High School Math Contest

University of South Carolina

February 4th, 2017

Problem 1. If $(x-y)^2 = 121$ and $(x+y)^2 = 169$, what is xy?

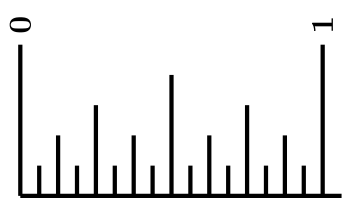
(a) 11 (b) 12 (c) 13 (d) 24 (e) 48

Problem 2. Suppose the function $g(x) = f(x) + x^3$ is even and $f(-10) = 2017$. What is the value of $f(10)$?

(a) -2017 (b) 17 (c) 1000 (d) 2017 (e) None of the above

Problem 3. A 12 inch ruler has tick marks every sixteenth of an inch: at each inch (including at the beginning and the end) it has a tick mark which is $\frac{5}{10}$ inches long; at each half inch a tick mark $\frac{4}{10}$ inches long; at each quarter inch a tick mark $\frac{3}{10}$ inches long; at each eighth inch a tick mark $\frac{2}{10}$ inches long; at each sixteenth inch a tick mark $\frac{1}{10}$ inches long.

The following diagram illustrates the first portion of the ruler:



What is the total length of all the tick marks on the 12 inch ruler?

(a) $\frac{371}{10}$ (b) $\frac{372}{10}$ (c) $\frac{373}{10}$ (d) $\frac{376}{10}$ (e) $\frac{377}{10}$

Problem 4. If you know that $f(x)$ is a quadratic polynomial with $f(2) = 8$, $f(3) = 15$, and $f(4) = 26$, what is the sum of the coefficients of f?

(a) 1 (b) 3 (c) 4 (d) 5 (e) 6

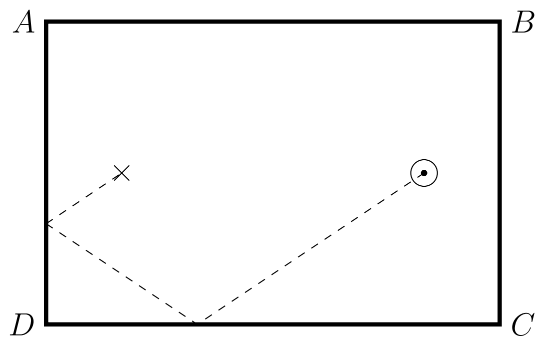
Problem 5. An equilateral triangle and a regular hexagon have perimeters of the same length. If the area of the triangle is 2 square units, what is the area of the hexagon?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 6

Problem 6. A real number θ satisfies $\cos\theta = \tan\theta$. What is the value of $\dfrac{1}{\sin\theta} + \cos^4\theta$?

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

Problem 7. A white billiard ball is on a 4 foot by 6 foot billiard table, with its center one foot away from a four foot wall, and in the middle of the two longer walls. You aim to reach the position symmetrically opposite.



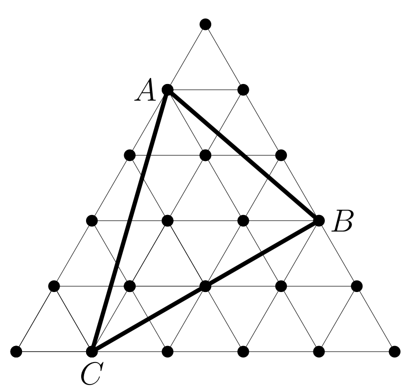
You hit the white ball so that it travels in a straight line to side $\overline{CD}$, bounces off this side so that the angle of reflection equals the angle of incidence, travels in another straight line to side $\overline{AD}$, bounces off this side so that the angle of reflection again equals the angle of incidence, and again travels in a straight line, now to its final position with its center exactly one foot from the opposite side of the table and in the middle of the two longer sides.

How far in feet does the center of the white ball travel from its initial position to its final position?

(a) $3 + 2\sqrt{3}$ (b) $4\sqrt{3}$ (c) $2\sqrt{13}$ (d) $2\sqrt{14}$ (e) 7.5

Problem 8. In the figure below, the large equilateral triangle is formed by 25 smaller equilateral triangles, each with an area of 1cm^2. What is the area of triangle $\triangle ABC$ in cm^2?

(a) 10 (b) 10.5 (c) 11 (d) 11.25 (e) 11.5



Problem 9. How many positive integer n satisfy the inequality $1! + 2! + 3! + 4! + n! \leq n^3$?

(a) 1 (b) 2 (c) 4 (d) 5 (e) None of the above

Problem 10. Suppose a and b are two different real numbers, and the function $f(x) = x^2 + ax + b$ satisfies $f(a) = f(b)$. What is the value of $f(2)$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Problem 11. For $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, let $A = (\cos\theta)^{\cos\theta}$, $B = (\sin\theta)^{\cos\theta}$, $C = (\cos\theta)^{\sin\theta}$. Which of the following is true?

(a) $A < B < C$ (b) $A < C < B$ (c) $B < A < C$ (d) $B < C < A$ (e) $C < A < B$

Problem 12. Find $a \cdot b$, given that

$$a = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{\cdots}}}}$$

$$b = \sqrt{9 - \sqrt{9 - \sqrt{9 - \sqrt{\cdots}}}}$$

(a) $\frac{3}{2}(\sqrt{26} - 2)$ (b) $\frac{3}{2}(\sqrt{37} - 1)$ (c) $4\sqrt{5} - 2\sqrt{2}$ (d) $3\sqrt{6} + 1$ (e) None of the above

Problem 13. How many integers n satisfy $n^4 + 6n < 6n^3 + n^2$?

(a) 3 (b) 4 (c) 5 (d) 6 (e) None of the above

Problem 14. What is the range of the following function?

$$f(x) = \frac{x+1}{x^2+1}$$

(a) $[\frac{1-\sqrt{2}}{4}, \frac{1+\sqrt{2}}{4}]$ (b) $[\frac{1-\sqrt{2}}{3}, \frac{1+\sqrt{2}}{3}]$ (c) $[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}]$ (d) $[1 - \sqrt{2}, 1 + \sqrt{2}]$ (e) None of the above

Problem 15. Let n be the smallest integer such that the sum of the digits of the sum of the digits of n is 11. What is the sum of the digits of the sum of the digits of $n + 1$?

(a) 3 (b) 4 (c) 7 (d) 8 (e) 12

Problem 16. Let a, b, c, and d be integers satisfying

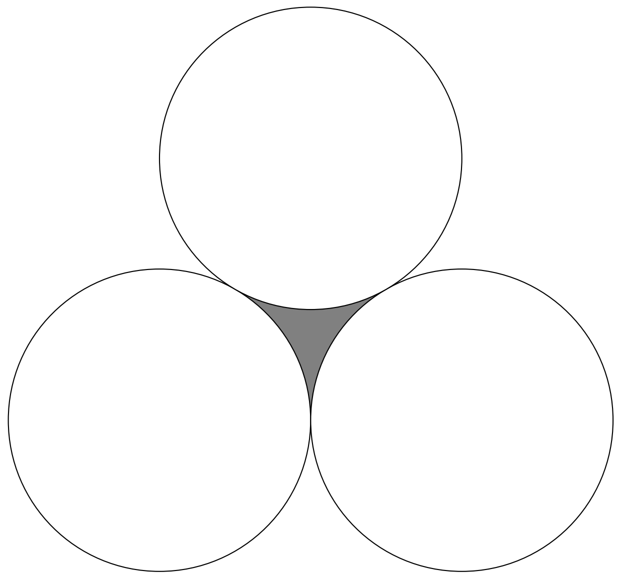
$$a \log_{10} 2 + b \log_{10} 3 + c \log_{10} 5 + d \log_{10} 7 = 2017.$$

What is $a + b + c + d$?

(a) 2016 (b) 2017 (c) 5012 (d) 5034 (e) None of the above

Problem 17. Given three circles of radius 2, tangent to each other as shown in the following diagram, what is the area for the shaded region?

(a) $\sqrt{3}-\frac{\pi}{2}$ (b) $4\sqrt{3}-2\pi$ (c) $\frac{4\sqrt{3}-\pi}{3}$ (d) $2\sqrt{6}-\pi$ (e) None of the above



Problem 18. What is the value of

$$\log_2(27) \cdot \log_3(25) \cdot \log_5(32)$$

to the nearest integer?

(a) 23 (b) 29 (c) 30 (d) 32 (e) 36

Problem 19. In this problem, a and b represent real numbers. Given that the maximum value and the minimum value of the function $f(t) = a \sin t + b \cos t$ differ by exactly 10, what is the largest possible value of $a + b$?

(a) $4\sqrt{2}$ (b) 6 (c) 7 (d) $5\sqrt{2}$ (e) None of the above

Problem 20. There are 10 sticks in a bag. The length of each stick is an integer (measured in inches). It is not possible to make a triangle out of any three sticks from that bag. What is the shortest length (in inches) the longest stick could possibly be?

(a) 11 (b) 19 (c) 55 (d) 101 (e) 1023

Problem 21. Triangle $\triangle ABC$ has sides $\overline{AB}$ of length 6, $\overline{BC}$ of length 3, and $\overline{AC}$ of length 5. The angle bisector of $\angle C$ intersects $\overline{AB}$ in point D. What is the length of the line segment $\overline{AD}$?

(a) 3 (b) $\frac{13}{4}$ (c) $\frac{7}{2}$ (d) $\frac{15}{4}$ (e) 4

Problem 22. For each positive integer n, write $\sigma(n)$ for the sum of the positive divisors of n. As n ranges through all positive integers up to 100, what is the largest value of $\sigma(n)$?

(a) 217 (b) 243 (c) 252 (d) 256 (e) 291

Problem 23. Let x, y and z be real numbers in $[0, 1]$. What is the maximum possible value of $\sqrt{|x-y|} + \sqrt{|y-z|} + \sqrt{|z-x|}$?

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2 (e) $1+\sqrt{2}$

Problem 24. The equation $x^3 - 4x^2 - 11x + a$ has three roots x_1, x_2, and x_3. If $x_1 = x_2 + x_3$, what is a?

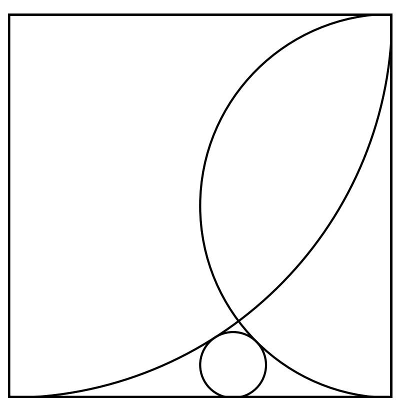
(a) 24 (b) 28 (c) 30 (d) 32 (e) 36

Problem 25. Suppose that $t = \sqrt{(x^2-2)^2 + (x-5)^2} + \sqrt{(x^2-3)^2 + x^2}$. As x ranges over all real numbers, what is the minimum value of t?

(a) $\sqrt{10}$ (b) $\sqrt{20}$ (c) $\sqrt{26}$ (d) $\sqrt{52}$ (e) There is no minimum

Problem 26. In a square with side length 1, a quarter circle is drawn with center at the top left corner and passing through the top right and bottom left corners, and a half circle is drawn with the right side of the square as its diameter. (See the accompanying diagram.)

A smaller circle is drawn tangent to the quarter circle, the half circle, and the bottom edge (as shown). Determine its radius.



(a) $\frac{9}{5} - \sqrt{3}$ (b) $\frac{3}{2} - \sqrt{2}$ (c) $3 - 2\sqrt{2}$ (d) $\frac{\sqrt{2}}{2} - \frac{1}{2}$ (e) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

Problem 27. What is the value of $\cos^2 10^\circ + \cos^2 50^\circ - \sin 40^\circ \sin 80^\circ$?

(a) $-\frac{3}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$ (e) $\frac{3}{4}$

Problem 28. What is the sum of all integers n that satisfy $\frac{1}{4} < \sin\frac{\pi}{n} < \frac{1}{3}$?

(a) 19 (b) 21 (c) 30 (d) 33 (e) 36

Problem 29. A standard die is rolled until a six rolls. Each time a six does not roll, a fair coin is tossed and a running tally of the number of heads minus the number of tails tossed is kept.

For example, if the die rolls are 5, 2, 1, 6 and the tosses are H, H, T, then the running tally is 1, 2, 1.

Find the probability that the absolute value of this running tally never equals 3.

(a) $\frac{289}{414}$ (b) $\frac{7}{10}$ (c) $\frac{81}{112}$ (d) $\frac{3}{4}$ (e) $\frac{5}{6}$

Problem 30. In $\triangle ABC$, if we have $\sin A = 2017 \sin B \sin C$ and $\cos A = 2017 \cos B \cos C$, what is the value of $\tan A$?

(a) 1 (b) $\frac{2018}{2017}$ (c) 2017 (d) 2018 (e) $2017\frac{\sqrt{2}}{2}$