

# High School Math Contest

University of South Carolina

January 31st, 2015

**Problem 1.** The figure below depicts a rectangle divided into two perfect squares and a smaller rectangle. If the dimensions of this smallest rectangle are proportional to those of the largest rectangle, and the squares each have side length 1, what is the length of a long side of the largest rectangle?



- (a)  $2\sqrt{3} - 1$       (b)  $1 + \sqrt{2}$       (c)  $\frac{1+\sqrt{5}}{2}$       (d)  $\frac{\sqrt{5}-1}{2}$       (e)  $8(\sqrt{3} - \sqrt{2})$

**Problem 2.** It takes three lumberjacks three minutes to saw three logs into three pieces each. How many minutes does it take six lumberjacks to saw six logs into six pieces each? Assume each cut takes the same amount of time, with one lumberjack assigned to each log.

- (a) 3      (b) 6      (c)  $7\frac{1}{2}$       (d) 12      (e) 15

**Problem 3.** What is the number of points  $(x, y)$  at which the parabola  $y = x^2$  intersects the graph of the function  $y = 1/(1 + x^2)$ ?

- (a) 0      (b) 1      (c) 2      (d) 3      (e) 4

**Problem 4.** Consider the set  $A = \{a_1, a_2, a_3, a_4\}$ . If the set of all possible sums of any three different elements from  $A$  is the set  $B = \{-1, 3, 5, 8\}$ , then what is the set  $A$ ?

- (a)  $\{-1, 2, 3, 5\}$       (b)  $\{-3, -1, 0, 2\}$       (c)  $\{-3, 1, 2, 5\}$       (d)  $\{-3, 0, 2, 6\}$       (e)  $\{-1, 0, 2, 4\}$

**Problem 5.** Four cards are laid out in front of you. You know for sure that on one side of each card is a single number, and on the other side of each card is a single geometric shape. The same number or the same geometric shape might be found on more than one of these four cards.

You see (on the top side) respectively a 2, a 5, a triangle, and a square. Your friend says: "Every card with a square has a 4 on the other side." Your task is to determine whether your friend is correct by choosing some of the cards to be flipped over. The chosen cards will only be flipped after you have made your choice(s). What is the fewest number of cards you can choose if you want to know for sure whether or not your friend is correct?

- (a) 0                      (b) 1                      (c) 2                      (d) 3                      (e) 4

**Problem 6.** If in the formula  $y = Ax/(B + Cx)$  we have that  $x$  is positive and increasing, while  $A$ ,  $B$  and  $C$  are positive constants, then what happens to  $y$  as  $x$  increases?

- (a)  $y$  increases  
(b)  $y$  decreases  
(c)  $y$  remains constant  
(d)  $y$  increases, then decreases  
(e)  $y$  decreases, then increases

**Problem 7.** If the larger base of an isosceles trapezoid equals a diagonal and the smaller base equals an altitude, what is the ratio of the smaller base to the larger base?

- (a)  $2/5$                       (b)  $3/5$                       (c)  $2/3$                       (d)  $3/4$                       (e)  $4/5$

**Problem 8.** What is the maximum value of the following function?

$$f(x) = \frac{\sin^3 x \cos x}{\tan^2 x + 1}$$

- (a)  $1/8$                       (b)  $1/4$                       (c)  $1/3$                       (d)  $1/2$                       (e) 1

**Problem 9.** A fair die is rolled 6 times. Let  $p$  be the probability that each of the six faces on the die appears exactly once among the six rolls. Which of the following is correct?

- (a)  $p \leq 0.02$     (b)  $0.02 < p \leq 0.04$     (c)  $0.04 < p \leq 0.06$     (d)  $0.06 < p \leq 0.08$     (e)  $p > 0.08$

**Problem 10.** Starting with an equilateral triangle, you inscribe a circle in the triangle, and then inscribe an equilateral triangle inside the circle. You then repeat this process four more times, each time inscribing a circle and then an equilateral triangle in the smallest triangle constructed up to that point, so that you end up drawing five triangles in addition to the one you started with.

What is the ratio of the area of the largest triangle to the area of the smallest triangles?

- (a) 32                      (b) 243                      (c) 1024                      (d) 59049                      (e) 1048576

**Problem 11.** For  $x > 0$ , how many solutions does the equation  $\log_{10}(x + \pi) = \log_{10} x + \log_{10} \pi$  have?

- (a) 0                      (b) 1                      (c) 2                      (d) more than 2 but finitely many                      (e) infinitely many

**Problem 12.** What is the range of the following function?

$$f(x) = \frac{\sqrt{x^2 + 1}}{x - 1}$$

- (a)  $(-\infty, -1) \cup [-\frac{\sqrt{2}}{2}, +\infty)$   
(b)  $(-\infty, -1] \cup [\frac{\sqrt{2}}{2}, +\infty)$   
(c)  $(-\infty, -1] \cup (1, +\infty)$   
(d)  $(-\infty, -\frac{\sqrt{2}}{2}] \cup (1, +\infty)$   
(e)  $(-\infty, \frac{\sqrt{2}}{2}] \cup (1, +\infty)$

**Problem 13.** What is the units digit of  $2^{2015}$ ?

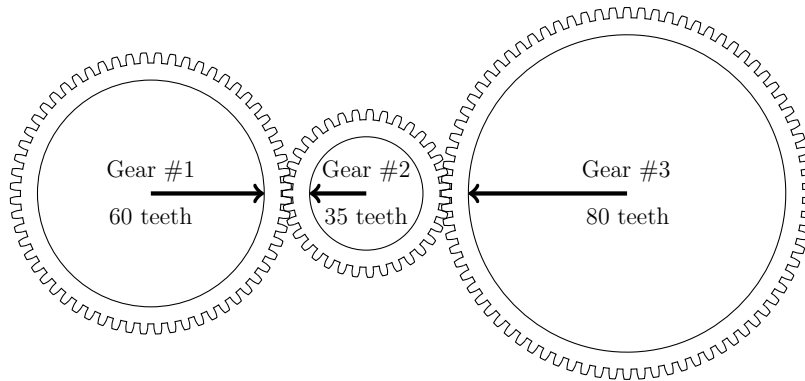
- (a) 0                      (b) 2                      (c) 4                      (d) 6                      (e) 8

**Problem 14.** Suppose that you answer every question on this test randomly. What is the probability that you will get every question wrong?

The answers below are not necessarily exact; choose the number which is closest to the exact probability.

- (a) 1/30                      (b) 0.0128                      (c) 0.00124                      (d) 0.0000321                      (e) 0.000000000000719

**Problem 15.** How many rotations of Gear #1 are required before all three gears return to the position shown, with the arrows lined up again and pointing in the same directions as before?



- (a) 28                      (b) 70                      (c) 175                      (d) 1680                      (e) 168000

**Problem 16.** If  $x^2 + xy + y^2 = 84$  and  $x - \sqrt{xy} + y = 6$ , then what is  $xy$ ?

- (a) 16                      (b) 25                      (c) 36                      (d) 49                      (e) 64

**Problem 17.** A bug is flying on a three-dimensional grid and wants to fly from  $(0, 0, 0)$  to  $(2, 2, 2)$ . It flies a distance of 1 unit at each step, parallel to one of the coordinate axes. How many paths can the bug choose which take only six steps?

- (a) 6                      (b) 24                      (c) 78                      (d) 90                      (e) 114

**Problem 18.** You and your partner went to a dinner party in which there were four other couples.

After the dinner was over, you asked everyone except yourself: “How many people did you shake hands with tonight?” To your surprise, no two people gave the same number, so that someone did not shake any hands, someone else shook only one person’s hand, a third person only shook two people’s hands, and so on.

Assume nobody shook hands with their own partner or with themselves. How many people did your partner shake hands with that evening at the party?

- (a) 0                      (b) 2                      (c) 3                      (d) 4                      (e) 6

**Problem 19.** The  $n$ -th string number,  $\text{string}(n)$ , is formed by writing the numbers 1 to  $n$  after each other in order. For instance,  $\text{string}(1) = 1$ ,  $\text{string}(2) = 12$ ,  $\text{string}(7) = 1234567$ , and  $\text{string}(12) = 123456789101112$ . What is the remainder when  $\text{string}(2015)$  is divided by 6?

- (a) 1                      (b) 2                      (c) 3                      (d) 4                      (e) 5

**Problem 20.** How many integer triples  $(x, y, z)$  satisfy the following equation?

$$x^2 + y^2 + z^2 = 2xyz$$

- (a) 0                      (b) 1                      (c) 2                      (d) 25                      (e) infinitely many

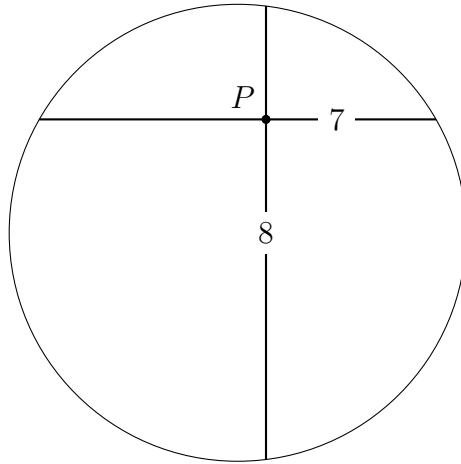
**Problem 21.** If  $f(x) = a + bx$ , what are the real values of  $a$  and  $b$  such that

$$f(f(f(1))) = 29,$$

$$f(f(f(0))) = 2?$$

- (a)  $a = 2/13, b = 3$    (b)  $a = 1, b = 3$    (c)  $a = 3, b = 2/13$    (d)  $a = 3, b = 1$    (e) there are none

**Problem 22.** Two perpendicular chords of a circle intersect at point  $P$ . One chord is 7 units long, divided by  $P$  into segments of length 3 and 4, while the other chord is divided into segments of length 2 and 6. What is the diameter of the circle?



- (a)  $\sqrt{56}$                       (b)  $\sqrt{61}$                       (c)  $\sqrt{65}$                       (d)  $\sqrt{75}$                       (e)  $\sqrt{89}$

**Problem 23.** In triangle  $\triangle ABC$ ,  $AB = 7$ ,  $BC = 5$ , and  $AC = 6$ . Locate points  $P_1, P_2, P_3$  and  $P_4$  on  $\overline{BC}$  so that the side is divided into 5 equal segments, each of length 1. Let  $q_k = AP_k$  for  $k \in \{1, 2, 3, 4\}$ . What is  $q_1^2 + q_2^2 + q_3^2 + q_4^2$ ?

- (a) 142                      (b) 150                      (c) 155                      (d) 160                      (e) 168

**Problem 24.** Let  $m$  and  $n$  be two positive integers.

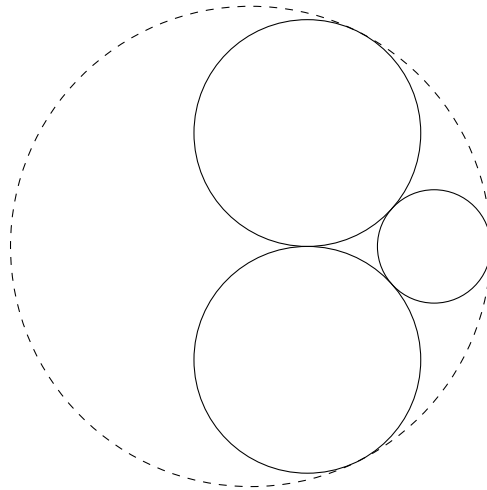
**Statement A:**  $m^2 + n^2$  is divisible by 8.

**Statement B:**  $m^3 + n^3$  is divisible by 16.

Which of the following must be true?

- (a) A is necessary but not sufficient for B.
- (b) A is not necessary but is sufficient for B.
- (c) A is necessary and sufficient for B.
- (d) A is neither necessary nor sufficient for B.
- (e) None of the above.

**Problem 25.** Two circles of radius 1 and one circle of radius  $1/2$  are drawn on a plane so that each of them is touching the other two at one point as shown below. What is the radius of the largest circle (dashed) tangent to all three of these circles?



- (a)  $1 + \frac{\sqrt{5}}{2}$
- (b)  $\sqrt{5}$
- (c)  $2(\sqrt{5} - 1)$
- (d)  $\frac{1}{3} + \sqrt{5}$
- (e)  $\frac{6}{5}\sqrt{5}$

**Problem 26.** What is the value of the following product?

$$2^{2015} \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{16}\right) \cos\left(\frac{\pi}{32}\right) \cdots \cos\left(\frac{\pi}{2^{2015}}\right)$$

- (a)  $2^{-2015} \sin\left(\frac{\pi}{2^{2014}}\right)$
- (b)  $\tan\left(\frac{\pi}{2^{2014}}\right)$
- (c)  $2 \csc\left(\frac{\pi}{2^{2015}}\right)$
- (d)  $4 \sec\left(\frac{\pi}{2^{2014}}\right)$
- (e)  $2^{2014} \cot\left(\frac{\pi}{2^{2015}}\right)$

**Problem 27.** Let  $k$  be a positive integer. Let  $\{a_1, a_2, \dots, a_k\}$  be a set of integers that satisfies the following three conditions:

- (1)  $0 < a_1 < 21$ ,      (2)  $a_n < a_{n+1} < a_n + 11$ , for  $1 \leq n < k$ ,      (3)  $a_k = 2015$ .

Considering all possible choices of  $k$  and the set  $\{a_1, a_2, \dots, a_k\}$  as above, what is the smallest possible value of the sum  $a_1 + a_2 + \dots + a_k$ ?

- (a)  $2015 \cdot 100$       (b)  $1015 \cdot 201$       (c)  $1010 \cdot 202$       (d)  $2030 \cdot 101$       (e)  $2030 \cdot 102$

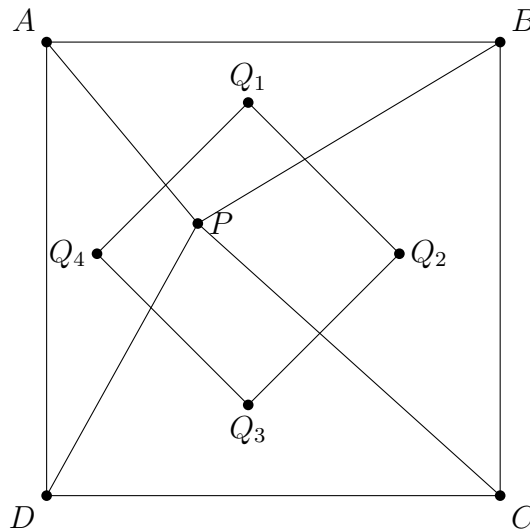
**Problem 28.** If  $a > 0$  and  $b > 0$  are real numbers satisfying

$$\frac{1}{a} + \frac{1}{b} \leq 2\sqrt{2} \quad \text{and} \quad (a - b)^2 = 4(ab)^3,$$

then what is the value of  $\log_a b$ ?

- (a)  $-2$       (b)  $-1$       (c)  $0$       (d)  $1$       (e)  $2$

**Problem 29.** Let  $P$  be a point in a square  $ABCD$ . Dissect the square with the four triangles  $\triangle PAB$ ,  $\triangle PBC$ ,  $\triangle PCD$  and  $\triangle PDA$ . Let  $Q_1, Q_2, Q_3$  and  $Q_4$  be the respective centroids of these triangles. It is a fact that  $Q_1Q_2Q_3Q_4$  forms another square.



Find  $\text{Area}(Q_1Q_2Q_3Q_4) / \text{Area}(ABCD)$ .

- (a)  $\sqrt{2}/5$       (b)  $1/4$       (c)  $2/9$       (d)  $1/3$       (e) it depends on the location of  $P$

**Problem 30.** Four prisoners are numbered 1 through 4. They are informed by the jail warden that each of them, in turn, will be taken to a room with four boxes labeled 1 through 4.

The numbers 1 through 4 are written on four slips of paper, one number per slip. These slips are placed into the boxes at random, one slip per box.

Each prisoner may look inside at most two of the boxes. If *all* of the prisoners see their own number, then all of them will be pardoned. If *any* prisoner does not see his/her own number, then all of the prisoners will be executed.

The prisoners may freely talk and coordinate a strategy beforehand, but once they begin they have no way of communicating with each other (including by adjusting the boxes, flipping the lights on or off, etc.)

Assuming the prisoners use an optimal strategy, what is the probability that the prisoners will go free?

(a)  $1/16$

(b)  $1/9$

(c)  $3/8$

(d)  $5/12$

(e)  $1/2$