

JANUARY/FEBRUARY SOLUTIONS

PROBLEM

Let

$$b(x) = (x - a)^2, \quad 0 \leq x \leq 1, \quad 0 \leq a \leq 1.$$

Denote the mean value of the function $b(x)$ on the interval from s to t by the formula

$$M(s, t) = \frac{1}{t - s} \int_s^t b(x) \, dx, \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 1.$$

Prove that

$$M(s, t) \geq \frac{b(t)}{4} \text{ for all } 0 \leq s \leq 1, \quad 0 \leq t \leq 1.$$

SOLUTION

After integration, we get

$$\begin{aligned} M(s, t) &= \frac{(t - a)^3 - (s - a)^3}{3(t - s)} \\ &= \frac{[(t - a) - (s - a)][(t - a)^2 + (t - a)(s - a) + (s - a)^2]}{3(t - s)} \\ &= \frac{(t - a)^2 + (t - a)(s - a) + (s - a)^2}{3} \end{aligned}$$

Completing the square, we have

$$\begin{aligned} M(s, t) &= \frac{1}{3} \left(\frac{(t - a)^2}{4} + (t - a)(s - a) + (s - a)^2 + \frac{3(t - a)^2}{4} \right) \\ &= \frac{1}{3} \left(\left(\frac{t - a}{2} \right) + (s - a) \right)^2 + \frac{(t - a)^2}{4} \geq \frac{(t - a)^2}{4}. \end{aligned}$$

CORRECT SOLUTIONS

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