1. Medium: The integer n is a solution to $8^n + 4^n + 2^n + 1 = 4368$. Then $\cos(\frac{n\pi}{3}) =$

   a) -1 b)$-\sqrt{2}/2$ c) $-1/2$ d) 0 e) $\sqrt{3}/2$

   Solution: c)

2. Easy: The smallest zero point of the function $f = t\cos(t) + \sqrt{3}t\sin(t), \quad t > 0$, is at

   a) $\pi/2$ b) $2\pi/3$ c) $3\pi/4$ d) $5\pi/6$ e) $11\pi/6$

   Solution: d)

$$f = \sqrt{4t^2}(\frac{t}{\sqrt{t^2+3t^2}}\cos(t) + \frac{\sqrt{3}t}{\sqrt{t^2+3t^2}}\sin(t)) = 2t(\sin(t+\frac{\pi}{6})).$$

   This function has zero at $5\pi/6, 11\pi/6$.

3. Medium: If $X + Y = 8, \quad \log_2(XY) = 2$. Then $Y =$

   a) $4 \pm 2\sqrt{3}$ b) $5 \pm 3\sqrt{2}$ c) $8 \pm 2\sqrt{3}$ d) $6 \pm 6\sqrt{2}$ e) $6 \pm 4\sqrt{5}$

   Solution: a)

$$2^x = X, 2^y = Y, 2^{x+y} + 2^{2y} = 8 \times 2^y \Rightarrow (2^y)^2 - 8(2^y) + 4 = 0 \Rightarrow 2^y = 4 \pm 2\sqrt{3}.$$

4. Medium: The sum $S = \sum_{i=1}^{14} \frac{i}{2^i} = \frac{1}{2} + \frac{2}{2^2} + ... + \frac{14}{2^{14}}$ is equal to

   a) 1 b) $\frac{255}{128}$ c) $\frac{1023}{512}$ d) $\frac{2047}{1024}$ e) $\frac{4095}{4096}$.

   Solution: d)

5. Medium: Find the first three correct decimal of the following expression

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}}$$

   a) 0.388 b) 0.395 c) 0.414 d) 0.424 e) 0.435

   Solution: c) $x \approx \frac{1}{2+x}$. $x^2 + 2x - 1 = 0$. So $x = \frac{-2\pm\sqrt{4+4}}{2} = \frac{-2\pm2\sqrt{2}}{2}$. So $x = -1 + \sqrt{2}$.

6. Medium: What is the sum of the squares of the integer solutions of the equation

$$x^{x^2-5x+6} = 1.$$

   a) 13 b) 14 c) 15 d) 16 e) 17

   Solution: c) Roots are $x = 1, 2, 3$, and -1.

7. Easy: Solve

$$6^{1/x} + 3 \cdot 9^{1/x} = 2 \cdot 4^{1/x}.$$

   (a) -3 (b) -2 (c) $-\frac{2}{3}$ (d) -1 (e) 0

   Solution: d)

8. Easy: Find the units digit of $5^{2023} + 6^{2023} + 9^{2023}$.
(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Answer: b) 5+6+1 = 12, so 2.

9. Medium: The equation

$$\frac{1}{a} + \frac{1}{b} = \frac{n}{a+b} \tag{1}$$

has a solution for some positive real numbers a, b if n is at least
(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution: e)

10. George wants to travel 2 miles at an average speed of 60 *mph*. George takes his time for the first mile and goes only 30 *mph*. How fast must he travel the second mile to average 60 *mph*?
(a) 90 *mph* (b) 105 *mph* (c) 120 *mph* (d) 150 *mph* (e) None of the above.

Solution: e) Answer: you can't. You already took 2 minutes to travel the first mile; you need to travel the second mile in no time at all. Not even the speed of list if fast enough!

11. Easy:

Enrollment in the math club of a high school attended by several girls and boys doubled after the sponsor started bringing pizza to each meeting, and attendance by girls jumped by 50%. Question: What can be said about the percentage of female students attending this club?
(a) Decreased by 25% (b) Stayed the same (c) Increased by 50% (d) Increased by 75% (e) Increased by 100%

Solution: a)

ANSWER: decreased 25%. Ask the same question about the boys and the answer is that it cannot be determined from the information given, I believe.

12. Easy: Find the product of the solutions for the equation:

$$25^{5x} = 125^{x^2+1}$$

a) -1 b) 0 c) $\frac{1}{3}$ d) 1 e) 3.

Solution: d)

13. Easy: The sum of the roots of $x^4 - 5x^2 + 4 = 0$ is
(a) -4 (b) -2 (c) 0 (d) 3 (e) 5

Solution: c.

14. Hard: Find the sum of the first 2023 terms of the sequence

$$\frac{\pi}{4}, \quad \arctan\left(\frac{1}{3}\right), \quad \arctan\left(\frac{1}{7}\right), \quad \ldots \quad , \quad \arctan\left(\frac{1}{n^2+n+1}\right), \quad \ldots$$

a) $\pi/3$ b) arctan(1011) c) arctan(2023) d) arctan(4044) e) $\pi/2$.

Solution: c)

$$\arctan\left(\frac{1}{n^2+n+1}\right) = \arctan(n+1) - \arctan(n). \tag{2}$$

15. Easy: What is

$$\left(e^{3\ln(x^2)}\right)^3?$$

a) $\ln(27x^2)$ b) x^6 c) $27x^6$ d) x^{18} e) e^{9x^2}.

Solution: d)

16. Medium: The $n!$ is the product of the first n natural numbers, i.e., $n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n$. Then

$$\frac{2023!}{2021! + 2022!}$$

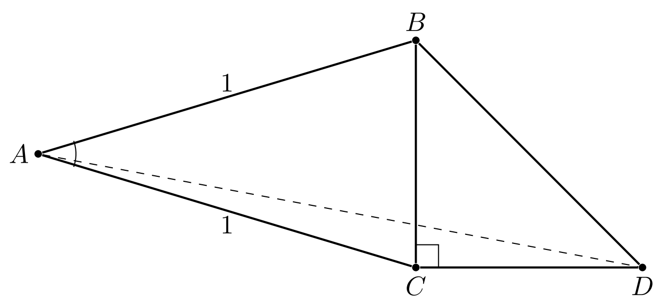
is equal to

a) 1 b) 2021 c) 2022 d) 2023 e) 4046.

Solution: c)

17. Hard: The ΔABC is isosceles with $AB = AC = 1$ and $\angle A = 30°$. Let ΔBCD be an isosceles right triangle with $\angle C = 90°$, not overlapping with ΔABC. Find the length of AD.

(a) $\frac{\sqrt{6}}{2}$ (b) $\sqrt{3} - 1$ (c) $\sqrt{4 - \sqrt{3}}$ (d) $\sqrt{4 + \sqrt{3}}$ (e) $\sqrt{3} + 1$



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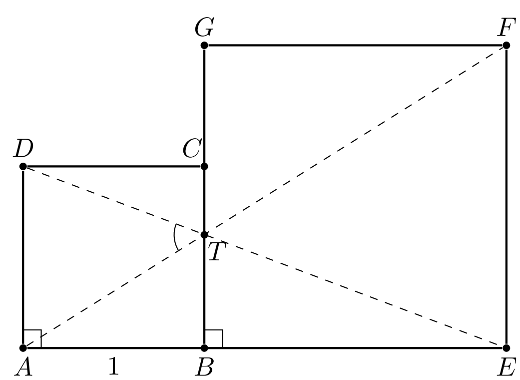
Solution: c)

18. Hard: Let $[ABCD]$ and $[BEFG]$ be squares with A, B and E colinear points in this order and C and F on the same side of the line AE, and let $AB = 1$. Find the maximum of the $\tan(\angle ATD)$, where T is at the intersection of AF and DE.

(a) 1 (b) 4/3 (c) $\sqrt{3}$ (d) 2 (e) 7/3



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Solution: b)

19. Medium: The year is 2023. A time traveler can jump into the future or the past only a number of years equal with a divisor of 2023. He wants to attend the Gettysburg Address of President Lincoln in 1863. What is the smallest number of jumps that he should take? We assume that there are no waiting time between consecutive jumps and the actual event.

    a) 2 b) 3 c) 4 d) 5 e) 6

    Solution: c)

20. A large software development company employs 100 computer programmers. Of them, 45 are proficient in Java, 30 in C#, 20 in Python, six in C# and Java, one in Java and Python, five in C# and Python, and just one programmer is proficient in all three languages above. Determine the number of computer programmers that are not proficient in any of these three languages.
    (a) 14 (b) 16 (c) 18 (d) 20 (e) 22

    Answer: (b) This is a standard inclusion/exclusion problem (or Venn diagram problem). The answer is 16.

21. Let $a = 799$ and $N = 80718$. Find a positive integer n so that the remainder of na, when divided by N, is equal to 1.
    (a) 1012 (b) 2023 (c) 2122 (d) 4043 (e) 4243

    Answer: e)

22. Easy: One night, a 6-feet tall person is standing 20 feet from a 30 feet high light. How long is the person's shadow?
    (a) 3 feet (b) 4 feet (c) 5 feet (d) 6 feet (e) 7 feet

    Answer: (c) Use similar triangles. Let x be the length of the person's shadow. Observe that

    $$\frac{x}{6} = \frac{x+20}{30}.$$

    It follows that the shadow is 5 feet long.

23. Hard: Suppose z_1, z_2, z_3 are the three roots of $z^3 - 18z - 8 = 0$. Simplify

    $$A = \frac{(z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2}{200}.$$

    (a) 108 (b) 150 (c) 216 (d) 312. (e) 427

    Solution: (a) **Siming: Here we set $q = 6, r = 4$ in the original problem. Please double check!**

    Original problem: Suppose z_1, z_2, z_3 are the three roots of $z^3 - 3qz - 2r = 0$. Simplify

    $$A = \frac{(z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2}{q^3 - r^2}.$$

    (a) 108 (b) $z_1^2 + z_2^2 + z_3^2$ (c) $z_1 z_2 z_3$ (d) $z_1 z_2 + z_2 z_3 + z_3 z_1$. (e) 27

    Possible solution: if you believe the fraction A is a constant you can let the roots $(z_1, z_2, z_3) = (-1, 0, 1)$ so that the polynomial equation is $z^3 - z^2 = 0$. Thus $q = \frac{1}{3}$ and $r = 0$. So $q^3 - r^2 = \frac{1}{27}$. Thus $(-1-0)^2(1-0)^2(1-(-1))^2 = 4 = A\frac{1}{27}$, so $A = 4(27) = 108$.

24. Medium: Which of the following is nearest to the value of

$$\sqrt{2021 \cdot 2023^2 \cdot 2025 + 4}\ ?$$

(a) $2023^2 - 2$ (b) $2023^2 - 1$ (c) 2023^2 (d) $2023^2 + 1$ (e) $2023^2 + 2$

Solution: The answer is (a). Let $u = 2023^2$. Then

$$2021 \cdot 2025 = (2023 - 2) \cdot (2023 + 2) = u - 4.$$

So

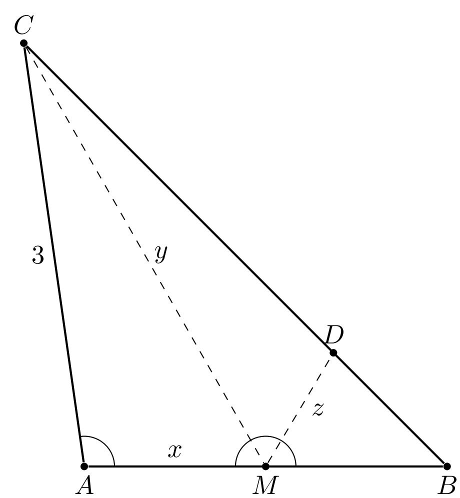
$$\sqrt{2021 \cdot 2023^2 \cdot 2025 + 4} = \sqrt{u(u-4) + 4} = \sqrt{u^2 - 4u + 4} = \sqrt{(u-2)^2} = u - 2.$$

Thus, the exact value of $\sqrt{2021 \cdot 2023^2 \cdot 2025 + 4}$ is $u - 2 = 2023^2 - 2$. ■

Comment. This problem becomes easier than intended if we ask for the exact value. In particular, (b) and (d) can be eliminated because they are odd and (c) can be eliminated fairly easily as well.

25. Medium:
The ΔABC has $AC = 3$ and $\angle A = 105°$. Let M be the midpoint of AB and $\angle AMC = \angle CMD = \angle BMD = 60°$, where D lies on BC. If $AM = x$, $CM = y$, and $DM = z$, find $xy + yz + zx$.

(a) $\dfrac{\sqrt{6} - \sqrt{2}}{2}$ (b) $\dfrac{\sqrt{6} + \sqrt{2}}{2}$ (c) 3 (d) $2\sqrt{3} + 2$ (e) 6



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Solution: e)