State High School Mathematics Tournament

Round 2 – University of South Carolina

February 3, 2018

Question 2-1

Given that

$$x + y + 2z = 3,$$
$$x + 2y + z = 4,$$
$$2x + y + z = 5,$$

what is $x + y + z$?

Solution 2-1

Answer. 3, with $x = 2$, $y = 1$, $z = 0$.

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Solution 2-1

Answer. 3, with $x = 2$, $y = 1$, $z = 0$.
Add all three equations to get

$$4x + 4y + 4z = 12,$$

and divide by 4.

A unique circle goes through the following three points:

Question 2-2

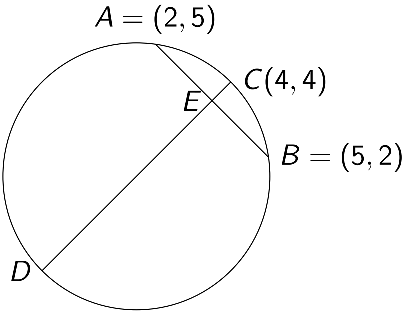
A unique circle goes through the following three points:

$$(2, 5), (4, 4), (5, 2).$$

What is its diameter?

Solution 2-2

Answer: $5\sqrt{2}$.



Solution 2-2

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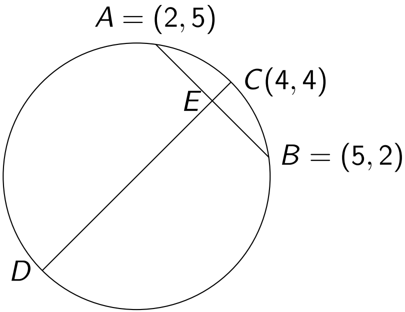

$\overline{AB} \perp \overline{CD}$ at $E = (3.5, 3.5)$, with $\overline{AE} = \overline{BE} = \frac{3}{2}\sqrt{2}$ and $\overline{CE} = \frac{1}{2}\sqrt{2}$.

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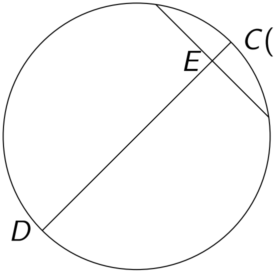

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$\overline{AE} \cdot \overline{BE} = \overline{CE} \cdot \overline{DE}$, so $\overline{DE} = \frac{9}{2}\sqrt{2}$.

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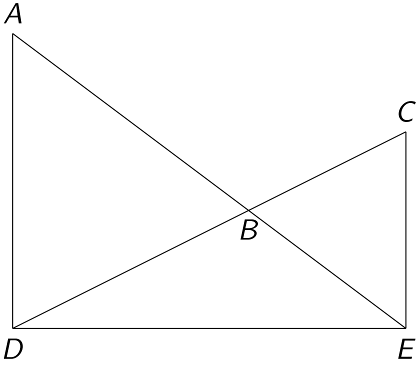
Question 2-3

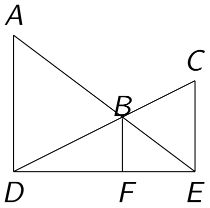
In the figure, $\overline{AD}$ and $\overline{CE}$ are perpendicular to $\overline{DE}$; $\overline{AD} = 5$, $\overline{DE} = 3$, and $\overline{CE} = 4$.
Find the area of $\triangle BDE$.



(not drawn to scale)

Solution 2-3

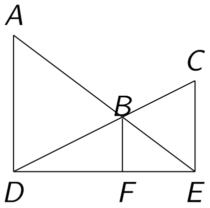
Answer: 10/3. Drop a perpendicular from B to DE:



We have $\frac{EF}{BF} = \frac{ED}{AD} = \frac{3}{5}$ and $\frac{DF}{BF} = \frac{DE}{CE} = \frac{3}{4}$.

Solution 2-3

Answer: 10/3. Drop a perpendicular from B to DE:



We have $\frac{EF}{BF} = \frac{ED}{AD} = \frac{3}{5}$ and $\frac{DF}{BF} = \frac{DE}{CE} = \frac{3}{4}$. So EF and DF are in a $4:5$ ratio, and since $DE = 3$ we have $EF = \frac{4}{3}$ and $DF = \frac{5}{3}$.

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Solution 2-3

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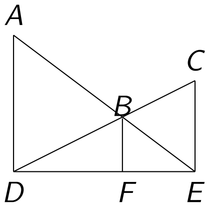

We have $\frac{EF}{BF} = \frac{ED}{AD} = \frac{3}{5}$ and $\frac{DF}{BF} = \frac{DE}{CE} = \frac{3}{4}$. So EF and DF are in a $4:5$ ratio, and since $DE = 3$ we have $EF = \frac{4}{3}$ and $DF = \frac{5}{3}$. So $BF = \frac{5}{3}EF = \frac{20}{9}$, and the area of $\triangle DBE$ is

$$\frac{1}{2} \cdot 3 \cdot \frac{20}{9} = \frac{10}{3}.$$

Hint. We have

$$10^{11} = 100000000000 = 23 \cdot 4347826087 - 1.$$

Question 2-4

Hint. We have

$$10^{11} = 100000000000 = 23 \cdot 4347826087 - 1.$$

The fraction $\frac{1}{23}$ can be written as a repeating decimal

$$\frac{1}{23} = 0.\overline{0434782608695652173913},$$

where the 22 digits under the bar repeat infinitely many times.

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$$10^{11} = 100000000000 = 23 \cdot 4347826087 - 1.$$

The fraction $\frac{1}{23}$ can be written as a repeating decimal

$$\frac{1}{23} = 0.\overline{0434782608695652173913},$$

where the 22 digits under the bar repeat infinitely many times. What is the sum of these 22 digits?

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Solution 2-4

Answer. 99.

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$$\frac{1}{23} = 0.\overline{04347826086{\color{blue}95652173913}},$$

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$$\frac{1}{23} = 0.\overline{0434782608695652173913},$$

$$\frac{22}{23} = 0.\overline{9565217391304347826086},$$

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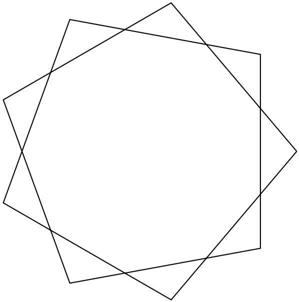
$$\frac{1}{23} = 0.\overline{0434782608695652173913},$$

$$\frac{22}{23} = 0.\overline{9565217391304347826086},$$

$$\frac{1}{23} + \frac{22}{23} = 0.\overline{9999999999999999999999}$$

Question 2-5

The following figure consists of nine line segments:



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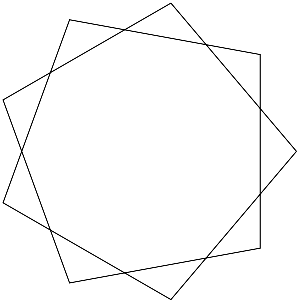

All of the triangles in the picture are congruent. What is the largest angle in any of these triangles?

Solution 2-5

Answer. $\frac{5}{9}\pi$ or 108°.

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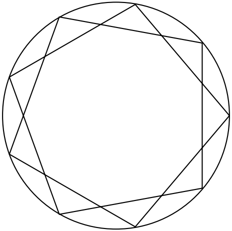
The figure is symmetric, and can be inscribed in a circle:



Each of these angles is subtended by an arc consisting of $\frac{5}{9}$ of the circle, hence of measure $\frac{5}{9} \cdot 2\pi$.

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Solution 2-5

Answer. $\frac{5}{9}\pi$ or 108°.
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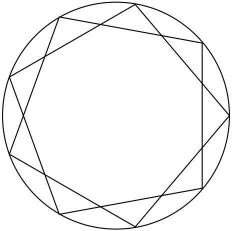

Each of these angles is subtended by an arc consisting of $\frac{5}{9}$ of the circle, hence of measure $\frac{5}{9} \cdot 2\pi$.

Oops! $\frac{5}{9}\pi = 100^\circ$. Fortunately, a student found and pointed out the mistake on the spot.

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How many digits are in the base 10 number 20^{18}?

Solution 2-6

Answer: 24.

Solution. We have

$$20^{18} = 262144000000000000000000,$$

which is 2^{18} with 18 zeroes after it.

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which is 2^{18} with 18 zeroes after it.

$$2^{18} = 2^{10}2^{8} = 1024 \cdot 256 \sim 1000 \cdot 250 = 250000,$$

with six digits, and $18 + 6 = 24$.

Question 7

What is the last digit of 3^{2018}?

Solution 7

Answer. 9.

Solution. Notice that $3^4 = 81$, with last digit 1.

Solution 7

Answer. 9.

Solution. Notice that $3^4 = 81$, with last digit 1. Since

$$3^{2018} = 3^{4 \cdot 504 + 2} = (81)^{504} \cdot 9,$$

the last digit of 3^{2018} is $1^{504} \cdot 9 = 9$.

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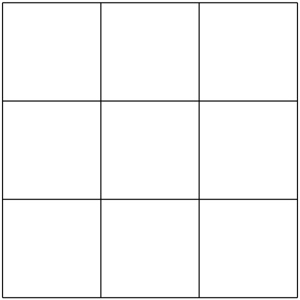
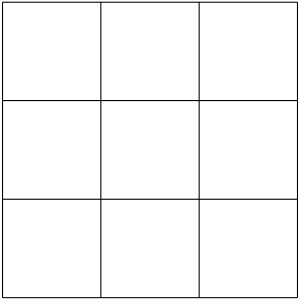
Question 8

Consider (again) a Rubik's cube, where each of the six faces has sixteen *corner points*, illustrated by the intersections of the line segments as follows:



Question 8

Consider (again) a Rubik's cube, where each of the six faces has sixteen *corner points*, illustrated by the intersections of the line segments as follows:



How many corner points are there on the cube total?

Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

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Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;
- 8 are shared with one other face, and $8 \cdot 3 = 24$;
- 4 are shared with two other faces, and $4 \cdot 2 = 8$.

Solution 8

Answer. 56.

Solution. On each face, there are 16 corner points. Of these:

- 4 are on that face alone, and $4 \cdot 6 = 24$;
- 8 are shared with one other face, and $8 \cdot 3 = 24$;
- 4 are shared with two other faces, and $4 \cdot 2 = 8$.

$$24 + 24 + 8 = 56.$$

Question 9

The squares of three consecutive positive integers are added, to obtain 770.
What is the smallest of these integers?

Solution 9

Answer. 15,

$$15^2 + 16^2 + 17^2 = 225 + 256 + 289 = 770.$$

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Solution 9

Answer. 15,

$$15^2 + 16^2 + 17^2 = 225 + 256 + 289 = 770.$$

Note that if n denotes the *middle* number, we have

$$(n-1)^2+n^2+(n+1)^2 = (n^2-2n+1)+n^2+(n^2+2n+1) = 3n^2+2,$$

so $3n^2 = 768$, $n^2 = 256$, and $n = 16$.

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Question 10

You flip two coins. One is fair; the other is weighted and is more likely to come up heads than tails.

If the probability of flipping at least one heads is 80%, what is the probability of flipping both heads?

Solution 10

Answer. $\frac{3}{10}$.

Solution. Let p be the probability that the weighted coin comes up heads.
The probability of flipping no heads is

$$\frac{1}{2}(1-p)=\frac{1}{5},$$

so $1-p=\frac{2}{5}$ and $p=\frac{3}{5}$. The probability of flipping two heads is thus

$$\frac{1}{2}\times\frac{3}{5}=\frac{3}{10}.$$

Question 11

What is

$$1 - 2 + 3 - 4 + 5 - \cdots + 2017 - 2018?$$

Solution 11

Answer. -1009. Write it as

$$(1-2)+(3-4)+(5-6)+\cdots+(2017-2018),$$

which is -1 added 1009 times.

Question 12

There are unique integers a and b for which

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$$(1+\sqrt{5})^3 = a + b\sqrt{5}.$$

What is $a+b$?

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Solution 12

Answer. 24.

Solution 12

Answer. 24. We have

$$(1+\sqrt{5})^3 = 1 + 3\sqrt{5} + 3(\sqrt{5})^2 + (\sqrt{5})^3 = 16 + 8\sqrt{5}.$$