1. (20pts)
Suppose that $g : [a, b] \mapsto [a, b]$ is continuous on the real interval $[a, b]$ and is a contraction in the sense that there exists a constant $\lambda \in (0, 1)$ such that

$$|g(x) - g(y)| \leq \lambda |x - y|, \forall x, y \in [a, b]$$  \hspace{1cm} (1)

Prove that there exists a unique fixed point in $[a, b]$ and that the fixed point iteration $x_{n+1} = g(x_n)$ converges to it for any $x_0 \in [a, b]$. And prove that the error is reduced by a factor of at least $\lambda$ from each iteration to the next.

2. (20pts)
(a) Construct Polynomial $p_3$ that passes the points $(1, 2), (2, 10), (-1, -3)$

(b) Construct Polynomial $p_3$ that passes the points $(1, 0), (2, -3)$ and $p_3'(1) = 1, p_3'(2) = -1$.

(c) Construct orthogonal Polynomial of degree 0, 1, 2 on the interval $(0, 1)$ with the weight $w(x) = -\ln(x)$.

3. (20pts)
(1) Determine the nodes and weights in the 2-node Gaussian quadrature formula

$$\int_0^\infty f(x)e^{-x}dx = w_1f(x_1) + w_2f(x_2)$$ \hspace{1cm} (2)

(2) Construct the polynomial of best approximation of degree 2 in the $L^2$ norm to the function $f(x) = \sin x$ over $[-\pi/2, \pi/2]$. 
4. (20pts) 

\[ T_k = \begin{pmatrix} a_1 & b_2 & \cdots & a_k \\ c_2 & a_2 & b_3 & \cdots \\ c_3 & a_3 & \ddots & \cdots \\ \vdots & \vdots & \ddots & b_k \\ c_k & a_k & \cdots & a_1 \end{pmatrix} \]

The characteristic polynomial of \( T_k \) is given by \( p_k(\lambda) = \det(\lambda I - T_k) \)

(1) Define \( p_k \) in terms of \( p_{k-1} \) and \( p_{k-2} \).

(2) Show that if \( c_j b_j > 0 \) for \( j = 2, \cdots, k \), then \( p_k(\lambda) = 0 \) only has real roots. (hint: find a real similarity transformation that symmetrizes \( T_k \)).

5. (20pts) (1) Prove: \( n \times n \) matrix is normal if and only if it has \( n \) orthonormal eigenvectors.

(2) Prove that if the matrix \( A \) has orthogonal columns, then \( Ax = b \) has a unique least squares solution, and find this unique solution.

6. (20pts) Use finite difference method to find the coefficient matrix for Poisson equation of 1D

\[ -\frac{d^2v(x)}{dx^2} = f(x), 0 < x < 1 \]

with boundary condition \( v(0) = v(1) = 0 \);

and 2D Poisson equation

\[ -\frac{d^2v(x,y)}{dx^2} - \frac{d^2v(x,y)}{dy^2} = f(x,y), 0 < x < 1, 0 < y < 1 \]

with boundary condition \( v(x,y) = 0 \) on the boundary square.