High School Math Contest

University of South Carolina

January 31st, 2015

Problem 1. The figure below depicts a rectangle divided into two perfect squares and a smaller rectangle. If the dimensions of this smallest rectangle are proportional to those of the largest rectangle, and the squares each have side length 1, what is the length of a long side of the largest rectangle?

(a) $2\sqrt{3} - 1$ (b) $1 + \sqrt{2}$ (c) $\frac{1+\sqrt{5}}{2}$ (d) $\frac{\sqrt{5}-1}{2}$ (e) $8(\sqrt{3} - \sqrt{2})$

Answer: (b)

Solution: Let x be the width of the smaller rectangle. Then $\frac{1}{x} = 2 + x$, giving $x^2 + 2x - 1 = 0$. The quadratic formula gives $x = -1 + \sqrt{2}$ as the positive solution. The length of the largest rectangle is then $2 + x = 1 + \sqrt{2}$. □

Problem 2. It takes three lumberjacks three minutes to saw three logs into three pieces each. How many minutes does it take six lumberjacks to saw six logs into six pieces each? Assume each cut takes the same amount of time, with one lumberjack assigned to each log.

(a) 3 (b) 6 (c) $7\frac{1}{2}$ (d) 12 (e) 15

Answer: (c)

Solution #1. To cut three logs into three pieces each requires six cuts (two for each log). If three lumberjacks work for three minutes each, then they work for a total of nine minutes, and so we see that it requires 1.5 minutes for each cut.

To cut six logs into six pieces each requires 30 cuts (five for each log) and so 45 minutes total. Since six lumberjacks are working the total time required is $\frac{45}{6} = 7.5$ minutes. □

Solution #2. Since the number of lumberjacks is the same as the number of logs, the amount of time required is proportional to the number of cuts required for each log. To cut into six pieces requires $\frac{5}{2}$ times as many cuts as to cut into three pieces, so the time required is $\frac{5}{2}$ as much, or 7.5 minutes. □

Problem 3. What is the number of points (x, y) at which the parabola $y = x^2$ intersects the graph of the function $y = 1/(1 + x^2)$?

(a) 0   (b) 1   (c) 2   (d) 3   (e) 4

Answer: (c)

Solution: Setting the two equations equal gives $x^4 + x^2 - 1 = 0$ and $y^2 + y - 1 = 0$. When the quadratic formula is applied to the second equation, the non-negative solution is $(-1 + \sqrt{5})/2$. The negative solution is useless since $y = x^2$ gives only non-negative y for real x. The two square roots of $(-1 + \sqrt{5})/2$ give the two points of intersection. $\square$

Problem 4. Consider the set $A = \{a_1, a_2, a_3, a_4\}$. If the set of all possible sums of any three different elements from A is the set $B = \{-1, 3, 5, 8\}$, then what is the set A?

(a) $\{-1, 2, 3, 5\}$ (b) $\{-3, -1, 0, 2\}$ (c) $\{-3, 1, 2, 5\}$ (d) $\{-3, 0, 2, 6\}$ (e) $\{-1, 0, 2, 4\}$

Answer: (d)

Solution #1. There are four 3-element subsets of A, and if they are listed, each of the elements $a_i (i = 1, 2, 3, 4)$ appears three times. Since B has 4 elements, no two sums are the same. So we have $3(a_1 + a_2 + a_3 + a_4) = -1 + 3 + 5 + 8 = 15$, i.e. $a_1 + a_2 + a_3 + a_4 = 5$.

Therefore, the four elements of A are $5 - (-1) = 6$, $5 - 3 = 2$, $5 - 5 = 0$, and $5 - 8 = -3$. So the answer is (d). $\square$

Solution #2. No three distinct elements of (a), (c) or (e) can give -1, and no three distinct elements of (b) can give 8. This leaves $\{-3, 0, 2, 6\}$, which is easily seen to work; in particular, only 5 requires the use of three nonzero elements. $\square$

Problem 5. Four cards are laid out in front of you. You know for sure that on one side of each card is a single number, and on the other side of each card is a single geometric shape. The same number or the same geometric shape might be found on more than one of these four cards.

You see (on the top side) respectively a 2, a 5, a triangle, and a square. Your friend says: "Every card with a square has a 4 on the other side." Your task is to determine whether your friend is correct by choosing some of the cards to be flipped over. The chosen cards will only be flipped after you have made your choice(s). What is the fewest number of cards you can choose if you want to know for sure whether or not your friend is correct?

(a) 0   (b) 1   (c) 2   (d) 3   (e) 4

Answer: (d)

Solution: Clearly you must flip over the square, to see if there is a 4 on the other side. You must also flip over the 2 and the 5: if either has a square on the other side, it contradicts your friend's statement. There is no need to flip over the triangle: the other side will have some number, and it doesn't matter whether it's a 4 or not. $\square$

Problem 6. If in the formula $y = Ax/(B + Cx)$ we have that x is positive and increasing, while A, B and C are positive constants, then what happens to y as x increases?

   (a) y increases

   (b) y decreases

   (c) y remains constant

   (d) y increases, then decreases

   (e) y decreases, then increases

Answer: (a)

Solution: Divide numerator and denominator by x, so that
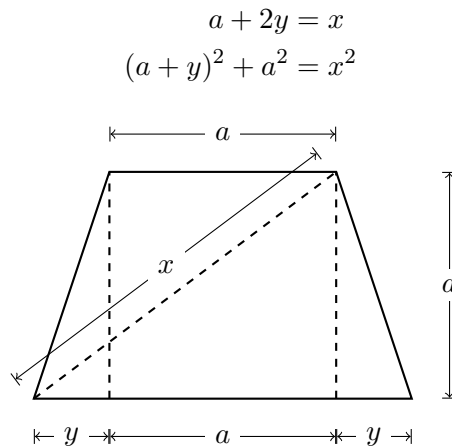
$$y = \frac{Ax}{B + Cx} = \frac{A}{\frac{B}{x} + C}$$

As x increases, the denominator on the right decreases while the numerator stays constant, so y increases. $\square$

Problem 7. If the larger base of an isosceles trapezoid equals a diagonal and the smaller base equals an altitude, what is the ratio of the smaller base to the larger base?

(a) 2/5   (b) 3/5   (c) 2/3   (d) 3/4   (e) 4/5

Answer: (b)

Solution: From the diagram below, it follows that:

$$a + 2y = x$$
$$(a + y)^2 + a^2 = x^2$$

Substituting x from the first equation in the second gives $y^2 + 2ay + 2a^2 = a^2 + 4y^2 + 4ay$, which reduces to $3y^2 + 2ay - a^2 = 0$. Solving gives $y = a/3$. The larger base can then be expressed in terms of the smaller as $a + 2y = \frac{5}{3}a$. $\square$

Problem 8. What is the maximum value of the following function?

$$f(x) = \frac{\sin^3 x \cos x}{\tan^2 x + 1}$$

(a) 1/8   (b) 1/4   (c) 1/3   (d) 1/2   (e) 1

Answer: (a)

Solution: Using the trigonometric identities $\tan^2 x + 1 = \sec^2 x$ and $1/\sec x = \cos x$, we get $f(x) = \sin^3 x \cos^3 x$. Using the identity $\sin x \cos x = \frac{1}{2}\sin 2x$, we get $f(x) = \frac{1}{8}\sin^3 2x$. Since the sine function varies between -1 and 1, the maximum value is $1/8$. $\quad\square$

Problem 9. A fair die is rolled 6 times. Let p be the probability that each of the six faces on the die appears exactly once among the six rolls. Which of the following is correct?

(a) $p \le 0.02$ (b) $0.02 < p \le 0.04$ (c) $0.04 < p \le 0.06$ (d) $0.06 < p \le 0.08$ (e) $p > 0.08$

Answer: (a)

Solution: On the second roll, the outcome has a 5/6 probability of being different from that in the first roll. The probabilities of the outcomes of rolls number 3, 4, 5 and 6 being different from the previous rolls are, respectively, 4/6, 3/6, 2/6 and 1/6. Canceling where possible gives

$$\frac{5}{6} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6} = \frac{5}{324}.$$

Since $5 < 2 \cdot 3.24$, the answer must be (a). $\quad\square$

Problem 10. Starting with an equilateral triangle, you inscribe a circle in the triangle, and then inscribe an equilateral triangle inside the circle. You then repeat this process four more times, each time inscribing a circle and then an equilateral triangle in the smallest triangle constructed up to that point, so that you end up drawing five triangles in addition to the one you started with.

    What is the ratio of of the area of the largest triangle to the area of the smallest triangles?

(a) 32   (b) 243   (c) 1024   (d) 59049   (e) 1048576

Answer: (c)

Solution: Each time you draw a new triangle, it has its vertices at the midpoints of the three sides of the old triangle. The new triangle is similar to the old triangle, and indeed this process divides the old triangle into four smaller triangles which are congruent and which each have area 1/4 of the old triangle. The answer is therefore $4^5 = 1024$. $\quad\square$

Problem 11. For $x > 0$, how many solutions does the equation $\log_{10}(x+\pi) = \log_{10} x + \log_{10} \pi$ have?

(a) 0   (b) 1   (c) 2   (d) more than 2 but finitely many   (e) infinitely many

Answer: (b)

Solution: Raising 10 to both sides and using the law $n^{b+c} = n^b \cdot n^c$ gives $x + \pi = \pi x$ which is readily solved to give the unique solution $x = \pi/(\pi - 1)$. $\square$

Problem 12. What is the range of the following function?

$$f(x) = \frac{\sqrt{x^2 + 1}}{x - 1}$$

(a) $\left(-\infty, -1\right) \cup \left[-\frac{\sqrt{2}}{2}, +\infty\right)$

(b) $\left(-\infty, -1\right] \cup \left[\frac{\sqrt{2}}{2}, +\infty\right)$

(c) $\left(-\infty, -1\right] \cup \left(1, +\infty\right)$

(d) $\left(-\infty, -\frac{\sqrt{2}}{2}\right] \cup \left(1, +\infty\right)$

(e) $\left(-\infty, \frac{\sqrt{2}}{2}\right] \cup \left(1, +\infty\right)$

Answer: (d)

Solution: Let

$$y = f(x+1) = \frac{\sqrt{(x+1)^2 + 1}}{x} = \frac{\sqrt{x^2 + 2x + 2}}{x}.$$

If $x > 0$, we have

$$y = \sqrt{1 + \frac{2}{x} + \frac{2}{x^2}} > 1.$$

This can be made close to 1 by taking x large, and arbitrarily large by taking x small.

If $x < 0$, we have

$$y = -\sqrt{1 + \frac{2}{x} + \frac{2}{x^2}} = -\sqrt{2\left(\frac{1}{x} + \frac{1}{2}\right)^2 + \frac{1}{2}} \leq -\frac{\sqrt{2}}{2}.$$

Now y can be any number less than or equal to $-\sqrt{2}/2$ because the expression $\frac{1}{x} + \frac{1}{2}$ can be any number less than $\frac{1}{2}$.

Now $f(x)$ and $f(x+1)$ have the same range, so the answer is (d). $\square$

Problem 13. What is the units digit of 2^{2015}?

(a) 0   (b) 2   (c) 4   (d) 6   (e) 8

Answer: (e)

Solution: The units digit of $2^1, 2^2, 2^3,$ and 2^4 are, respectively, $2, 4, 8,$ and 6. These four digits repeat in the same order all through the positive integers with increasing powers of 2, so that when the exponent is divisible by 4, the units digit is 6. The last year divisible by 4 was 2012, and so 2^{2015} is three steps into the next cycle, and so ends in 8. □

Problem 14. Suppose that you answer every question on this test randomly. What is the probability that you will get every question wrong?

The answers below are not necessarily exact; choose the number which is closest to the exact probability.

(a) 1/30   (b) 0.0128   (c) 0.00124   (d) 0.0000321   (e) 0.00000000000719

Answer: (c)

Solution: Each question has five answers, so the probability of answering each incorrectly is 0.8. Therefore, the correct answer is 0.8^{30}.
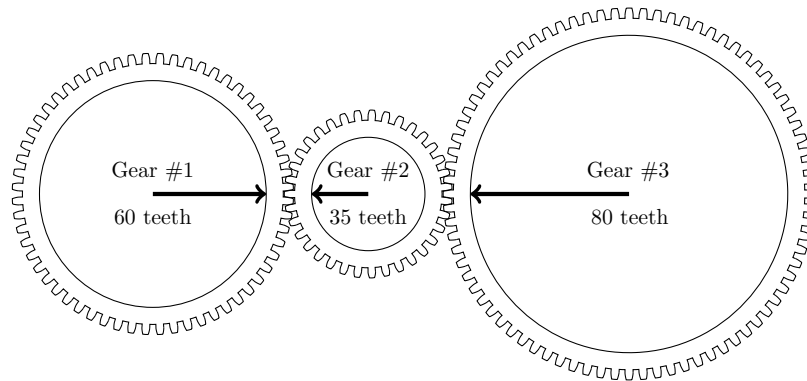
There are several ways you might estimate 0.8^{30}. Since the answers are all very different from one another, you don't have to be too precise. One way is to observe that $0.8^3 = 0.512$, and so the answer is $(0.512)^{10}$ which is about $(1/2)^{10}$. Since 2^{10} is about 1000, the answer is roughly 1/1000.

Alternatively, you might compute powers of 0.8, only keeping a couple of digits of accuracy after each step, and observing that after about 10 steps the answer is close to 0.1. Therefore, after 30 steps the answer is close to 0.001.

A third possibility is to observe that $\sqrt{0.1} = \frac{\sqrt{10}}{10}$ is a little bigger than 0.3.

Since $0.8^5 = .32768$, you see that 0.8^{10} will be close to 0.1. Then proceed as before. □

Problem 15. How many rotations of Gear #1 are required before all three gears return to the position shown, with the arrows lined up again and pointing in the same directions as before?

(a) 28   (b) 70   (c) 175   (d) 1680   (e) 168000

Answer: (a)

Solution: The least common multiple of 60, 35, and 80 is 1680. Dividing this by 60 gives 28. □

Problem 16. If $x^2 + xy + y^2 = 84$ and $x - \sqrt{xy} + y = 6$, then what is xy?

(a) 16   (b) 25   (c) 36   (d) 49   (e) 64

Answer: (a)

Solution: Rewrite the second equation as $x + y = 6 + \sqrt{xy}$. Squaring then gives $x^2 + y^2 + 2xy = 36 + xy + 12\sqrt{xy}$. Subtracting this equation from $x^2 + xy + y^2 = 84$ and simplifying gives $48 = 12\sqrt{xy}$, thus $xy = 16$. □

Problem 17. A bug is flying on a three-dimensional grid and wants to fly from $(0,0,0)$ to $(2,2,2)$. It flies a distance of 1 unit at each step, parallel to one of the coordinate axes. How many paths can the bug choose which take only six steps?

(a) 6   (b) 24   (c) 78   (d) 90   (e) 114

Answer: (d)

Solution: Say that the cube is oriented so that you go two spaces up, two to the right, and two forward. Write U for up, R for right, and F for forward. Then the answer is equal to the number of ways to rearrange the letters of $UURRFF$.

There are $\frac{6!}{2 \cdot 2 \cdot 2} = \frac{720}{8} = 90$ ways to do this: 6! ways to rearrange letters, but the U's, the R's, and the F's can each be swapped (or not), so we must divide by 8. □

Problem 18. You and your partner went to a dinner party in which there were four other couples.

After the dinner was over, you asked everyone except yourself: "How many people did you shake hands with tonight?" To your surprise, no two people gave the same number, so that someone did not shake any hands, someone else shook only one person's hand, a third person only shook two people's hands, and so on.

Assume nobody shook hands with their own partner or with themselves. How many people did your partner shake hands with that evening at the party?

(a) 0   (b) 2   (c) 3   (d) 4   (e) 6

Answer: (d)

Solution: The maximum possible number of people whose hands anyone shook is 8, and the only way 9 people can give different answers if if all integers from 0 through 8 are included.

Let A be the one who shook the hands of 8 people. Since everyone shook hands with A, except for A's partner and A him- or herself, then A's partner has to be the one who shook 0 people's hands.

Similar reasoning shows that the partner of the one who shook 7 people's hands, shook only 1 person's hand (specifically, A's hand). The partner of the one who shook 6 (resp. 5) people's hands, shook 2 (resp. 3) people's hands.

All these people are distinct from yourself, and also from your partner. This leaves 4 for the answer given by your partner.

Incidentally, with this information, it is possible to deduce that you also shook hands with 4 people—the ones who shook 8, 7, 6 and 5 people's hands. $\square$

Problem 19. The n-th string number, string(n), is formed by writing the numbers 1 to n after each other in order. For instance, string(1) = 1, string(2) = 12, string(7) = 1234567, and string(12) = 123456789101112. What is the remainder when string(2015) is divided by 6?

(a) 1   (b) 2   (c) 3   (d) 4   (e) 5

Answer: (c)

Solution: First of all, note that string(2015) is odd because it ends in a 5. So we must test what the remainder is after division by 3.

For any integer n, write sum(n) for the sum of the digits of n. Then, the divisibility test for 3 tells us that the remainder after dividing n by 3 is the same as the remainder after dividing sum(n) by 3.

So, the remainder after dividing string(2015) by 3 is the same as the sums of the remainders after dividing n by 3, summed over all n between $n = 1$ and $n = 2015$. So, the problem reduces to finding the remainder after

$$1 + 2 + \cdots + 2015 = \frac{2015 \cdot 2016}{2}$$

is divided by 3, and 3 | 2016. So the answer is (c). $\square$

Problem 20. How many integer triples (x, y, z) satisfy the following equation?

$$x^2 + y^2 + z^2 = 2xyz$$

(a) 0   (b) 1   (c) 2   (d) 25   (e) infinitely many

Answer: (b)

Solution: Since the left-hand side is even, either all of x, y, and z, or exactly one of x, y and z are even. If only one term is even, then the right side is divisible by 4 and the left side leaves a remainder of 2 after dividing by 4. So this can't be.

Therefore, assume that $x = 2x_1, y = 2y_1, z = 2z_1$ for integers z_1, z_2, and z_3. Plugging these equations in, we obtain $4x_1^2 + 4y_1^2 + 4z_1^2 = 16x_1y_1z_1$, or $x_1^2 + y_1^2 + z_1^2 = 4x_1y_1z_1$. This is nearly the same as our original equation! By an argument identical to the one above, all of x_1, y_1 and z_1 must again be even.

The pattern keeps repeating, and so we conclude that x, y, and z are all evenly divisible by an arbitrary power of 2. Therefore they are all zero. $\square$

Problem 21. If $f(x) = a + bx$, what are the real values of a and b such that

$$f(f(f(1))) = 29,$$
$$f(f(f(0))) = 2?$$

(a) $a = 2/13, b = 3$ (b) $a = 1, b = 3$ (c) $a = 3, b = 2/13$ (d) $a = 3, b = 1$ (e) there are none
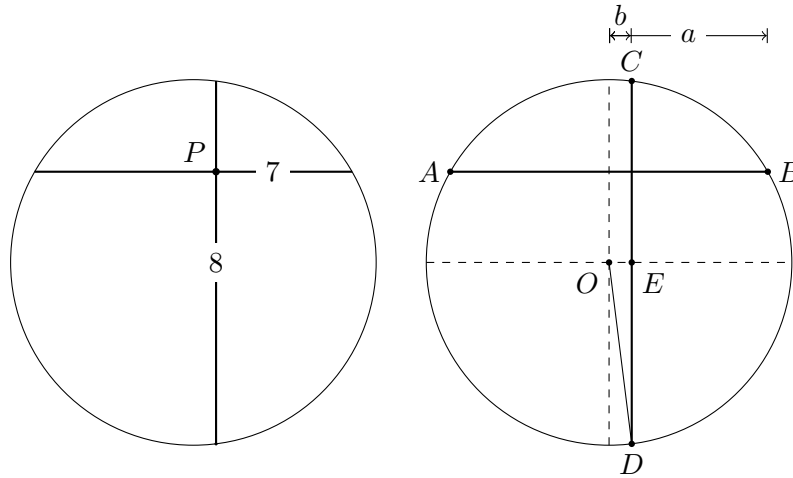
Answer: (a)

Solution: $f(0) = a, f(f(0)) = a + ab, f(f(f(0))) = a + ab + ab^2 = 2, f(1) = a + b, f(f(1)) = a + ab + b^2, f(f(f(1))) = a + ab + ab^2 + b^3 = 29$, thus $2 + b^3 = 29$, $b = 3$, and from $a + ab + ab^2 = 2$, we get $a = 2/13$. $\square$

Problem 22. Two perpendicular chords of a circle intersect at point P. One chord is 7 units long, divided by P into segments of length 3 and 4, while the other chord is divided into segments of length 2 and 6. What is the diameter of the circle?

(a) $\sqrt{56}$ (b) $\sqrt{61}$ (c) $\sqrt{65}$ (d) $\sqrt{75}$ (e) $\sqrt{89}$

Answer: (c)

Solution: Label points A, B, C, D, E and the center of the circle, O, in the diagram below (right), draw diameters parallel to both chords, and assign distances a, b as depicted.
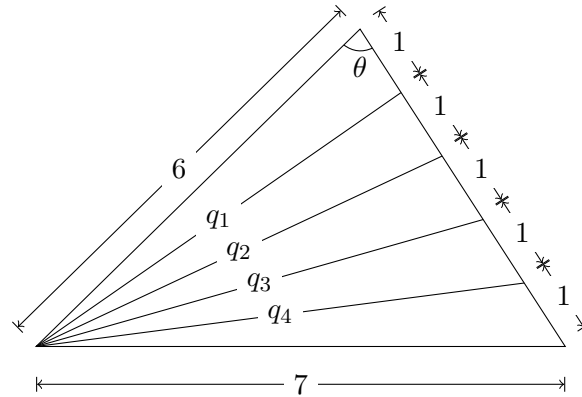

Since $\overline{CD}$ is the longer chord of length 8, the distance DE must equal 4. Since $\overline{AB}$ is the shorter chord of length 7, it must be that $a = 3$ and $a + b = 3.5$. This gives $b = 0.5$. By the Pythagorean Theorem, it must be that $r = OD = \sqrt{65}/2$, and the answer is (c). $\qquad\square$

Problem 23. In triangle $\triangle ABC$, $AB = 7$, $BC = 5$, and $AC = 6$. Locate points P_1, P_2, P_3 and P_4 on $\overline{BC}$ so that the side is divided into 5 equal segments, each of length 1. Let $q_k = AP_k$ for $k \in \{1, 2, 3, 4\}$. What is $q_1^2 + q_2^2 + q_3^2 + q_4^2$?

(a) 142　　　　　　(b) 150　　　　　　(c) 155　　　　　　(d) 160　　　　　　(e) 168

Answer: (b)

Solution: We are going to use the law of cosines on five of the triangles below:

We have,

$$
\begin{aligned}
5^2 + 6^2 - 2 \cdot 5 \cdot 6 \cdot \cos\theta &= 7^2 && \text{largest triangle} \\
4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cdot \cos\theta &= q_4^2 && \text{triangle with sides 4,6 and } q_4 \\
3^2 + 6^2 - 2 \cdot 3 \cdot 6 \cdot \cos\theta &= q_3^2 && \text{triangle with sides 3,6 and } q_3 \\
2^2 + 6^2 - 2 \cdot 2 \cdot 6 \cdot \cos\theta &= q_2^2 && \text{triangle with sides 2,6 and } q_2 \\
1^2 + 6^2 - 2 \cdot 1 \cdot 6 \cdot \cos\theta &= q_1^2 && \text{triangle with sides 1,6 and } q_1
\end{aligned}
$$

From the last four equations, we find that

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1^2 + 2^2 + 3^2 + 4^2 + 4 \cdot 6^2 - 2 \cdot 6 \cdot (1 + 2 + 3 + 4) \cdot \cos\theta$$

Solving for $\cos\theta$ in the first equation, and inserting that information in the previous identity, gives

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1^2 + 2^2 + 3^2 + 4^2 + 4 \cdot 6^2 - 2 \cdot 6 \cdot (1 + 2 + 3 + 4) \cdot \frac{5^2 + 6^2 - 7^2}{2 \cdot 5 \cdot 6} = 150. \ \square$$

Problem 24. Let m and n be two positive integers.

Statement A: $m^2 + n^2$ is divisible by 8.

Statement B: $m^3 + n^3$ is divisible by 16.

Which of the following must be true?

   (a) A is necessary but not sufficient for B.

   (b) A is not necessary but is sufficient for B.

   (c) A is necessary and sufficient for B.

   (d) A is neither necessary nor sufficient for B.

   (e) None of the above.

Answer: (b)

Solution: Let $m = 4k + x$ and $n = 4l + y$, where $0 \leq x, y \leq 3$ are integers. Then

$$m^2 + n^2 = (4k + x)^2 + (4l + y)^2 = x^2 + y^2 \mod 8,$$

So A is true if and only if $x^2 + y^2 = 0 \mod 8$ and it is easy to see that the latter holds if and only if $(x, y) = (0, 0)$ or $(x, y) = (2, 2)$. On the other hand,

$$m^3 + n^3 = (4k + x)^3 + (4l + y)^3 = 12kx^2 + 12ly^2 + x^3 + y^3 \mod 16,$$

So if $(x, y) = (0, 0)$ or $(x, y) = (2, 2)$, then B holds; that is, A implies B. However, there are other cases for
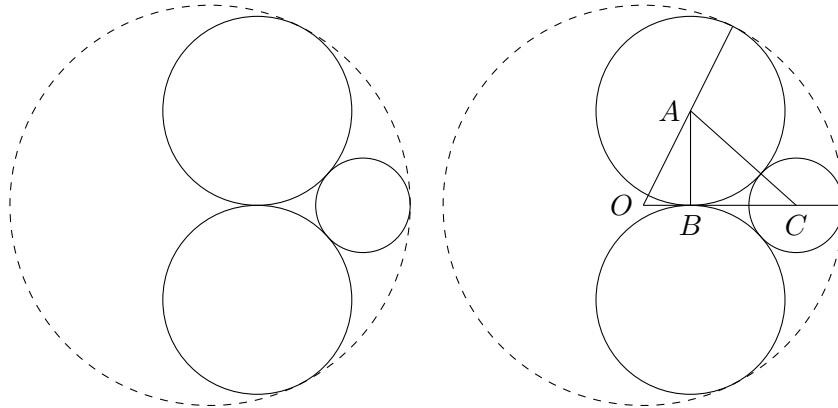
$$12kx^2 + 12ly^2 + x^3 + y^3 = 0 \mod 16,$$

such as $(k, l, x, y) = (0, 3, 1, 3)$. In fact, $1^3 + 15^3 = 3376 = 16 \cdot 211$ but $1^2 + 15^2 = 226 = 2 \cdot 113$. So the answer is: (b) A is not necessary but is sufficient for B. $\square$

Problem 25. Two circles of radius 1 and one circle of radius $1/2$ are drawn on a plane so that each of them is touching the other two at one point as shown below. What is the radius of the largest circle (dashed) tangent to all three of these circles?

(a) $1 + \frac{\sqrt{5}}{2}$ (b) $\sqrt{5}$ (c) $2(\sqrt{5} - 1)$ (d) $\frac{1}{3} + \sqrt{5}$ (e) $\frac{6}{5}\sqrt{5}$

Answer: (a)

Solution: In the diagram, label points O (the center of the largest circle), A (the center of one circle of radius 1), B (the intersection of the two circles of radius 1), and C (the center of the circle of radius $1/2$).

The points O and A are collinear with the point of tangency of the circle of radius 1 with the largest circle, and the radius can then be expressed as $OA + 1$.

The points O, B and C are collinear with the point of tangency of the circle of radius $1/2$ with the largest circle. The radius can also be expressed as $OB + BC + 1/2$.

Note that $AC = 1 + 1/2 = 3/2$; by the Pythagorean Theorem in triangle $\triangle ABC$, we have $BC = \sqrt{5}/2$.

This gives two equations with two variables:

$$OA + 1 = OB + \frac{1 + \sqrt{5}}{2} \qquad \text{the two radii have the same length}$$

$$OA^2 = OB^2 + 1 \qquad \text{Pythagorean Theorem on triangle } \triangle OAB$$

Solving for OA, for instance, gives us that $r = OA + 1 = 1 + \frac{\sqrt{5}}{2}$. $\square$

Problem 26. What is the value of the following product?

$$2^{2015} \cos\left(\tfrac{\pi}{4}\right) \cos\left(\tfrac{\pi}{8}\right) \cos\left(\tfrac{\pi}{16}\right) \cos\left(\tfrac{\pi}{32}\right) \cdots \cos\left(\tfrac{\pi}{2^{2015}}\right)$$

(a) $2^{-2015} \sin\left(\tfrac{\pi}{2^{2014}}\right)$ (b) $\tan\left(\tfrac{\pi}{2^{2014}}\right)$ (c) $2 \csc\left(\tfrac{\pi}{2^{2015}}\right)$ (d) $4 \sec\left(\tfrac{\pi}{2^{2014}}\right)$ (e) $2^{2014} \cot\left(\tfrac{\pi}{2^{2015}}\right)$

Answer: (c)

Solution: The key is to repeatedly use the equality:

$$\cos x = \frac{\sin 2x}{2 \sin x}$$

The product then equals:

$$2^{2015} \frac{\sin(\pi/2)}{2\sin(\pi/4)} \cdot \frac{\sin(\pi/4)}{2\sin(\pi/8)} \cdots \frac{\sin(\pi/2^{2014})}{2\sin(\pi/2^{2015})}$$

$$= \frac{2^{2015}}{2^{2014}} \frac{\sin(\pi/2)}{\sin(\pi/2^{2015})} = 2\frac{1}{\sin(\pi/2^{2015})} = 2 \csc\frac{\pi}{2^{2015}} \qquad \square$$

Problem 27. Let k be a positive integer. Let $\{a_1, a_2, \ldots, a_k\}$ be a set of integers that satisfies the following three conditions:

    (1) $0 < a_1 < 21$,   **(2)** $a_n < a_{n+1} < a_n + 11$, for $1 \leq n < k$,   **(3)** $a_k = 2015$.

Considering all possible choices of k and the set $\{a_1, a_2, \ldots, a_k\}$ as above, what is the smallest possible value of the sum $a_1 + a_2 + \cdots + a_k$?

(a) $2015 \cdot 100$ (b) $1015 \cdot 201$ (c) $1010 \cdot 202$ (d) $2030 \cdot 101$ (e) $2030 \cdot 102$

Answer: (b)

Solution: We want to minimize the value of S. We have $a_k = 2015$. We will minimize S by taking a_{k-1} as small as possible, that is, $a_{k-1} = 2005$. Similarly we take $a_{k-2} = 1995$ and so on, until we get to $a_{k-200} = 15$. At this point a_{k-200} satisfies condition (1), so we can set $k = 201$ and make 15 the first term.

Therefore, the smallest possible value of S is

$$15 + 25 + 35 + \cdots + 2015.$$

There are 201 terms and their average value is $\frac{15+2015}{2} = 1015$, so the answer is (b). $\square$

Problem 28. If $a > 0$ and $b > 0$ are real numbers satisfying

$$\frac{1}{a} + \frac{1}{b} \leq 2\sqrt{2} \quad \text{and} \quad (a-b)^2 = 4(ab)^3,$$

then what is the value of $\log_a b$?

(a) -2 \qquad\qquad (b) -1 \qquad\qquad (c) 0 \qquad\qquad (d) 1 \qquad\qquad (e) 2
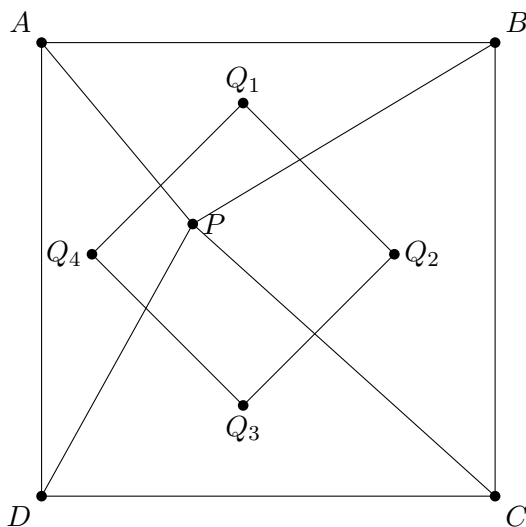
Answer: (b)

Solution: From $\frac{1}{a} + \frac{1}{b} \leq 2\sqrt{2}$, we have $0 < a + b \leq 2\sqrt{2}ab$, then $0 < (a+b)^2 \leq 8(ab)^2$. So we get $4(ab)^3 = (a-b)^2 = (a+b)^2 - 4ab \leq 8(ab)^2 - 4ab$.

Now let us focus on $4(ab)^3 \leq 8(ab)^2 - 4ab$. Since $ab > 0$, we may divide both sides of previous inequality by $4ab$ and rearrange the inequality, giving $(ab)^2 - 2ab + 1 \leq 0$ and then $(ab - 1)^2 \leq 0$, which can only be true if $ab = 1$. So $b = a^{-1}$, i.e., $\log_a b = -1$.

To see that there really are such a and b, replace $\leq$ with $=$ in the now-simplified formula $(a + b) \leq 2\sqrt{2}$ and use $b = 1/a$ to get $a^2 + 1 = 2\sqrt{2}a$. Now use the quadratic formula to get the two solutions $a = \sqrt{2} + 1, b = \sqrt{2} - 1$ and $a = \sqrt{2} - 1, b = \sqrt{2} + 1$. It is easy to check that each solution satisfies the two formulas in the problem. $\qquad\square$

Problem 29. Let P be a point in a square $ABCD$. Dissect the square with the four triangles $\triangle PAB$, $\triangle PBC$, $\triangle PCD$ and $\triangle PDA$. Let Q_1, Q_2, Q_3 and Q_4 be the respective centroids of these triangles. It is a fact that $Q_1Q_2Q_3Q_4$ forms another square.

Find Area$(Q_1Q_2Q_3Q_4)/$ Area$(ABCD)$.

(a) $\sqrt{2}/5$ \qquad (b) $1/4$ \qquad (c) $2/9$ \qquad (d) $1/3$ \qquad (e) it depends on the location of P

Answer: (c)

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Solution: The point Q_1 can be expressed as $(A+B+P)/3$, and the point Q_2 can be expressed as $(B+C+P)/3$. In that case, the vector $\overrightarrow{Q_1Q_2}$ can be expressed as $(A-C)/3$; that is, a vector parallel to the diagonal of the square, with length one-third of said diagonal. The same argument applies to all the other three sides of the polygon $Q_1Q_2Q_3Q_4$, thus proving that it is indeed a square, and the ratio to the area of the largest square is $(\sqrt{2}/3)^2 = 2/9$. □

Problem 30. Four prisoners are numbered 1 through 4. They are informed by the jail warden that each of them, in turn, will be taken to a room with four boxes labeled 1 through 4.

The numbers 1 through 4 are written on four slips of paper, one number per slip. These slips are placed into the boxes at random, one slip per box.

Each prisoner may look inside at most two of the boxes. If *all* of the prisoners see their own number, then all of them will be pardoned. If *any* prisoner does not see his/her own number, then all of the prisoners will be executed.

The prisoners may freely talk and coordinate a strategy beforehand, but once they begin they have no way of communicating with each other (including by adjusting the boxes, flipping the lights on or off, etc.)

Assuming the prisioners use an optimal strategy, what is the probability that the prisoners will go free?

(a) 1/16   (b) 1/9   (c) 3/8   (d) 5/12   (e) 1/2

Answer: (d)

Solution: The answer is $\frac{5}{12}$. The prisoners should adopt the following strategy. First, each should look in the box matching his or her own number. If that doesn't contain the slip of paper with the prisoner's number, then the prisoner should look in the box matching the slip of paper he/she did see.

There are 24 possible ways in which the slips of paper can be ordered, and the prisoners survive in any of the following scenarios: 1234, 1243, 1324, 1432, 2134, 2143, 3214, 3412, 4231, 4321. The cases where the number in the first box is 1 are easy to determine. If n is in the first box and $n \neq 1$, then 1 must be in the nth box or all will die. Then, the other two slips of paper may be switched or in their respective boxes, and in either case the prisoners will live. Therefore, there are two possibilities for each $n > 1$.) Therefore the prisoners have a $\frac{5}{12}$ chance of survival by adopting this strategy.

Now, we need to prove that we cannot do any better. (Of course, coming up with the above strategy makes 5/12 an extremely good guess, so under time pressure you might prefer to skip this step.)

Regardless of strategy, there is a 1/4 chance that prisoner 1 finds his/own number on the first attempt, and a 1/4 chance on the second attempt. In particular, 1/2 is an absolute upper bound for the probability of any strategy working!

Moreover, suppose that prisoner 1 finds his/her number on the first attempt. Then, the fact that prisoner 1 lives implies nothing about the positions of slips 2, 3, and 4, beyond the fact that none of them are in the first box. These three slips are equally likely to be in any permutation. Therefore, Prisoner 2 has at most a 2/3 chance of finding his/her own number.

Therefore, the probability that prisoners 1 and 2 both live is at most $\frac{1}{4} \times \frac{2}{3}$ (the scenario described above) plus $\frac{1}{4}$ (the probability that Prisoner 1 finds his/her number on the second attempt), which is $\frac{5}{12}$. We can therefore conclude that the strategy above is optimal. □