Admission-to-Candidacy Exam

Foundations of Computational Mathematics

August 10, 2016

Name: ....

Show your work! Write the solution of different problems on separate pages. Write your initials and the number of the problem on the top right corner of each page.

Read carefully and sign the following statement:

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code. As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam. I understand that, should it be determined that I used any unauthorized assistance or otherwise violate the University's Honor Code that I will receive a failing grade for this exam and be referred to the academic Dean and the Office of Academic Integrity for additional disciplinary action.

Signature:

1. A modification of the Newton's method for approximating the root ξ of the equation $f(x) = 0$ finds the next approximation x_{n+1} as the zero of the linear interpolation polynomial to f at the points x_n and $x_n^\star$, where $x_n^\star := x_n + |f(x_n)|$. Analyze the rate of convergence of this method and find conditions on f and the initial approximations x_0 and x_1 that guarantee the convergence.

2. Given the values at the points $x-2h,\ x-h,\ x,\ x+h,\ x+2h$ of the function g, derive **two** formulas approximating the second derivative $g''(x)$ described by:

(a) has maximal degree of approximation assuming that $g(x)$ is sufficiently smooth;

(b) is based on least squares fitting by a second degree polynomial.

Comment on the optimal choice of h in each case assuming that the error of calculating g is ε.

3. Find the algebraic polynomials P_k of degrees $k = 0, 1, 2, 3$ that realize the best L_∞-approximation in the interval $[-1, 1]$ to the function $F(x)$ defined as $F(x) = x^2$ for $x \in [-1, 0]$ and $F(x) = x^2 - x$ for $x \in [0, 1]$.

4. Find the simple quadrature rule of highest degree of precision for estimating $\int_{-1}^{1} f(x)dx$ in terms of the values of f at $-1/2, 0$, and $1/2$. Give a complete convergence analysis for the corresponding composite quadrature rule and comment on the effect of the calculation error assuming that the error of calculating f is ε.

5. Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Determine SVD of A.

(b) Determine QR factorization of A.

(c) What is the orthogonal projector P onto range(A), and what is the image under P of the vector $(1, 2, 3)^*$?

6. Estimate the condition numbers of the problem of finding a unit vector $x \in \mathbb{C}^n$ such that the product Fx has the maximal possible ℓ_2-norm for a given matrix $F \in \mathbb{C}^{m\times n}$. What is the condition number of the problem of finding the value of the maximal ℓ_2-norm of the product of F with a unit vector?

7. Either propose a backward stable method or prove that no such method exist for the problem of finding $x \in \mathbb{C}^m$ that satisfies $A^2x = b$ for a given non-singular matrix $A \in \mathbb{C}^{m\times m}$ and a vector $b \in \mathbb{C}^m$.

8. A real symmetric $m \times m$ matrix A has eigenvalues $\lambda_1 \geq 8$ and $\lambda_2 \in (2, 3)$ while all the other eigenvalues are much smaller: $|\lambda_j| \leq \frac{1}{8}$ for $j = 3, 4, ..., m$. Describe an iterative algorithm for finding λ_2 and the corresponding eigenvector v_2. Give an estimate how much the approximations of λ_2 and v_2 improve after each iteration.