathrm{BV}(Q)$ and each $n \ge 1$, we have*

$$\gamma_n(f) \le C_1 \frac{\mathrm{V}_Q(f)}{n} \tag{8.8}$$

with $C_1 = 36C_1'$ and $C_1' := 216\sqrt{5} + 72\sqrt{3}$.

Proof. We can assume that f is continuous and piecewise continuously differentiable on Q. We can also assume that $\mathrm{V}_Q(f) = 1$ since the general case then follows by scaling. We shall show that there is a set $\Lambda_n \subset \mathcal{D}$ such that

(i) $|\Lambda_n| \le 6 \cdot 2^n$, $n = 1, 2, \dots$,

(ii) $|c_I| \le C_1' 2^{-n}$, $I \notin \Lambda_n$,

where in (ii), c_I is any of the three Haar coefficients associated to I. It is easy to see that this implies (8.8).

We shall use constructions of trees similar to that in §6. We shall also use the abbreviated notation $\mathrm{V}(S) := \mathrm{V}_S(f)$ for any set S in the algebra of rings. For each $m = 1, 2, \dots$, let $\mathcal{T}_m$ denote the collection of all cubes $I \in \mathcal{D}$ for which $\mathrm{V}(I) \ge 2^{-m}$. The cubes in $\mathcal{T}_m$ form a tree. Note also that the tree $\mathcal{T}_m$ is contained in the tree $\mathcal{T}_{m+1}$ and we can obtain $\mathcal{T}_{m+1}$ from $\mathcal{T}_m$ by growing $\mathcal{T}_m$.

We shall give each cube $I \in \mathcal{D}$ an index $m(I)$ as follows. We consider the four children of $J_i \subset I$, $i = 1, 2, 3, 4$. We can write $\mathrm{V}(J_i) = 2^{-m_i+\epsilon_i}$, where m_i is a nonnegative integer (or $m_i = \infty$) and $0 \le \epsilon_i < 1$. We define $m(I)$ as the second smallest of the four numbers m_i. Another way to describe $m(I)$ (when it is finite) is that it is the smallest integer m such that I has at least two of its children in $\mathcal{T}_m$. Note also that if I has index m then

$I \in \mathcal{T}_{m-1}$ and I has at least two children in $\mathcal{T}_m$. We have remarked in §6 that for any tree the number of branching cubes (i.e. cubes with at least two children in the tree) does not exceed the number of final cubes. Since the final leaves of $\mathcal{T}_m$ are disjoint and on each final cube I, $\mathrm{V}(I) \geq 2^{-m}$, it follows that there are at most 2^m cubes I in $\mathcal{D}$ with index m.

We shall also define a distance between two dyadic cubes $J \subset I$. This distance is the difference of the dyadic levels of J and I, i.e.

$$d(I,J) = \frac{1}{2}(\log_2 |I| - \log_2 |J|).$$

We fix $n > 0$ and define for all $0 \leq m \leq n$ the set $\mathcal{A}_m$ consisting of the cubes I in $\mathcal{T}_n$ which contain a cube J with index $m = m(J)$ which satisfies $d(I,J) \leq 2(n-m)$. We thus have

$$|\mathcal{A}_m| \leq [2(n-m)+1]2^m, \quad m = 0, 1, \dots, n. \tag{8.9}$$

Defining $\Lambda_n := \cup_{m=0}^n \mathcal{A}_m$, it follows that

$$|\Lambda_n| \leq \sum_{m=0}^{n} [2(n-m)+1]2^m \leq 6 \cdot 2^n - 1. \tag{8.10}$$

so that (i) is satisfied.

To prove (ii), let $I \in \mathcal{D}$ be a cube not in Λ_n. We consider two cases. The first case is when $I \notin \mathcal{T}_n$. In this case $\mathrm{V}(I) < 2^{-n}$. Let (as before) $a_I := a_I(f)$ be the average of f on I. By Remark 4.2, we have for any of the three coefficients c_I,

$$|c_I| \leq |\int_I (f(x) - a_I) H_I(x)\, dx| \leq \|f - a_I\|_{L_2(I)} \leq V(I) < 2^{-n}. \tag{8.11}$$

Hence, we have verified (ii) in this case.

Consider now the remaining case when $I \in \mathcal{T}_n$. We define a chain of cubes $I = I_0 \supset I_1 \supset \cdots \supset I_r$ as follows: given that I_j has been defined, we define I_{j+1} as the child of I_j in $\mathcal{T}_n$ on which f has largest variation. The chain terminates when I_r is a final leave in $\mathcal{T}_n$. Let $K_j := I_j \setminus I_{j+1}$, $j = 0, \dots, r-1$, and $K_r := I_r$. The three children J different from I_{j+1} all satisfy $\mathrm{V}(J) \leq 2^{-m(I_j)+1}$. It follows from the additivity of V that

$$\mathrm{V}(K_j) \leq 6 \cdot 2^{-m(I_j)}, \quad j = 0, \dots, r-1. \tag{8.12}$$

We can now estimate any of the three Haar coefficients c_I as follows. We define

$$g := \sum_{j=0}^{r} a_{K_j} \varphi_{K_j}, \tag{8.13}$$

where

$$a_{K_j} := \frac{1}{|K_j|} \int_{K_j} f(x)\, dx. \tag{8.14}$$

We let H_I denote the Haar functions associated to I and c_I. Then,

$$\begin{aligned}|c_I| &= \left|\int_{I_0} f(x)H_I(x)\,dx\right| \\ &\le |I_0|^{-1/2}\int_{I_0}|f(x)-g(x)|\,dx + \left|\int_{I_0} g(x)H_I(x)\,dx\right| \\ &=: \eta_1+\eta_2.\end{aligned}$$

We can estimate η_1 by using Theorem 4.1 and the Cauchy-Schwarz inequality. This gives

$$\begin{aligned}\eta_1 &\le |I_0|^{-1/2}\sum_{j=0}^{r}\|f-g\|_{L_1(K_j)} \le |I_0|^{-1/2}\sum_{j=0}^{r}\|f-g\|_{L_2(K_j)}|K_j|^{1/2} \\ &\le 6\sqrt{3}|I_0|^{-1/2}\sum_{j=0}^{r}\mathrm{V}(K_j)|K_j|^{1/2} \le 6\sqrt{3}\sum_{j=0}^{r}2^{-j}\,\mathrm{V}(K_j).\end{aligned}$$

We now show a similar estimate for η_2. Since g is a constant on each ring K_j we get

$$\begin{aligned}\eta_2 &\le |I_0|^{-1/2}\int_{I_1}|g(x)-a_{K_0}|\,dx = |I_0|^{-1/2}\sum_{j=1}^{r}\int_{K_j}|g(x)-a_{K_0}|\,dx \\ &= |I_0|^{-1/2}\sum_{j=1}^{r}|a_{K_j}-a_{K_0}||K_j| \le |I_0|^{-1/2}\sum_{j=1}^{r}|K_j|\sum_{\mu=1}^{j}|a_{K_\mu}-a_{K_{\mu-1}}|.\end{aligned}$$

We now change the order of summation to find

$$\eta_2 \le |I_0|^{-1/2}\sum_{\mu=1}^{r}|a_{K_\mu}-a_{K_{\mu-1}}|\sum_{j=\mu}^{r}|K_j| \le |I_0|^{-1/2}\sum_{\mu=1}^{r}|a_{K_\mu}-a_{K_{\mu-1}}||I_\mu|.$$

For each μ, the set $K := K_\mu \cup K_{\mu-1}$ is a ring and if a is the average of f over K, then

$$\begin{aligned}|a_{K_\mu}-a_{K_{\mu-1}}| &\le |a_{K_\mu}-a| + |a_{K_{\mu-1}}-a| \\ &\le \frac{1}{|K_\mu|}\int_{K_\mu}|f(x)-a|\,dx + \frac{1}{|K_{\mu-1}|}\int_{K_{\mu-1}}|f(x)-a|\,dx \\ &\le \frac{1}{|K_\mu|}\int_{K}|f(x)-a|\,dx \\ &\le |K|^{1/2}|K_\mu|^{-1}\|f-a\|_{L_2(K)} \le 6\sqrt{3}\sqrt{5}|K_\mu|^{-1/2}\,\mathrm{V}(K).\end{aligned}$$

Since $|I_0|^{-1/2}|K_\mu|^{-1/2}|I_\mu| \le \frac{2}{\sqrt{3}}2^{-\mu}$, we obtain

$$\eta_2 \le 12\sqrt{5}\sum_{\mu=1}^{r}\left(V(K_\mu)+V(K_{\mu-1})\right)2^{-\mu}. \tag{8.15}$$

This together with the estimate of η_1 shows that

$$|c_I| \le (18\sqrt{5}+6\sqrt{3})\sum_{j=0}^{r} 2^{-j}V(K_j) = \sum_{k=0}^{n} S_k, \tag{8.16}$$

where S_k consists of that portion of the sum on the right side of (8.16) corresponding to the terms for which $m(I_j)=k$. Then, as we have shown earlier, $V(K_j)\le 6\cdot 2^{-k}$ for each such j. Also, I_j is at a distance $> 2(n-k)$ from I because of the definition of $\mathcal{A}_k$ and Λ_n. Hence,

$$S_k \le (108\sqrt{5}+36\sqrt{3}) \sum_{\nu=2(n-k)+1}^{\infty} 2^{-\nu-k} = (108\sqrt{5}+36\sqrt{3})2^{-2n+k}. \tag{8.17}$$

We now return to (8.16) to find that

$$|c_I| \le (108\sqrt{5}+36\sqrt{3})\sum_{k=0}^{n} 2^{-2n+k} \le (216\sqrt{5}+72\sqrt{3})2^{-n}. \tag{8.18}$$

Thus, we have provided the desired estimate for these I as well. $\square$

Theorem 8.1 immediately yields a direct estimate for Haar thresholding. For this, we define two nonlinear operators associated to the Haar decomposition. Let f have mean value zero on Q and $f=\sum c_I^e H_I^e$. We define for $\epsilon>0$

$$\mathcal{H}_\epsilon f = \sum_{|c_I^e|>\epsilon} c_I^e H_I^e, \tag{8.19}$$

the thresholding of f at level ϵ, and for each positive integer N

$$\mathcal{G}_N f = \sum_{(I,e)\in E_N(f)} c_I^e H_I^e, \tag{8.20}$$

the best approximation of f from Σ_N^w: the set $E_N(f)$ contains the indices of the N largest Haar coefficients c_I^e of f. In the case of ties in the size of the coefficients we make an arbitrary assignment to the set $E_N(f)$ in order to remove the ambiguity.

Theorem 8.2. *If $f\in \mathrm{BV}$ has mean value zero on Q, we have*

$$\|f-\mathcal{H}_\epsilon f\|_{L_2(Q)} \le C_2[\epsilon \mathrm{V}_Q(f)]^{1/2}, \tag{8.21}$$

and

$$\inf_{g\in\Sigma_N^w} \|f-g\|_{L_2(Q)} = \|f-\mathcal{G}_N f\|_{L_2(Q)} \le C_3 N^{-1/2}\,\mathrm{V}_Q(f). \tag{8.22}$$

with $C_2 = 2\sqrt{C_1}$ *and* $C_3 = C_1$ *with* C_1 *the constant of Theorem 8.1.*

Proof. If $\epsilon \geq V_Q(f)$, then (8.11) and (8.12) follow trivially from the embedding theorem (Theorem 4.1 and Remark 4.2). We can therefore assume $V_Q(f) > \epsilon$ in going further. For each n, let $\gamma_n := \gamma_n(f)$ denote the n-th largest Haar coefficient of f in absolute value and for each $k = 0, 1, \ldots$, let $\Lambda_k := \{n : \gamma_n \leq 2^{-k}\epsilon\}$. We then have

$$\begin{aligned} \|f - \mathcal{H}_\epsilon f\|_{L_2(Q)}^2 &= \sum_{n \in \Lambda_0} \gamma_n^2 = \sum_{k \geq 0} \sum_{n \in \Lambda_k \setminus \Lambda_{k+1}} \gamma_n^2 \\ &\leq \epsilon^2 \sum_{k \geq 0} 2^{-2k} |\Lambda_k \setminus \Lambda_{k+1}|. \end{aligned} \tag{8.23}$$

For each $n \in \Lambda_k \setminus \Lambda_{k+1}$, we have $\gamma_n > 2^{-k-1}\epsilon$ and hence from Theorem 8.1, $|\Lambda_k \setminus \Lambda_{k+1}| \leq C_1 \mathrm{V}_Q(f) 2^{k+1}/\epsilon$. Using this in (8.23) we arrive at (8.21).

For (8.22), we have from Theorem 8.1,

$$\|f - \mathcal{G}_N f\|_{L_2(Q)}^2 = \sum_{n \geq N+1} \gamma_n^2 \leq C_1^2 \mathrm{V}_Q(f)^2 \sum_{n \geq N+1} n^{-2} \leq C_1^2 \mathrm{V}_Q(f)^2 N^{-1}. \quad \square$$

9. Minimization of the K and U-functionals by Haar thresholding.

We shall now show that Haar thresholding provides near minimizers for (1.1) and (1.2). For this, we shall thus prove a stability result concerning the nonlinear operators that we have introduced in the previous section.

Theorem 9.1. *The operators* $\mathcal{G}_N$ *and* $\mathcal{H}_\epsilon$ *satisfy for all* $\epsilon > 0$, $N > 0$ *and* $f \in \mathrm{BV}(Q)$,

$$\mathrm{V}_Q(\mathcal{G}_N f) \leq C_4 \mathrm{V}_Q(f), \tag{9.1}$$

and

$$\mathrm{V}_Q(\mathcal{H}_\epsilon f) \leq C_4 \mathrm{V}_Q(f), \tag{9.2}$$

with $C_4 = 10 + 28\sqrt{2}(18\sqrt{3} + C_3)$ *and* C_3 *the constant of Theorem 8.3.*

Proof. Clearly, it suffices to prove (9.1) since $\mathcal{H}_\epsilon f = \mathcal{G}_N f$ for some $N = N(\epsilon)$. Let g be a best approximation to f from Σ_N^r. We can write $g = P_{\mathcal{P}} f$ with $\mathcal{P}$ the partition associated to g. Recall that each element of Σ_N^r is in Σ_{2N}^c and also $\mathcal{G}_N f$ is in Σ_{4N}^c. Therefore, we have

$$\begin{aligned} \mathrm{V}_Q(\mathcal{G}_N f) &\leq \mathrm{V}_Q(g) + \mathrm{V}_Q(\mathcal{G}_N f - g) \\ &\leq 10 \mathrm{V}_Q(f) + \frac{28}{\sqrt{3}} (6N)^{1/2} \|\mathcal{G}_N f - g\|_{L_2(Q)} \\ &\leq 10 \mathrm{V}_Q(f) + 28\sqrt{2} N^{1/2} [\|f - g\|_{L_2(Q)} + \|f - \mathcal{G}_N f\|_{L_2(Q)}] \\ &\leq [10 + 28\sqrt{2}(18\sqrt{3} + C_3)] \mathrm{V}_Q(f), \end{aligned}$$

where we have used Theorem 5.1 to estimate $\mathrm{V}_Q(g)$ and the inverse estimate (3.9) for Σ_N^c as well as the direct estimates (6.21) and (8.22) in the estimate of $V_Q(\mathcal{G}_N f - g)$. $\square$

Remark 9.1 The stability of the Haar thresholding is a quite surprising result since the operation of discarding coefficients is in general not uniformly stable in BV (i.e. stable independently of the set of coefficients which is discarded). Also in the proof of this result, we have made use of our approximation results for Σ_N^r: a more direct proof of this stability is still to be found. Note that we also have used decompositions into rings to prove that the Haar coefficients of a BV function are in weak ℓ^1, leaving open the possibility of a more direct proof.

Theorem 9.2. *For each $N \geq 1$, and each $f \in L_2(Q)$, we have*

$$\|f - \mathcal{G}_N f\|_{L_2(Q)} + N^{-1/2} \operatorname{V}_Q(\mathcal{G}_N f) \leq C_5 K(f, N^{-1/2}), \tag{9.3}$$

with $C_5 = (112\frac{\sqrt{2}}{\sqrt{3}} + 1)C_3 + C_4$ with C_3 the constant of Theorem 8.3 and C_4 the constant of Theorem 9.1.

Proof. Let g be any function in $\mathrm{BV}(Q)$. Since $\mathcal{G}_N f$ is the best N term approximation to f, we have

$$\begin{aligned}
\|f - \mathcal{G}_N f\|_{L_2(Q)} &\leq \|f - \mathcal{G}_N g\|_{L_2(Q)} \\
&\leq \|f - g\|_{L_2(Q)} + \|g - \mathcal{G}_N g\|_{L_2(Q)} \\
&\leq \|f - g\|_{L_2(Q)} + C_3 N^{-1/2} \operatorname{V}_Q(g),
\end{aligned}$$

where the last inequality uses Theorem 8.3. The function $\mathcal{G}_N f - \mathcal{G}_N g$ is in Σ_{8N}^c. We can therefore use the Bernstein inequality (3.9) and Theorem 9.1 to obtain

$$\begin{aligned}
N^{-1/2} \operatorname{V}_Q(\mathcal{G}_N f) &\leq N^{-1/2} [\operatorname{V}_Q(\mathcal{G}_N f - \mathcal{G}_N g) + \operatorname{V}_Q(\mathcal{G}_N g)] \\
&\leq \frac{56\sqrt{2}}{\sqrt{3}} \|\mathcal{G}_N f - \mathcal{G}_N g\|_{L_2(Q)} + C_4 N^{-1/2} \operatorname{V}_Q(g) \\
&\leq \frac{112\sqrt{2}}{\sqrt{3}} \|f - \mathcal{G}_N g\|_{L_2(Q)} + C_4 N^{-1/2} \operatorname{V}_Q(g) \\
&\leq \frac{112\sqrt{2}}{\sqrt{3}} \|f - g\|_{L_2(Q)} + (\frac{112\sqrt{2}}{\sqrt{3}} C_3 + C_4) N^{-1/2} \operatorname{V}_Q(g).
\end{aligned}$$

Combining these two estimates, we obtain

$$\|f - \mathcal{G}_N f\|_{L_2(Q)} + N^{-1/2} \operatorname{V}_Q(\mathcal{G}_N f) \leq C_5 [\|f - g\|_{L_2(Q)} + N^{-1/2} \operatorname{V}_Q(g)]. \tag{9.4}$$

Taking an infimum over all $g \in \mathrm{BV}(Q)$ gives (9.3). □

Our next result concerns the minimization of the U-functional, i.e. problem (1.1). As in the case of the Besov space $B_1^1(L_1)$, a thresholding procedure, now in the Haar system, yields the approximate minimizer.

Theorem 9.3. *For each $\epsilon > 0$, and each $f \in L_2(Q)$, we have*

$$\|f - \mathcal{H}_\epsilon f\|_{L_2(Q)}^2 + \epsilon \operatorname{V}_Q(\mathcal{H}_\epsilon(f)) \leq C_6 U(f, \epsilon), \tag{9.5}$$

with $C_6 = C_4 + 112C_2^2 + 4C_1 + 2$ *and* C_1 *the constant of Theorem 8.2,* C_2 *the constant of Theorem 8.3 and* C_4 *the constant of Theorem 9.1.*

Proof. Let g be any function in $\mathrm{BV}(Q)$. We first remark that we have

$$\|f - \mathcal{H}_\epsilon f\|^2_{L_2(Q)} \le \|f - \mathcal{H}_{2\epsilon} g\|^2_{L_2(Q)}. \tag{9.6}$$

Indeed, if the coefficient $c_I(f - \mathcal{H}_\epsilon f) = \langle f - \mathcal{H}_\epsilon f, H_I\rangle$ is non zero, then necessarily $|c_I(f)| \le \epsilon$ and $c_I(f - \mathcal{H}_\epsilon f) = c_I(f)$. For this coefficient, we either have $|c_I(g)| \le 2\epsilon$, in which case

$$c_I(f - \mathcal{H}_\epsilon f) = c_I(f) = c_I(f - \mathcal{H}_{2\epsilon} g), \tag{9.7}$$

or $|c_I(g)| > 2\epsilon$, in which case

$$|c_I(f - \mathcal{H}_{2\epsilon} g)| = |c_I(f) - c_I(g)| > \epsilon \ge |c_I(f - \mathcal{H}_\epsilon f))|. \tag{9.8}$$

In all cases the coefficients of $f - \mathcal{H}_{2\epsilon} g$ dominate those of $f - \mathcal{H}_\epsilon f$, so that (9.6) holds.
We thus have

$$\begin{aligned} \|f - \mathcal{H}_\epsilon f\|^2_{L_2(Q)} &\le 2\|f - g\|^2_{L_2(Q)} + 2\|g - \mathcal{H}_{2\epsilon} g\|^2_{L_2(Q)} \\ &\le 2\|f - g\|^2_{L_2(Q)} + 4C_2^2 \epsilon \, \mathrm{V}_Q(g). \end{aligned} \tag{9.10}$$

where we have used (8.21) of Theorem 8.3.
We now estimate the variation of $\mathcal{H}_\epsilon f$ as follows: using Theorem 9.1, we obtain

$$\begin{aligned} \mathrm{V}_Q(\mathcal{H}_\epsilon f) &\le \mathrm{V}_Q(\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g) + \mathrm{V}_Q(\mathcal{H}_\epsilon g) \\ &\le \mathrm{V}_Q(\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g) + C_4 \, \mathrm{V}_Q(g). \end{aligned} \tag{9.11}$$

We are left with estimating the variation of $\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g$. For this, we write

$$\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g = \mathcal{H}_\epsilon[\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g] + \tilde{\mathcal{H}}_\epsilon[\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g], \tag{9.12}$$

where for a function h, $\tilde{\mathcal{H}}_\epsilon h := h - \mathcal{H}_\epsilon h$ is the part of the Haar expansion of h corresponding to the coefficients which satisfy $|c_I(h)| \le \epsilon$. Using the inverse estimate (3.4) of Remark 3.1 and then (9.10), we have

$$\begin{aligned} \mathrm{V}_Q(\mathcal{H}_\epsilon[\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g]) &\le 8\,\epsilon^{-1}\|\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g\|^2_{L_2(Q)} \le 16\,\epsilon^{-1}[\|\mathcal{H}_\epsilon f - f\|^2_{L_2(Q)} + \|f - \mathcal{H}_\epsilon g\|^2_{L_2(Q)}] \\ &\le 16\,\epsilon^{-1}[2\|f - g\|^2_{L_2(Q)} + 4C_2^2 \epsilon V_Q(g) + 2\|f - g\|^2_{L_2(Q)} + 2\|g - \mathcal{H}_\epsilon g\|^2_{L_2(Q)}] \\ &\le 16\,\epsilon^{-1}[4\|f - g\|^2_{L_2(Q)} + 6[C_2]^2 \epsilon \, \mathrm{V}_Q(g)], \end{aligned}$$

where the last inequality again uses (8.21) of Theorem 8.3.
It remains to estimate the variation of $\tilde{\mathcal{H}}_\epsilon[\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g]$. For this, we remark that if $0 < |c_I(\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g)| \le \epsilon$, then necessarily $|c_I(g)| > \epsilon$. In other words, if we denote by $N_g(\epsilon)$ the number of coefficients of g above the threshold ϵ, we see that $\tilde{\mathcal{H}}_\epsilon[\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g]$ has

at most $N_g(\epsilon)$ non-zero coefficients. We can then use the inverse estimate (3.1) of Theorem 3.1 to obtain

$$\mathrm{V}_Q(\tilde{\mathcal{H}}_\epsilon[\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g]) \le 8[N_g(\epsilon)]^{1/2}\|\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g\|_{L_2(Q)}. \tag{9.13}$$

¿From Theorem 8.2, we have the estimate

$$N_g(\epsilon) \le C_1\epsilon^{-1}\,\mathrm{V}_Q(g). \tag{9.14}$$

Combined with (9.13), this gives

$$\begin{aligned}
\epsilon\,\mathrm{V}_Q(\tilde{\mathcal{H}}_\epsilon[\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g]) &\le 8\epsilon[C_1\epsilon^{-1}\,\mathrm{V}_Q(g)]^{1/2}\|\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g\|_{L_2(Q)}\\
&\le 4\epsilon[C_1\,\mathrm{V}_Q(g) + \epsilon^{-1}\|\mathcal{H}_\epsilon f - \mathcal{H}_\epsilon g\|^2_{L_2(Q)}]\\
&\le 4C_1\epsilon\,\mathrm{V}_Q(g) + 8\|f - \mathcal{H}_\epsilon f\|^2_{L_2(Q)} + 8\|f - H_\epsilon g\|^2_{L_2(Q)}\\
&\le 4C_1\epsilon\,\mathrm{V}_Q(g) + 16\|f-g\|^2_{L_2(Q)} + 32C_2^2\epsilon\,\mathrm{V}_Q(g) + 16\|f-g\|^2_{L_2(Q)} + 16\|g - H_\epsilon g\|^2_{L_2(Q)}\\
&\le 4C_1\epsilon\,\mathrm{V}_Q(g) + 32\|f-g\|^2_{L_2(Q)} + 32C_2^2\epsilon\,\mathrm{V}_Q(g) + 16C_2^2\epsilon\,\mathrm{V}_Q(g)\\
&\le 32\|f-g\|^2_{L_2(Q)} + (4C_1 + 48C_2^2)\epsilon\,\mathrm{V}_Q(g),
\end{aligned}$$

where we have used (9.10) and (8.21) of Theorem 8.3.

Combining all our estimates we obtain

$$\|f - \mathcal{H}_\epsilon f\|^2_{L_2(Q)} + \epsilon\,\mathrm{V}_Q(\mathcal{H}_\epsilon(f)) \le 98\|f-g\|^2_{L_2(Q)} + (C_4 + 148C_2^2 + 4C_1)\epsilon\,\mathrm{V}_Q(g), \tag{9.15}$$

which gives (9.5) by taking the infimum over all $g \in \mathrm{BV}$. □

10. Interpolation spaces between L_2 and BV.

As a by product of our results, we shall obtain several results concerning interpolation spaces between $L_2(Q)$ and $\mathrm{BV}(Q)$. For each $0 < \alpha < 1$ and $0 < q \le \infty$, let $\mathcal{A}^\alpha_q(L_2(Q))$ denote the set of functions $f \in L_2(Q)$ such that

$$|f|_{\mathcal{A}^\alpha_q(L_2(Q))} := \|(N^\alpha\sigma_N(f))\|_{\ell^*_q(\mathbb{Z}_+)} < \infty \tag{10.1}$$

where $\sigma_N(f) = \inf_{g\in\Sigma_N}\|f-g\|_{L_2(Q)}$, Σ_N is any of the three families Σ^w_N, Σ^r_N or Σ^c_N, and with ℓ^*_q the ℓ_q norm with respect to Haar measure:

$$\|(a_n)\|_{\ell^*_q} := \begin{cases} (\sum_{n=1}^\infty |a_n|^q\frac{1}{n})^{1/q}, & 0 \le q < \infty\\ \sup_{n\ge1}|a_n|, & q = \infty.\end{cases}$$

Then, it follows from the Jackson and Bernstein estimates, which were proved throughout the paper for these different families of approximation spaces, that

$$\mathcal{A}^\alpha_q(L_2(Q)) = (L_2(Q), \mathrm{BV}(Q))_{\alpha,q}, \quad 0 < \alpha < 1,\ 0 < q \le \infty \tag{10.2}$$

with equivalent norms, where $(L_2(Q), \mathrm{BV}(Q))_{\alpha,q}$ are the real interpolation spaces for the pair $(L_2(Q), \mathrm{BV}(Q))$ (see [DL, Chapter 5] for the definition of interpolation spaces and for the general mechanism relating these with approximation spaces, through Jackson and Bernstein estimates).

Moreover, it was shown in [DP] that

$$\mathcal{A}^\alpha_q(L_2(Q)) = (L_2(Q), B^1_1(L_1(Q)))_{\alpha,q}, \tag{10.3}$$

in the case of the particular family Σ^c_N.

We thus obtain the following corollary to our results, where the second statement exploits the known interpolation results for Besov spaces (see [T] or [DP1]).

Corollary 10.1. *We have*

$$(L_2(Q), \mathrm{BV}(Q))_{\alpha,q} = (L_2(Q), B_1^1(L_1(Q)))_{\alpha,q}, \quad 0 < \alpha < 1,\ 0 < q \leq \infty \tag{10.4}$$

and in particular

$$(L_2(Q), \mathrm{BV}(Q))_{\alpha,q} = B_\alpha^q(L_q(Q)), \quad 0 < \alpha < 1,\ 1/q = 1/2 + \alpha/2. \tag{10.5}$$

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