1. Given the following data for a function \( f : \mathbb{R} \to \mathbb{R} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-1</td>
<td>2</td>
<td>15</td>
</tr>
</tbody>
</table>

(a) Construct the quadratic interpolation polynomial \( p_2(x) \) which interpolates the data.
(b) If the function being interpolated was in fact \( f(x) = x^3 + 2x^2 - 1 \), derive a tight upper bound on the error in using \( p_2(x) \) as an approximation to \( f(x) \) on \([0, 2]\).

2. This problem concerns orthogonal polynomials and Gaussian quadratures.
(a) Find \( \{p_0, p_1, p_2\} \) such that \( p_i \) is a polynomial of degree \( i \) and these polynomials are orthogonal to each other on \([0, \infty)\) with respect to the weight function \( w(x) = e^{-x} \).
(b) Find the points and weights \( \{(x_i, w_i)\}_{i=1}^{2} \) of the 2-point Gaussian quadrature

\[
\int_{0}^{\infty} f(x) e^{-x} dx \approx w_1 f(x_1) + w_2 f(x_2).
\]

3. Consider the following Runge-Kutta method for solving the initial value problem \( y' = f(t, y), y(0) = y_0 \) where \( h \) is the time step size:

\[
y_{n+1} = y_n + \alpha hf(t_n, y_n) + \frac{h}{2} f(t_n + \beta h, y_n + \beta hf(t_n, y_n)).
\]

(a) For what values of \( \{\alpha, \beta\} \) is the method consistent?
(b) For what values of \( \{\alpha, \beta\} \) is the method stable?
(c) For what values of \( \{\alpha, \beta\} \) is the method most accurate?

4. Consider the 3-step Adams-Bashforth method,

\[
y_{n+1} = y_n + h \left[ \frac{23}{12} f(t_n, y_n) - \frac{4}{3} f(t_{n-1}, y_{n-1}) + \frac{5}{12} f(t_{n-2}, y_{n-2}) \right]
\]

for solving the initial value problem \( y' = f(t, y), y(0) = y_0 \).
(a) Derive this method based on \( y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y) dt \) and the polynomial interpolation approximation of \( f \) on \( t_n, t_{n-1}, t_{n-2} \).
(b) Determine the order of accuracy of this linear multistep method.
(c) Is the method convergent? Justify your answer.

5. This problem concerns condition numbers and system stability.
(a) Let \( A \) be an \( n \times n \) nonsingular matrix. We consider the solution of the linear
system $Ax = b$. Suppose we have an approximate solution $x^*$ to the exact solution $x$ of this system, and let $r = b - Ax^*$ be the residual. Prove

$$\frac{\|x - x^*\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}$$

where $\| \cdot \|$ is any vector norm, and $\kappa(A)$ is the condition number of $A$ with respect to the induced matrix norm.

(b) For the matrix

$$A = \begin{bmatrix} 5.4 & 0.6 & 2.2 \\ 0.6 & 6.4 & 0.5 \\ 2.2 & 0.5 & 4.7 \end{bmatrix},$$

compute an upper bound for the condition number $\kappa_2(A)$, using the estimates of the eigenvalues by the Gershgorin Circle Theorem.

6. Prove that every Hermitian, positive definite matrix $A$ (i.e., $x^*Ax > 0$ for all $x \neq 0$) has a unique Cholesky factorization (i.e., $A = R^*R$ with $r_{jj} > 0$).

7. Compute one step of the QR algorithm (for computing eigenvalues) with the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

(a) Without shift.
(b) With shift $\mu = 1$.

8. Let $A$ be a real symmetric positive definite matrix and given a linear system of equations $Ax = b$. Consider an iterative solution strategy of the form

$$x_{k+1} = x_k + \alpha_k r_k,$$

where $x_0$ is arbitrary, $r_k = b - Ax_k$ is the residual and $\alpha_k$ is a scalar parameter to be determined.

(a) Derive an expression for $\alpha_k$ such that $\|r_{k+1}\|_2$ is minimized as a function of $\alpha_k$. Is this expression always well-defined and nonzero?

(b) Show that with this choice

$$\frac{\|r_k\|_2}{\|r_0\|_2} \leq \left(1 - \frac{\lambda_{\min}(A)}{\lambda_{\max}(A)}\right)^{k/2}$$

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimal and maximal eigenvalues of $A$ respectively.