Team Round – University of South Carolina Math Contest, 2018

1. This is a team round. You have one hour to solve these problems as a team, and you should submit one set of answers for your team as a whole. Working together is, of course, encouraged.

2. Submit one answer per question, on the attached sheet. If you change your answer, make sure your earlier answer is clearly crossed out.

3. All answers should be in the form of a number – simplified as much as possible.

   Use your best judgment – for example, 1.75, $\frac{7}{4}$, $\left(\frac{31}{18}\right)^7$, $\frac{\sqrt{2}}{2}$, and $\frac{1}{\sqrt{2}}$ will all be considered to be in simplified form. $\sin(\pi/4)$, $\frac{\sqrt{8}}{4}$, and $5 + \frac{9}{5}$ will not be.

4. The questions are difficult. Do not be discouraged if you don't get them right away.

5. The competition consists of two parts. The first part has 10 independent questions, and the second part has 5 questions on a single theme. The questions in the second part are related, and working on earlier questions will help you with later questions in this part.

   There is also a 'take home question'. It's not part of the competition, and it doesn't 'count' for anything. But you might be interested in trying to solve it after you get home. You may submit solutions to `thorne@math.sc.edu` for an evaluation.

6. Your room will have a whiteboard or chalkboard, which you are encouraged to use. Markers or chalk, erasers, and blank paper will be provided by the proctors. (You should bring your own pencil or pen.) Please let your proctor know if your markers don't work or if you need additional markers, chalk, or paper.

7. No calculators, books, notes, or other tools are allowed.

8. An exam proctor will either be stationed in your room, or will circulate and frequently stop by your room. Please let them know if you have any questions about the exam, or if you require additional materials.

GOOD LUCK!

Answer Sheet for Team Round

Your School/Team Name: ______________________________

1. ______________________

2. ______________________

3. ______________________

4. ______________________

5. ______________________

6. ______________________

7. ______________________

8. ______________________

9. ______________________

10. ______________________

11. ______________________

12. ______________________

13. ______________________

14. ______________________

15. ______________________

Part I.

1. Suppose that x, y, z are real numbers satisfying $x + y = 4$ and $|z + 1| = xy + 2y - 9$.

   What is $x + 2y + 3z$?

2. Let H be a regular hexagon with side length 1. Circles of radius 1 are drawn centered at each of the vertices of H.

   Let A be the subset of points of H which are in or on at least three of the circles. What is the area of A?

3. If n is a positive integer, $\sigma(n)$ denotes the sum of the positive integer divisors of n. For example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

   It is true that $\sigma(\sigma(88)) = \sigma(\sigma(90)) = 546$, and there is exactly one positive integer $n \leq 100$ for which $\sigma(\sigma(n)) > 546$. What is $\sigma(\sigma(n))$ for this value of n?

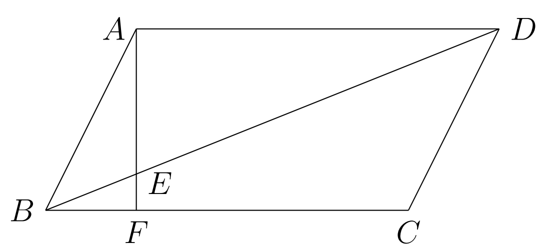
4. Simplify:
$$\frac{1}{\log_2(2018!)} + \frac{1}{\log_3(2018!)} + \frac{1}{\log_4(2018!)} + \cdots + \frac{1}{\log_{2018}(2018!)}.$$

5. Suppose that x and y are two real numbers satisfying $x + y = 3$ and $\frac{1}{x+y^2} + \frac{1}{x^2+y} = \frac{1}{2}$.

   What is $x^5 + y^5$?

6. In the (infinite) decimal expansion of $1/999998$, what is the 98th digit after the decimal point?

7. In the parallelogram $ABCD$ depicted below, we have $\angle ABC = 72°$, $AF \perp BC$, AF intersects BD at E, and $\overline{ED} = 2\overline{AB}$. Compute the measure of $\angle AED$.



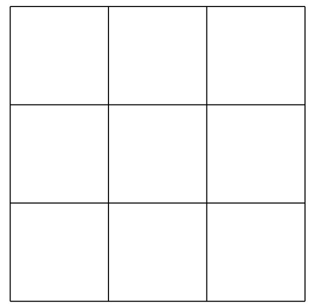
8. You and one opponent play a game as follows. There is a pile of pennies, and on your turn you must take exactly one, two, or six of them. The player forced to take the last penny loses.

   Call a positive integer n *good* if, when the pile has n pennies in it and it is your turn, you can force a win. For example, 7 is good: you may take 6 pennies and thereby force your opponent to take the last one.

   How many positive integers $n \leq 2018$ are good?

9. How many positive integers $n \leq 2018$ can be written in the form $a^3 + b^3$, where a and b are positive integers?

10. Consider a Rubik's cube, where each of the six faces has sixteen *corner points*, illustrated by the intersections of the line segments as follows:



How many distinct (nonzero) distances are there between pairs of corner points?

Part II. The following questions concern the game *Let 'em Roll*, which appears on the TV show *The Price Is Right*.

You are given five six-sided dice, and you can win up to \$7,500 in cash, or a \$15,000 car. Each die has a picture of a car on three faces, and has dollar amounts of \$500, \$1,000, and \$1,500 on the other three faces. At the end of the game, if all five dice have a car showing face up, you win the car. If any dice show a monetary amount, you win the total amount of money showing on the dice (with cars counting as nothing).

You get three rolls[1] of the dice. After the first and second rolls, you may either accept the results and end the game, or continue. If you continue, the dice showing a car are taken out of play, and you reroll only those dice showing money.

11. What is the probability you win the car on the first roll of the dice?

12. If you play for the car, what is the probability you win the car on the first or second roll of the dice?

13. If you play for the car, what is the probability you win it?

14. Suppose that after two rounds you have not rolled any cars, but all five dice show \$1,500. You give up the \$7,500 in prize money and reroll all five dice on the third round.

    What is the *expected value* of your roll – that is, the average amount you expect to win, counting the car as equivalent to \$15,000 in cash? *(Answer as an exact number of dollars and cents.)*

15. Let $S \subseteq \{0, 1, 2, 3, 4, 5\}$ be the set of integers n for which the following statement is true:

    Suppose that after two rounds you have rolled $5 - n$ dice showing cars, and n dice each showing \$1,500. If you choose to reroll the n dice showing \$1,500, then the expected value of your roll is larger than $1500n$.

    What is the sum of the elements of S?

Take Home Question. Don't work on this now, it's not part of the competition. If you solve it after you get home, you may submit your solution to `thorne@math.sc.edu` for an evaluation.

Describe and justify, as well as you can, an optimal or near-optimal strategy for playing this game. Assume that you value the car equivalently to \$15,000 cash, and play to maximize your expected value.

In particular, under what situations does it make sense not to reroll?

[1] On the actual game show, you get three rolls only if you price some small grocery items correctly. Assume that you do.