Throughout this examination the term measurable refers to the Lebesgue measure $m$ on the real line. Integrals with respect to Lebesgue measure will be denoted by $\int f$. Problems 1-8 are worth 10 points each. Each part of problem 9 is worth 4 points.

1. The function $\Omega : C[0,1] \to C[0,1]$ is defined by

$$\Omega(\phi)(x) := \int_0^x \phi(t)\,dt \quad (\phi \in C[0,1], x \in [0,1]).$$

Prove that $\Omega$ is not a contraction mapping, but that $\Omega^2 := \Omega \circ \Omega$ is a contraction mapping.

2. Let $E$ be a compact set in a metric space $(X, \rho)$. Prove that there are points $a, b \in E$ such that

$$\rho(a, b) = \sup_{x, y \in E} \rho(x, y).$$

3. Does the series

$$\sum_{n=1}^{\infty} \frac{x}{n + n^3 x^3}$$

converge uniformly on $[0, \infty)$?

4. Let $f$ be measurable on $[0, 1]$.

(i) Prove that the condition

$$(1) \quad \lim_{k \to \infty} km(\{x : |f(x)| > k\}) = 0$$

is a necessary condition for $f \in L_1([0, 1])$.

(ii) Give an example showing that (1) is not a sufficient condition for $f \in L_1([0, 1])$.

5. Let $f$ be integrable on $[0, 2]$. Prove that

$$\lim_{h \to 0^+} \int_0^1 |f(x + h) - f(x)|\,dx = 0.$$
6. Let \( \{f_n\} \) be a sequence of nonnegative measurable functions on \((-\infty, \infty)\) such that \( f_n \to f \) a.e., and suppose that \( \int f_n \to \int f < \infty \). Prove that for each measurable set \( E \) we have \( \int_E f_n \to \int_E f \).

7. If \( \gamma \) is the positively oriented unit circle, compute

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{e^z - e^{-z}}{z^4} \, dz.
\]

8. Prove Fatou’s Lemma: If \( \{f_n\} \) is a sequence of nonnegative measurable functions and \( f_n(x) \to f(x) \) almost everywhere on a measurable set \( E \), then

\[
\int_E f \leq \liminf_{n \to \infty} \int_E f_n.
\]

9. Mark each of the following statements as True or False. In order to obtain points you have to provide proofs or counterexamples to justify your answers.

(a) Let \( \{x_n\} \) be a sequence of positive numbers. Show that

\[
\limsup_{n \to \infty} (x_1 \cdots x_n)^{1/n} \leq \limsup_{n \to \infty} x_n.
\]

(b) The trigonometric series

\[
\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}
\]

is the Fourier series of a continuous \( 2\pi \)-periodic function.

(c) Let \( f \) be absolutely continuous on \([0, 1]\) with \( f' \in L_p([0, 1]), \ 1 < p < \infty \). There is a constant \( C \) such that

\[
|f(b) - f(a)| \leq C|b - a|^{1/p}
\]

for all \( a, b \in [0, 1] \).

(d) Let \( f \) be a function which is regular (holomorphic) on a closed disc \( D \), (i.e. regular on a region of the complex plane which contains the closed disc \( D \)). If \( |f| \) is constant on the boundary of \( D \) then \( f \) is constant.

(e) There exist a non-empty compact set \( A \), and a non-empty closed set \( B \) of the complex plane such that \( A \cap B = \emptyset \) and \( \inf\{|a - b| : a \in A, \ b \in B\} = 0 \).