MARCH/APRIL 2012 SOLUTIONS

1) SUM OF PERMUTATIONS

By permuting the digits 4, 5, 6, 7, and 8, we can form 120 five-digit numbers. What is the sum of all such numbers?

SOLUTION

Let S be the sum of all such numbers. Each four digit number can be express as

$$a_4a_3a_2a_1a_0 = a_4 \cdot 10{,}000 + a_3 \cdot 1{,}000 + a_2 \cdot 100 + a_1 \cdot 10 + a_0$$

There are exactly 24 numbers such that the first digit a_4 is 4. Similarly, there are exactly 24 numbers such that a_4 is 5, 6, 7, or 8. Also there are 24 numbers in which a_3 is 4, 5, 6, 7, or 8, and analogous statements can be made for digits a_2, a_1, and a_0. Therefore the sum of all these numbers is

$$\begin{aligned}&24\cdot(4+5+6+7+8)\cdot 10{,}000 + 24\cdot(4+5+6+7+8)\cdot 1{,}000 +\\ &24\cdot(4+5+6+7+8)\cdot 100 + 24\cdot(4+5+6+7+8)\cdot 10 +\\ &24\cdot(4+5+6+7+8)\cdot 1 = (24)\cdot(30)\cdot(11{,}111) = \mathbf{7{,}999{,}920}\end{aligned}$$

2) CONTINUOUS FUNCTIONS

Find a family of real-valued continuous functions defined on $(-\infty,\infty)$ such that

$$f(x+y) = f(x) + f(y) + f(x)f(y).$$

SOLUTION

Set $y=0$ in the above equation. Then $f(x) = f(x) + f(0) + f(x)f(0)$, or $0 = f(0)(1+f(x))$ which implies $f(x) = -1$, or $0 = f(0)$. Although $f(x) = -1$ is a solution, it is not a family of functions. Restricting the search for solutions within the class of differentiable functions, we note that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(h)+f(x)f(h)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}(1+f(x))$$

Since $f(0)=0$, we have $f'(x)=f'(0)(1+f(x))$. This is a separable ordinary differential equation of first order which has solution $\ln(1+f(x))=Ax+C$ where C is an arbitrary constant, and $A=f'(0)$. Using $f(0)=0$ again yields $C=0$. Letting $a=e^A$ the function can be expressed as $f(x)=a^x-1$,

Another way to solve this problem is to let $g(x)=f(x)+1$. Substituting into the given expression yields $g(x+y)=g(x)g(y)$ which holds for an exponential function of the form $g(x)=a^x$ where $a > 0$, and $a\neq 1$. Therefore $f(x)=a^x-1$.

3) SOLVE THE EQUATION

Find all real solutions of the following:

$$\frac{\sqrt{x}}{\sqrt[3]{x}}-\frac{\sqrt[3]{x}}{\sqrt[4]{x}}=2\cdot$$

SOLUTION

Observe that the equation can be expressed as $x^{1/6}-x^{1/12`}=2$. Let $u=x^{1/12`}$. Then the given equation is $u^2-u=2$, and solving for u yields solutions -1 and 2. Now $u=-1$ corresponds to $x=1$ which does not satisfy the original equation. However, $u=2$ which corresponds to $x=4096$ does satisfy the initial equation, and is the only solution.

4) FINDING DISTANCE WITH LIMITED DATA

Two motor boats on opposite shores a river start moving toward each other but at different speeds. (Neglect other factors, such as acceleration, turn-around time and current.) When they pass each other the first time, they are 700 yards from one shoreline. They continue to the opposite shore, then turn around and start moving toward each other again. When they pass the second time they are 300 yards from the other shoreline. Their speeds although different remain constant. How wide is the river?

SOLUTION TO 4)

This problem can be solved using algebra, but an interesting way to solve this problem using only basic arithmetic follows: When the boats meet the first time, the combined distance they've traveled must be equal to the width of the river. When they meet the second time, the combined distance they've traveled must be equal to three times the width of the river. So when the boats meet the second time, each one separately has traveled three times as far as when they met the first time. One boat has traveled 700 yards from one shoreline when they meet the first time, so it has traveled 2100 yards when they meet the second time. So 2100 must be 300 yards more than the width of the river since that same boat had continued to the opposite shore to met the other boat 300 yards from the other shoreline. The river is 2100 – 300 = **1800** yards wide.

CORRECT SOLUTIONS

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