Team Round – University of South Carolina Math Contest, 2025

1. This is a team round. You have one hour to solve these problems as a team, and you should submit one set of answers for your team as a whole. Working together is, of course, encouraged.

2. Submit one answer per question, on the attached sheet. If you change your answer, make sure your earlier answer is clearly crossed out.

3. Drawings are not necessarily drawn to scale.

4. All answers should be in the form of a number – simplified as much as possible.

   Use your best judgment – for example, 1.75, $\frac{7}{4}$, $\left(\frac{31}{18}\right)^7$, $\frac{\sqrt{2}}{2}$, and $\frac{1}{\sqrt{2}}$ will all be considered to be in simplified form. $\sin(\pi/4)$, $\frac{\sqrt{8}}{4}$, and $5+\frac{9}{5}$ will not be.

5. There are ten questions, all independent from each other. They are difficult. Do not be discouraged if you don't get them right away.

6. Some of the solutions involve some trial and error. When cleverness fails, persistence may work.

7. Your room will have a whiteboard or chalkboard, which you are encouraged to use. Markers or chalk, erasers, and blank paper will be provided by the proctors. (You should bring your own pencil or pen.) Please let your proctor know if your markers don't work or if you need additional markers, chalk, or paper.

8. No calculators, cell phones, books, notes, or other tools are allowed.

9. Please let the exam proctor know if you have any questions about the exam, or if you require additional materials.

GOOD LUCK!

Answer Sheet for Team Round

Your School/Team Name: ______________________________

1. ______________________

2. ______________________

3. ______________________

4. ______________________

5. ______________________

6. ______________________

7. ______________________

8. ______________________

9. ______________________

10. ______________________

Answers for Team Round

1. 54
2. 151/406
3. 19584
4. 3
5. 49500
6. 2525
7. 104
8. 6
9. 89101
10. 2033

1. How many positive integers $n \leq 2025$ are perfect squares, perfect cubes, or perfect fourth powers?

   Answer. 54.

   Since $2025 = 45^2$, there are 45 perfect squares.

   Since $12^3 < 2025 < 13^3$, there are 12 perfect cubes.

   Every perfect fourth power is also a perfect square, so we don't need to count them again.

   Finally, $1^6 = 1$, $2^6 = 64$, and $3^6 = 729$ are both squares and cubes, so we shouldn't double count them. So the answer is $45 + 12 - 3 = 54$.

2. The picture shows the game *Spelling Bee* from the game show *The Price is Right*.

   The contestant has chosen three cards at random from a total of thirty – of which eleven read 'C', eleven read 'A', six read 'R', and two read 'CAR'. The contestant wins a car if the letters C, A, and R all appear on any combination of the cards he chose.

   What is the probability the contestant has won the car?

   Answer. 151/406.

   The contestant wins it they choose one of the two 'CAR' cards, or if they choose a 'C', an 'A', and an 'R'. Note that these possibilities are mutually exclusive.

   Considering the contestant's three choices in order, there are $30 \cdot 29 \cdot 28$ total possibilities. Of these,

   - $11 \cdot 11 \cdot 6 \cdot 6$ include the letters 'C', 'A', and 'R';
   - $28 \cdot 27 \cdot 26$ *don't* include either of the 'CAR' cards (including those possibilities counted above).

   Therefore, the total probability is

$$\frac{30 \cdot 29 \cdot 28 - 28 \cdot 27 \cdot 26 + 11 \cdot 11 \cdot 6 \cdot 6}{30 \cdot 29 \cdot 28}.$$

   If you were hoping the arithmetic would be less messy, sorry to disappoint. That simplifies to 151/406.

3. Amy is 16 years old today, and she is excited to have the opportunity to take the South Carolina High School Math Contest when her age and the year are both squares!

   If she takes the exam on the same day every year, and she lives forever, then in how many years from now will she get the next such opportunity?

   Answer. 19584.

   Let n be Amy's age; then we are asking, for what n are n and $n + 2009$ both squares?

   Write $n = m^2$ and $n + 2009 = k^2$; then we have $k^2 - m^2 = (k - m)(k + m) = 2009$. We have the prime factorization $2009 = 7 \times 7 \times 41$, and $k - m$ and $k + m$ must be factors. Since $k - m$ is smaller, we have $k - m \in \{1, 7, 41\}$.

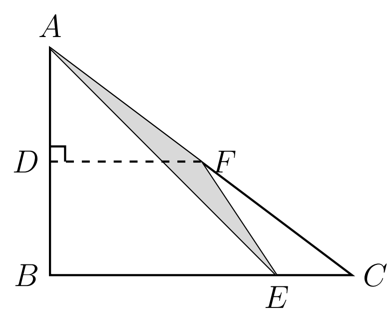
   If $k - m = 1$, then $k + m = 2009$, so that $m = 1004$ and Amy will be 1010025.

   If $k - m = 41$, then $k + m = 49$, so that $m = 4$. This corresponds to today.

   If $k - m = 7$, then $k + m = 287$, so that $m = 140$. This corresponds to the year 21609, in which Amy will be 19600 years old. She will have lots of opportunity to practice!

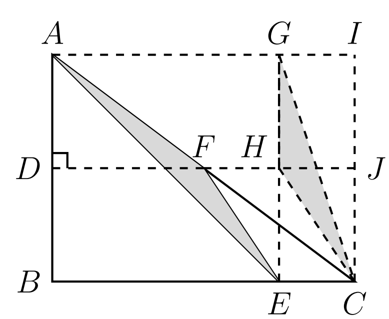
   So the answer is $19600 - 16 = 19584$.

4. In the right triangle ΔABC, DF is parallel to BC. If $\overline{AD} = 3$ and $\overline{CE} = 2$, then what is the area of the triangle ΔAEF?



   Answer. 3.

   Denote by S_Δ the area of the triangle Δ. As the picture below shows, we have $S_{\Delta AEC} = S_{\Delta GEC}$ and $S_{\Delta FEC} = S_{\Delta HEC}$. Thus $S_{\Delta AEF} = S_{\Delta AEC} - S_{\Delta FEC} = S_{\Delta GEC} - S_{\Delta HEC} = S_{\Delta GHC}$. By assumption, $S_{\Delta GHC} = \frac{1}{2} \times 2 \times 3 = 3$.



5. What is the sum of all three-digit numbers which read the same forwards and backwards?

   Don't count numbers with leading zeroes, e.g. 070*, as being three-digit numbers.*

   Answer. 49500.

   The sum is the total of all integers $100a + 10b + a = 101a + 10b$, where $1 \le a \le 9$ and $0 \le b \le 9$. We have

$$\sum_{a=1}^{9}\sum_{b=0}^{9}(101a + 10b) = 10 \cdot 101 \sum_{a=1}^{9} a + 9 \cdot 10 \sum_{b=0}^{9} b = 49500.$$

   Alternatively, and equivalently, notice that there are 90 integers being counted, and that their average is 550 (think: $5 \cdot 100 + 4.5 \cdot 10 + 5$), and $90 \cdot 550 = 49500$.

6. If $x + \frac{1}{x} = 5$, what is $x^5 + \frac{1}{x^5}$?

   Answer. 2525.

   We have

$$\begin{aligned} x + \frac{1}{x} &= 5 \\ x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} &= 125 \\ x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5} &= 3125. \end{aligned}$$

   Hence, $x^3 + \frac{1}{x^3} = 125 - 3 \cdot 5 = 110$, and so

$$x^5 + \frac{1}{x^5} = 3125 - 5 \cdot 110 - 10 \cdot 5 = 2525.$$

7. Simplify the expression $\sqrt{106\sqrt{104 \times 100 + 4} + 4}$.

   Answer. 104.

   For any integer n, we have $(n + 4) \times n + 4 = (n + 2)^2$.

   Therefore, the expression equals

$$\sqrt{106 \cdot 102 + 4} = \sqrt{(104)^2} = 104.$$

8. Find the closest integer to the sum

$$\sum_{n=1}^{2025} \frac{n^2}{2^n}.$$

   Answer. 6.

   The sum is equal to *exactly* 6. To show this (in brief), set S to be the sum and consider $2S - S$. That will reduce the sum to the form $\sum \frac{n}{2^n}$; then, do this trick again.

Alternatively, since an approximate answer will suffice, write

$$S = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \frac{36}{64} + \frac{49}{128} + \frac{64}{256} + \cdots$$

We compute that

$$\frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \frac{36}{64} + \frac{49}{128} + \frac{64}{256} = 5 + \frac{77}{128},$$

between 5.5 and 6. Since the last listed summand is $\frac{1}{4}$, and each summand that follows is less than 2/3 the previous one, the tail of the sum is less than

$$\frac{1}{4}\left(\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right) = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

Therefore, $S < 6.5$, and so the nearest integer is 6.

9. How many integers $n \geq 2$ satisfy

$$0 \leq \log_{2025}(n) - \log_n(2025) \leq \frac{5}{6}?$$

Answer. 89101.

If $x = \log_{2025}(n)$, the equation is equivalent to $0 \leq x - \frac{1}{x} \leq \frac{5}{6}$.

The function $x - \frac{1}{x}$ is increasing in x, and we have $x - \frac{1}{x} = 0$ when $x = 1$, and $x - \frac{1}{x} = \frac{5}{6}$ when $x = \frac{3}{2}$. Therefore, the equation is equivalent to $1 \leq x \leq \frac{3}{2}$, or $2025 \leq n \leq 2025^{3/2}$.

We have $2025^{3/2} = 2025 \cdot 45 = 91125$, so the answer is

$$91125 - 2025 + 1 = 89101.$$

10. Let

$$n = 2026 \times 2027 \times \cdots \times 4050.$$

What is the largest positive integer m for which $m! \mid n$?

Answer. 2033**.** This is *not* easy, and I don't know of any elegant systematic proof.

Probably the best way to proceed is to pick individual primes p, and figure out: For what values of m is the highest power of p dividing $m!$ less than or equal to the highest power of p dividing n?

It turns out that the smallest primes cause the most trouble, with $\frac{n}{2034!} = (\text{integer}) + \frac{5}{18}$. It is a not *too* terrible computation to show that $m = 2033$ is okay, regarding powers of 2 and 3 – whereas both 2 and 3 fail with $m = 2034$. (Note that these are the primes dividing the denominator of $\frac{5}{18}$.) So, although this falls short of a proof, this explains roughly why the answer is 2033.